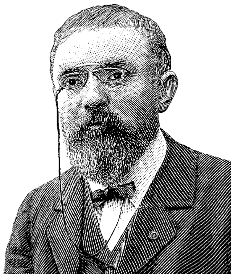
Volume 53, Number 3, 2017
ISSN 0246-0203



# ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES

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# PROBABILITÉS ET STATISTIQUES





Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques (ISSN 0246-0203), Volume 53, Number 3, August 2017. Published quarterly by Association des Publications de l'Institut Henri Poincaré.

POSTMASTER: Send address changes to Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998 USA.





Président et directeur de la publication : Cédric Villani
Périodicité : 4 n$^{os}$ / an

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1007–1050
DOI: 10.1214/16-AIHP744


# On U- and V-statistics for discontinuous Itô semimartingales

Mark Podolskij[a], Christian Schmidt[a] and Mathias Vetter[b]

[a]*Department of Mathematics, Aarhus University, Ny Munkegade 118, 69120 Aarhus, Denmark.*
*E-mail: mpodolskij@math.au.dk; cschmidt@math.au.dk*
[b]*Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Straße 4, 24118 Kiel, Germany.*
*E-mail: vetter@math.uni-kiel.de*

**Abstract.** In this paper we examine the asymptotic theory for U-statistics and V-statistics of discontinuous Itô semimartingales that are observed at high frequency. For different types of kernel functions we show laws of large numbers and associated stable central limit theorems. In most of the cases the limiting process will be conditionally centered Gaussian. The structure of the kernel function determines whether the jump and/or the continuous part of the semimartingale contribute to the limit.

**Résumé.** Dans cet article, nous étudions la théorie asymptotique de U-statistiques et de V-statistiques pour des semimartingales d'Itô discontinues qui sont observées à haute fréquence. Pour différents types de fonctions de noyaux, nous montrons des lois des grands nombres et des théorèmes de la limite centrale vers des lois stables. Dans la majorité des cas, le processus limite est conditionnellement centré Gaussien. La structure du noyau détermine si le la partie de sauts et/ou la partie continue de la semimartingale contribue à la limite.

*MSC:* Primary 60F05; 62F12; secondary 60G48; 60H05

*Keywords:* High frequency data; Limit theorems; Semimartingales; Stable convergence; U-statistics

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1051–1068
DOI: 10.1214/16-AIHP747


# On estimating the perimeter using the alpha-shape

Ery Arias-Castro[a] and Alberto Rodríguez-Casal[b]

[a]*Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, San Diego, CA 92093-0112, USA.*
*url: http://math.ucsd.edu/~eariasca*
[b]*Departamento de Estatística e Investigación Operativa, Facultade de Matemáticas, Universidade de Santiago de Compostela, Rúa Lope Gómez de Marzoa, s/n. Campus sur, 15782, Santiago de Compostela, A Coruña, Spain.*
*url: http://eio.usc.es/pub/alberto/*

**Abstract.** We consider the problem of estimating the perimeter of a smooth domain in the plane based on a sample from the uniform distribution over the domain. We study the performance of the estimator defined as the perimeter of the alpha-shape of the sample. Some numerical experiments corroborate our theoretical findings.

**Résumé.** Nous considérons le problème de l'estimation du périmètre d'un domaine à bord lisse dans le plan basé sur un échantillon tiré de la loi uniforme ayant pour support le domaine en question. Nous étudions la performance de l'estimateur défini par le périmètre de la forme-alpha (« alpha-shape ») de l'échantillon. Des expériences numériques confirment notre théorie.

*MSC:* 62G99; 60D05

*Keywords:* Perimeter estimation; $\alpha$-shape; $r$-convex hull; Rolling condition; Sets with positive reach

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1069–1107
DOI: 10.1214/16-AIHP748


# Horton self-similarity of Kingman's coalescent tree

Yevgeniy Kovchegov[a,1] and Ilya Zaliapin[b,2]

[a]Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA. E-mail: kovchegy@math.oregonstate.edu
[b]Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557-0084, USA. E-mail: zal@unr.edu

**Abstract.** The paper establishes Horton self-similarity for a tree representation of Kingman's coalescent process. The proof is based on a Smoluchowski-type system of ordinary differential equations that describes evolution of the number of branches of a given Horton–Strahler order in a tree that represents Kingman's $N$-coalescent, in a hydrodynamic limit. We also demonstrate a close connection between the combinatorial Kingman's tree and the combinatorial level set tree of a white noise, which implies Horton self-similarity for the latter.

**Résumé.** Cet article prouve l'auto-similarité à la Horton pour la représentation par arbres du processus de coalescence de Kingman. La preuve est basée sur un système d'équations différentielles ordinaires de type Smoluchowski décrivant, dans la limite hydrodynamique, l'évolution du nombre de branches d'un ordre de Horton–Strahler donné dans un arbre représentant le $N$-coalescent de Kingman. Nous prouvons aussi un lien étroit entre l'arbre de Kingman combinatoire et l'arbre combinatoire des ensembles de niveaux d'un bruit blanc, ce qui implique l'auto-similarité à la Horton de ce dernier.

*MSC:* Primary 60C05; secondary 82B99

*Keywords:* Coalescent; Kingman's coalescent; Horton–Strahler order; Horton self-similarity

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1108–1135
DOI: 10.1214/16-AIHP749


# The spans in Brownian motion

Steven Evans[1], Jim Pitman and Wenpin Tang

*Department of Statistics, University of California at Berkeley, 367 Evans Hall, Berkeley CA 94720-3860, USA.*
*E-mail: evans@stat.berkeley.edu; pitman@stat.berkeley.edu; wenpintang@stat.berkeley.edu*

**Abstract.** For $d \in \{1, 2, 3\}$, let $(B_t^d; t \geq 0)$ be a $d$-dimensional standard Brownian motion. We study the *$d$-Brownian span set* $\text{Span}(d) := \{t - s; B_s^d = B_t^d \text{ for some } 0 \leq s \leq t\}$. We prove that almost surely the random set $\text{Span}(d)$ is $\sigma$-compact and dense in $\mathbb{R}_+$. In addition, we show that $\text{Span}(1) = \mathbb{R}_+$ almost surely; the Lebesgue measure of $\text{Span}(2)$ is 0 almost surely and its Hausdorff dimension is 1 almost surely; and the Hausdorff dimension of $\text{Span}(3)$ is $\frac{1}{2}$ almost surely. We also list a number of conjectures and open problems.

**Résumé.** Pour $d \in \{1, 2, 3\}$, soit $(B_t^d; t \geq 0)$ un mouvement brownien standard $d$-dimensionnel. Nous étudions le *$d$-ensemble de portée brownienne* $\text{Span}(d) := \{t - s; B_s^d = B_t^d \text{ pour certains } 0 \leq s \leq t\}$. Nous prouvons que presque sûrement l'ensemble aléatoire $\text{Span}(d)$ est $\sigma$-compact et dense dans $\mathbb{R}_+$. De plus, nous montrons que $\text{Span}(1) = \mathbb{R}_+$ presque sûrement ; la mesure de Lebesgue de $\text{Span}(2)$ est 0 presque sûrement et sa dimension de Hausdorff est 1 presque sûrement ; et la dimension de Hausdorff de $\text{Span}(3)$ est $\frac{1}{2}$ presque sûrement. Nous listons aussi un certain nombre de conjectures et problèmes ouverts.

*MSC:* 28A78; 60J65

*Keywords:* Brownian span set; Random set; Energy method; Fractal projection; Hausdorff dimension; Multiple point; Self-intersection; Local time; Self-similar

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1136–1163
DOI: 10.1214/16-AIHP750


# Ergodicity for multidimensional jump diffusions with position dependent jump rate

Eva Löcherbach[a] and Victor Rabiet[b]

[a]*CNRS UMR 8088, Département de Mathématiques, Université de Cergy-Pontoise, France. E-mail: eva.loecherbach@u-cergy.fr*
[b]*LAMA - Laboratoire d'Analyse et de Mathématiques Appliquées, France. E-mail: victor.rabiet@ens-cachan.fr*

**Abstract.** We consider a jump type diffusion $X=(X_t)_t$ with infinitesimal generator given by

$$L\psi(x)=\frac{1}{2}\sum_{1\le i,j\le d}a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\,\partial x_j}+g(x)\nabla\psi(x)+\int_{\mathbb{R}^d}\big(\psi\big(x+c(z,x)\big)-\psi(x)\big)\gamma(z,x)\mu(\mathrm{d}z),$$

where $\mu$ is of infinite total mass. We prove Harris recurrence of $X$ using a regeneration scheme which is entirely based on the jumps of the process. Moreover we state explicit conditions in terms of the coefficients of the process allowing to control the speed of convergence to equilibrium in terms of deviation inequalities for integrable additive functionals.

**Résumé.** On considère une diffusion $X=(X_t)_t$, avec des sauts, correspondant au générateur infinitésimal suivant :

$$L\psi(x)=\frac{1}{2}\sum_{1\le i,j\le d}a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\,\partial x_j}+g(x)\nabla\psi(x)+\int_{\mathbb{R}^d}\big(\psi\big(x+c(z,x)\big)-\psi(x)\big)\gamma(z,x)\mu(\mathrm{d}z)$$

où $\mu$ est de masse totale infinie. On prouve ici la récurrence au sens de Harris de $X$ en utilisant un schéma de régénération entièrement basé sur les sauts du processus. De plus, on donnera des conditions explicites en terme de coefficients du processus $X$ permettant de contrôler la vitesse de convergence à l'équilibre en terme d'inégalités de déviations pour des fonctionnelles additives intégrables.

*MSC:* 60J55; 60J35; 60F10; 62M05

*Keywords:* Diffusions with jumps; Harris recurrence; Nummelin splitting; Continuous time Markov processes; Additive functionals

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1164–1180
DOI: 10.1214/16-AIHP751


# Transience in growing subgraphs via evolving sets

Amir Dembo[a], Ruojun Huang[b], Ben Morris[c] and Yuval Peres[d]

[a]*Department of Mathematics, Stanford University, Building 380, Sloan Hall, Stanford, CA 94305, USA*
[b]*Department of Statistics, Stanford University, Sequoia Hall, 390 Serra Mall, Stanford, CA 94305, USA*
[c]*Department of Mathematics, University of California at Davis, One Shields Ave, Davis, CA 95616, USA*
[d]*Theory Group, Microsoft Research, One Microsoft Way, Redmond, WA 98052, USA*

**Abstract.** We extend the use of random evolving sets to time-varying conductance models and utilize it to provide tight heat kernel upper bounds. It yields the transience of any uniformly lazy random walk, on $\mathbb{Z}^d$, $d \geq 3$, equipped with uniformly bounded above and below, independently time-varying edge conductances, of (effectively) non-decreasing in time vertex conductances, thereby affirming part of Conjecture 7.1 (Random walk in changing environment (2015) Preprint).

**Résumé.** Nous généralisons la méthode basée sur l'évolution aléatoire d'ensembles au cas de modèles de conductances variant avec le temps. Nous l'utilisons pour prouver des bornes supérieures sur le noyau de la chaleur. Ceci montre la transitivité de n'importe quelle marche aléatoire fainéante, dans $\mathbb{Z}^d$, $d \geq 3$, avec des conductances par arêtes (bornées uniformément supérieurement et inférieurement) variant indépendamment en temps en fonction des conductances par sites. Ceci répond partiellement à la Conjecture 7.1 (Random walk in changing environment (2015) Preprint).

*MSC:* Primary 60J10; secondary 60K37; 60K35

*Keywords:* Transience; Time in-homogeneous Markov chains; Heat kernel estimate; Growing sub-graphs; Conductance models; Evolving sets; Percolation

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1181–1195
DOI: 10.1214/16-AIHP752


# Overcrowding asymptotics for the $\mathrm{Sine}_\beta$ process

Diane Holcomb[a] and Benedek Valkó[b]

[a]Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA. E-mail: dianeholcomb@math.arizona.edu
[b]Department of Mathematics, University of Wisconsin – Madison, Madison, WI 53706, USA. E-mail: valko@math.wisc.edu

**Abstract.** We give overcrowding estimates for the $\mathrm{Sine}_\beta$ process, the bulk point process limit of the Gaussian $\beta$-ensemble. We show that the probability of having exactly $n$ points in a fixed interval is given by $e^{-\frac{\beta}{2}n^2\log(n)+\mathcal{O}(n^2)}$ as $n\to\infty$. We also identify the next order term in the exponent if the size of the interval goes to zero.

**Résumé.** Nous obtenons des résultats asymptotiques pour le surpeuplement du processus $\mathrm{Sine}_\beta$, le processus ponctuel limite dans le milieu du spectre de l'ensemble $\beta$-gaussien. Nous montrons que la probabilité d'observer $n$ points dans un interval fixé est donné par la formule $e^{-\frac{\beta}{2}n^2\log(n)+\mathcal{O}(n^2)}$ quand $n\to\infty$. Nous obtenons aussi une approximation jusqu'à l'ordre suivant lorsque la longueur de l'interval tend vers 0.

*MSC:* 60B20; 60F10; 15B52

*Keywords:* $\beta$-ensembles; Random matrices; Overcrowding

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1196–1240
DOI: 10.1214/16-AIHP753


# Long time dynamics and disorder-induced traveling waves in the stochastic Kuramoto model

E. Luçon[a] and C. Poquet[b,c]

[a]*Laboratoire MAP5 (UMR CNRS 8145), Université Paris Descartes, Sorbonne Paris Cité, F-75270 Paris, France.*
*E-mail: eric.lucon@paridescartes.fr*
[b]*Dipartimento di Matematica, Università di Roma Tor Vergata, I-00133 Roma, Italia*
[c]*Institut Camille Jordan (UMR CNRS 5208), Université Lyon 1, Université de Lyon, F-69622 Villeurbanne, France.*
*E-mail: poquet@math.univ-lyon1.fr*

**Abstract.** The aim of the paper is to address the long time behavior of the Kuramoto model of mean-field coupled phase rotators, subject to white noise and quenched frequencies. We analyse the influence of the fluctuations of both thermal noise and frequencies (seen as a disorder) on a large but finite population of $N$ rotators, in the case where the law of the disorder is symmetric. On a finite time scale $[0, T]$, the system is known to be self-averaging: the empirical measure of the system converges as $N \to \infty$ to the deterministic solution of a nonlinear Fokker–Planck equation which exhibits a stable manifold of synchronized stationary profiles for large interaction. On longer time scales, competition between the finite-size effects of the noise and disorder makes the system deviate from this mean-field behavior. In the main result of the paper we show that on a time scale of order $\sqrt{N}$ the fluctuations of the disorder prevail over the fluctuations of the noise: we establish the existence of disorder-induced traveling waves for the empirical measure along the stationary manifold. This result is proved for fixed realizations of the disorder and emphasis is put on the influence of the asymmetry of these quenched frequencies on the direction and speed of rotation of the system. Asymptotics on the drift are provided in the limit of small disorder.

**Résumé.** Le but de ce travail est d'étudier le comportement en temps long du modèle de Kuramoto, défini par un système de rotateurs en interaction de type champ-moyen, perturbé par un bruit blanc et possédant des fréquences aléatoires gelées. Nous analysons l'influence des fluctuations induites par le bruit et les fréquences (vues comme un désordre pour le modèle) sur une population de $N$ rotateurs ($N$ grand mais fini), dans le cas où la loi du désordre est symétrique. Sur un intervalle de temps borné $[0, T]$, le système est auto-moyennant: la mesure empirique du système converge pour $N \to \infty$ vers la solution déterministe d'une équation de Fokker–Planck non linéaire possédant une variété stable de solutions stationnaires synchronisées pour une interaction suffisamment grande. Sur une échelle de temps plus grande, les effets de taille finie dûs à la présence du bruit et du désordre induisent une déviation macroscopique du système par rapport à ce comportement de champ-moyen. Le résultat principal de cet article montre que, sur une échelle de temps d'ordre $\sqrt{N}$, les fluctuations induites par le désordre l'emportent sur celles données par le bruit: nous montrons que le désordre induit l'existence de fronts pour la dynamique de la mesure empirique se propageant le long de la variété stationnaire. Ce résultat est valide pour une réalisation gelée du désordre. L'accent est mis sur l'influence de l'asymétrie des fréquences sur la direction et la vitesse de propagation du front et nous donnons une asymptotique de cette vitesse dans la limite de faible désordre.

*MSC:* 60K35; 37N25; 82C26; 82C31; 82C44; 92B20

*Keywords:* Kuramoto synchronization model; Mean-field particle systems; Disordered models; Nonlinear Fokker–Planck PDE; Long time dynamics; Traveling waves; Stochastic partial differential equations

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1241–1279
DOI: 10.1214/16-AIHP754


# Spectra of nearly Hermitian random matrices

Sean O'Rourke[a] and Philip Matchett Wood[b,1]

[a]*Department of Mathematics, University of Colorado at Boulder, Boulder, CO 80309, USA. E-mail:* *sean.d.orourke@colorado.edu*
[b]*Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Dr., Madison, WI 53706, USA. E-mail:* *pmwood@math.wisc.edu*

**Abstract.** We consider the eigenvalues and eigenvectors of matrices of the form $\mathbf{M}+\mathbf{P}$, where $\mathbf{M}$ is an $n \times n$ Wigner random matrix and $\mathbf{P}$ is an arbitrary $n \times n$ deterministic matrix with low rank. In general, we show that none of the eigenvalues of $\mathbf{M}+\mathbf{P}$ need be real, even when $\mathbf{P}$ has rank one. We also show that, except for a few outlier eigenvalues, most of the eigenvalues of $\mathbf{M}+\mathbf{P}$ are within $n^{-1}$ of the real line, up to small order corrections. We also prove a new result quantifying the outlier eigenvalues for multiplicative perturbations of the form $\mathbf{S}(\mathbf{I}+\mathbf{P})$, where $\mathbf{S}$ is a sample covariance matrix and $\mathbf{I}$ is the identity matrix. We extend our result showing all eigenvalues except the outliers are close to the real line to this case as well. As an application, we study the critical points of the characteristic polynomials of nearly Hermitian random matrices.

**Résumé.** Nous considérons les valeurs et les vecteurs propres de matrices de la forme $\mathbf{M}+\mathbf{P}$, où $\mathbf{M}$ est une matrice de Wigner $n \times n$ et $\mathbf{P}$ est une matrice arbitraire déterministe $n \times n$ de rang petit. Nous montrons que, génériquement, aucune des valeurs propres de $\mathbf{M}+\mathbf{P}$ n'est réelle, même quand $\mathbf{P}$ a rang un. Nous montrons aussi que, sauf pour un petit nombre d'exceptions, la plupart des valeurs propres de $\mathbf{M}+\mathbf{P}$ sont à distance au plus $n^{-1}$ de la droite réelle, à des corrections d'ordre petit près. Nous montrons aussi un nouveau résultat qui quantifie les valeurs propres exceptionnelles pour des perturbations multiplicatives de la forme $\mathbf{S}(\mathbf{I}+\mathbf{P})$, où $\mathbf{S}$ est une matrice de covariance empirique et $\mathbf{I}$ est la matrice identité. Nous étendons à ce cas notre résultat montrant que toutes les valeurs propres sauf les valeurs propres exceptionnelles sont proches de la droite réelle. Comme application, nous étudions les points critiques du polynôme caractéristique de matrices aléatoires presque hermitiennes.

*MSC:* 60B20

*Keywords:* Random matrices; Perturbation; Random eigenvalues; Random eigenvectors; Wigner matrices; Sample covariance matrices; Outlier eigenvalues

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1280–1304
DOI: 10.1214/16-AIHP755


# Affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$ and multiparameter time changes

M. Emilia Caballero[a], José Luis Pérez Garmendia[b] and Gerónimo Uribe Bravo[a]

[a] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, Coyoacán, 04510. Ciudad de México, México. E-mail: marie@matem.unam.mx; geronimo@matem.unam.mx; url: www.matem.unam.mx/geronimo*

[b] *Departamento de Probabilidad y Estadística, IIMAS, Universidad Nacional Autónoma de México, Apartado Postal 20-126, 01000, Ciudad de México, México. E-mail: garmendia@sigma.iimas.unam.mx*

**Abstract.** We present a time change construction of affine processes with state-space $\mathbb{R}_+^m \times \mathbb{R}^n$. These processes were systematically studied in (*Ann. Appl. Probab.* **13** (2003) 984–1053) since they gather interesting classes of processes such as Lévy processes, continuous-state branching processes with immigration, and of the Ornstein–Uhlenbeck type. The construction is based on a (basically) continuous functional of a multidimensional Lévy process which implies that limit theorems for Lévy processes (both almost surely and in distribution) can be inherited to affine processes. The construction can be interpreted as a multiparameter time change scheme or as a (random) ordinary differential equation driven by discontinuous functions. In particular, we propose approximation schemes for affine processes based on the Euler method for solving the associated discontinuous ODEs, which are shown to converge.

**Résumé.** Nous présentons une construction des processus affines à valeurs dans $\mathbb{R}_+^m \times \mathbb{R}^n$ à partir de changement de temps. Ces processus ont été systématiquement étudiés dans (*Ann. Appl. Probab.* **13** (2003) 984–1053) car ils regroupent certaines classes intéressantes de processus tels que les processus de Lévy, les processus de branchement continu avec immigration et du type Ornstein–Uhlenbeck. La construction se base sur une fonctionnelle (presque) continue d'un processus de Lévy multidimensionnel, ce qui implique que les théorèmes limites pour les processus de Lévy (que ce soit presque sûrement ou en loi) peuvent être transmis aux processus affines. La construction peut être interprétée comme un changement de temps à plusieurs paramètres ou comme une équation différentielle ordinaire aléatoire dirigée par des fonctions discontinues. En particulier, on propose des schémas d'approximation pour les processus affines basés sur la méthode d'Euler pour résoudre les EDO discontinues associées, dont la convergence est démontrée.

*MSC:* 60J80; 60F17

*Keywords:* Lévy processes; Continuous-state branching processes with immigration; Ornstein–Uhlenbeck processes; Multiparameter time change

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1305–1340
DOI: 10.1214/16-AIHP756


# Large time asymptotics for the parabolic Anderson model driven by spatially correlated noise

Jingyu Huang[a], Khoa Lê[a] and David Nualart[b]

[a]*Mathematical Sciences Research Institute, Berkeley, CA 94720, USA. E-mail: jhuang@msri.org; khoa.le@ucalgary.ca*
[b]*Department of Mathematics, The University of Kansas, Lawrence, KS 66045, USA. E-mail: nualart@ku.edu*

**Abstract.** In this paper we study the linear stochastic heat equation, also known as parabolic Anderson model, in multidimension driven by a Gaussian noise which is white in time and it has a correlated spatial covariance. Examples of such covariance include the Riesz kernel in any dimension and the covariance of the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2}]$ in dimension one. First we establish the existence of a unique mild solution and we derive a Feynman–Kac formula for its moments using a family of independent Brownian bridges and assuming a general integrability condition on the initial data. In the second part of the paper we compute Lyapunov exponents, lower and upper exponential growth indices in terms of a variational quantity. The last part of the paper is devoted to study the phase transition property of the Anderson model.

**Résumé.** Dans cet article nous étudions l'équation de la chaleur linéaire stochastique multidimensionnelle, connue aussi comme model d'Anderson parabolique, perturbée par un bruit gaussien qui est blanc en temps et qui a une covariance corrélée en espace. Le noyau de Riesz en dimension quelconque et la covariance du mouvement Brownien fractionnaire avec paramètre de Hurst $H \in (\frac{1}{4}, \frac{1}{2}]$ en une dimension, sont des examples d'une telle covariance. D'abord, on établit l'existence d'une solution d'evolution unique et on obtient une formule de Feynman–Kac pour les moments de la solution, en utilisant une famille de ponts browniens indépendants et en supposant une condition générale d'intégrabilité sur la condition initiale. Dans la deuxième partie du travail nous calculons les exposants de Lyapunov et les exposants supérieur et inférieur de croissance exponentielle en fonction d'une quantité variationnelle. La dernière partie du travail est consacré à l'etude de la transition de phase pour le model d'Anderson.

*MSC:* 60G15; 60H07; 60H15; 60F10; 65C30

*Keywords:* Stochastic heat equation; Brownian bridge; Feynman–Kac formula; Exponential growth index; Phase transition

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1341–1370
DOI: 10.1214/16-AIHP757


# Extreme Value Laws for non stationary processes generated by sequential and random dynamical systems

Ana Cristina Moreira Freitas[a], Jorge Milhazes Freitas[b] and Sandro Vaienti[c]

[a]*Centro de Matemática & Faculdade de Economia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal. E-mail: amoreira@fep.up.pt; url: http://www.fep.up.pt/docentes/amoreira/*

[b]*Centro de Matemática & Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal. E-mail: jmfreita@fc.up.pt; url: http://www.fc.up.pt/pessoas/jmfreita/*

[c]*Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France. E-mail: vaienti@cpt.univ-mrs.fr; url: http://www.cpt.univ-mrs.fr/~vaienti/*

**Abstract.** We develop and generalise the theory of extreme value for non-stationary stochastic processes, mostly by weakening the uniform mixing condition that was previously used in this setting. We apply our results to non-autonomous dynamical systems, in particular to *sequential dynamical systems*, given by uniformly expanding maps, and to a few classes of *random dynamical systems*. Some examples are presented and worked out in detail.

**Résumé.** Nous développons et généralisons la théorie des valeurs extrêmes pour des processus stochastiques non-stationnaires, en affaiblissant la condition de mélange uniforme qui avait été utilisée auparavant. Nous appliquons nos résultats à des systèmes dynamiques non autonomes, en particulier aux systèmes dynamiques séquentiels engendrés par des applications dilatantes et à une large classe de systèmes dynamiques aléatoires. Quelques exemples sont présentés et calculés en détail.

*MSC:* 37A50; 60G70; 37B20; 37A25

*Keywords:* Non-stationarity; Extreme value theory; Hitting Times; Sequential dynamical systems; Random dynamical systems

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1371–1401
DOI: 10.1214/16-AIHP758


# Recurrence statistics for the space of interval exchange maps and the Teichmüller flow on the space of translation surfaces

Romain Aimino[a], Matthew Nicol[b] and Mike Todd[c]

[a]*Departamento de Matemética, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal. E-mail: romain.aimino@fc.up.pt; url: http://www.fc.up.pt/pessoas/romain.aimino/*

[b]*Department of Mathematics, University of Houston, Houston, Texas, USA. E-mail: nicol@math.uh.edu; url: http://www.math.uh.edu/~nicol/*

[c]*Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland. E-mail: m.todd@st-andrews.ac.uk; url: http://www.mcs.st-and.ac.uk/~miket/*

**Abstract.** In this paper we show that the transfer operator of a Rauzy–Veech–Zorich renormalization map acting on a space of quasi-Hölder functions is quasicompact and derive certain statistical recurrence properties for this map and its associated Teichmüller flow. We establish Borel–Cantelli lemmas, Extreme Value statistics and return time statistics for the map and flow. Previous results have established quasicompactness in Hölder or analytic function spaces, for example the work of M. Pollicott and T. Morita. The quasi-Hölder function space is particularly useful for investigating return time statistics. In particular we establish the shrinking target property for nested balls in the setting of Teichmüller flow. Our point of view, approach and terminology derive from the work of M. Pollicott augmented by that of M. Viana.

**Résumé.** Dans cet article, nous démontrons que l'opérateur de transfert de l'application de renormalisation de Rauzy–Veech–Zorich est quasi-compact sur l'espace des fonctions quasi-Hölder, et nous en déduisons plusieurs propriétés de récurrence statistiques pour cette application et le flot de Teichmüller associé. Nous établissons des lemmes de Borel–Cantelli, des statistiques des valeurs extrêmes et des temps de retour pour l'application et le flot. De précédents résultats ont établi la quasi-compacité dans des espaces de fonctions Hölder ou analytiques, comme par exemple les travaux de M. Pollicott ou de T. Morita. L'espace fonctionnel quasi-Hölder est particulièrement adapté pour analyser les propriétés de récurrence statistiques. En particulier, nous démontrons la propriétés des cibles rétrécissantes pour des boules imbriquées dans le cadre du flot de Teichmüller. Notre point de vue, approche et terminologie proviennent du travail de M. Pollicott ainsi que de celui de M. Viana.

*MSC:* 37A50; 37D40; 60G70

*Keywords:* Interval exchange map; Teichmüller flow; Rauzy–Veech–Zorich renormalisation map; Transfer operator; Borel–Cantelli lemmas; Extreme Value Laws; Return/hitting time statistics; Quasi-Hölder function space

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1402–1437
DOI: 10.1214/16-AIHP759


# The simple exclusion process on the circle has a diffusive cutoff window

Hubert Lacoin

*IMPA, Via Dona Castorina 110, Jardim Botanico - CEP 22460-320 Rio de Janeiro - Brasil. E-mail: lacoin@impa.br*

**Abstract.** In this paper, we investigate the mixing time of the simple exclusion process on the circle with $N$ sites, with a number of particle $k(N)$ tending to infinity, both from the worst initial condition and from a typical initial condition. We show that the worst-case mixing time is asymptotically equivalent to $(8\pi^2)^{-1}N^2\log k$, while the cutoff window is identified to be $N^2$. Starting from a typical condition, we show that there is no cutoff and that the mixing time is of order $N^2$.

**Résumé.** Nous analysons temps de mélange pour le processus d'exclusion simple sur un cercle de $N$ sommets, avec un nombre de particules $k(N)$ qui tend vers l'infini avec $N$, et partant de la pire configuration initiale possible. Nous étudions également le cas d'une configuration initiale typique. Nous montrons que le temps de mélange est asymptotiquement équivalent $(8\pi^2)^{-1}N^2\log k$, pour la pire condition initiale, et que la fenêtre de cutoff est d'ordre $N^2$. Dans le cas d'une condition initiale typique nous montrons qu'il n'y a pas de cutoff et que le temps de mélange est d'ordre $N^2$.

*MSC:* 82D60; 60K37; 82B44

*Keywords:* Markov chains; Mixing time; Particle systems; Cutoff Window

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1438–1457
DOI: 10.1214/16-AIHP760


# A functional limit theorem for irregular SDEs

Stefan Ankirchner[a,1], Thomas Kruse[b,1] and Mikhail Urusov[b,c,2]

[a]Institute for Mathematics, University of Jena, Ernst-Abbe-Platz 2, 07745 Jena, Germany. E-mail: s.ankirchner@uni-jena.de
[b]Faculty of Mathematics, University of Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany. E-mail: thomas.kruse@uni-due.de
[c]Steklov Mathematical Institute, Russian Academy of Sciences, ul. Gubkina 8, 119991 Moscow, Russia. E-mail: mikhail.urusov@uni-due.de

**Abstract.** Let $X_1, X_2, \dots$ be a sequence of i.i.d. real-valued random variables with mean zero, and consider the scaled random walk of the form $Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}$, where $a_N : \mathbb{R} \to \mathbb{R}_+$. We show, under mild assumptions on the law of $X_i$, that one can choose the scale factor $a_N$ in such a way that the process $(Y_{\lfloor Nt \rfloor}^N)_{t \in \mathbb{R}_+}$ converges in distribution to a given diffusion $(M_t)_{t \in \mathbb{R}_+}$ solving a stochastic differential equation with possibly irregular coefficients, as $N \to \infty$. To this end we embed the scaled random walks into the diffusion $M$ with a sequence of stopping times with expected time step $1/N$.

**Résumé.** Soit $X_1, X_2, \dots$ une suite de variables aléatoires indépendantes avec espérance $E(X_i) = 0$, et $Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}$ une marche aléatoire renormalisée avec une fonction $a_N : \mathbb{R} \to \mathbb{R}_+$. On montre, sous certaines conditions légères sur la loi de $X_i$, que l'on peut choisir le facteur $a_N$ d'une facon que $(Y_{\lfloor Nt \rfloor}^N)_{t \in \mathbb{R}_+}$ converge en loi, quand $N$ tend vers l'infini, vers une diffusion $(M_t)_{t \in \mathbb{R}_+}$ étant la solution d'une équation differentielle stochastique avec des coefficients irréguliers. À cet effet, nous plongeons la marche aléatoire renormalisée dans la diffusion $M$ par une suite de temps d'arrêt ayant un pas de temps avec espérance $1/N$.

*MSC:* 60F17; 60J60; 65C30

*Keywords:* Stochastic differential equations; Irregular diffusion coefficient; Weak law of large numbers for u.i. arrays; Weak convergence of processes; Skorokhod embedding problem

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1458–1497
DOI: 10.1214/16-AIHP761


# Excited random walks with Markovian cookie stacks

Elena Kosygina[a,1] and Jonathon Peterson[b,2]

[a]*Department of Mathematics, Baruch College, One Bernard Baruch Way, Box B6-230, New York, NY 10010, USA.*
*E-mail:* elena.kosygina@baruch.cuny.edu; *url:* www.baruch.cuny.edu/math/elenak/
[b]*Department of Mathematics, Purdue University, 150 N University Street, Lafayette, IN 47907, USA.*
*E-mail:* peterson@purdue.edu; *url:* www.math.purdue.edu/~peterson

**Abstract.** We consider a nearest-neighbor random walk on $\mathbb{Z}$ whose probability $\omega_x(j)$ to jump to the right from site $x$ depends not only on $x$ but also on the number of prior visits $j$ to $x$. The collection $(\omega_x(j))_{x\in\mathbb{Z}, j\ge 1}$ is sometimes called the "cookie environment" due to the following informal interpretation. Upon each visit to a site the walker eats a cookie from the cookie stack at that site and chooses the transition probabilities according to the "strength" of the cookie eaten. We assume that the cookie stacks are i.i.d. and that the cookie "strengths" within the stack $(\omega_x(j))_{j\ge 1}$ at site $x$ follow a finite state Markov chain. Thus, the environment at each site is dynamic, but it evolves according to the local time of the walk at each site rather than the original random walk time.

The model admits two different regimes, critical or non-critical, depending on whether the expected probability to jump to the right (or left) under the invariant measure for the Markov chain is equal to $1/2$ or not. We show that in the non-critical regime the walk is always transient, has non-zero linear speed, and satisfies the classical central limit theorem. The critical regime allows for a much more diverse behavior. We give necessary and sufficient conditions for recurrence/transience and ballisticity of the walk in the critical regime as well as a complete characterization of limit laws under the averaged measure in the transient case.

The setting considered in this paper generalizes the previously studied model with periodic cookie stacks [Excited random walk with periodic cookies (2014) Preprint]. Our results on ballisticity and limit theorems are new even for the periodic model.

**Résumé.** Nous considérons une marche aléatoire au plus proche voisin sur $\mathbb{Z}$ dont la probabilité $\omega_x(j)$ de sauter à droite du site $x$ ne dépend pas seulement de $x$ mais aussi du nombre $j$ de visites antérieures en $x$. La collection $(\omega_x(j))_{x\in\mathbb{Z}, j\ge 1}$ est parfois nommée « l'environnement cookie » à cause de l'interprétation suivante. À chaque visite d'un site le marcheur mange un cookie de la pile de cookie à ce site et choisi la probabilitéé de transition en fonction de la force du cookie qui a été mangé. Nous supposons que les piles de cookie sont i.i.d. et que la force des cookies à l'intérieur de la pile $(\omega_x(j))_{j\ge 1}$ au site $x$ est une chaine de Markov à espace d'états fini. Par conséquent l'environnement à chaque site est dynamique mais évolue en fonction du temps local de la marche à chaque site, plutôt que le temps propre de la marche aléatoire originale.

Le modèle admet deux régimes différents, critique ou non critique, dépendant du fait que la probabilité sous la mesure invariante de la chaine de Markov de sauter à droite (ou à gauche) est égale à $1/2$ ou non. Nous montrons que dans le régime non-critique la marche est toujours transiente, a une vitesse déchappement linéaire et satisfait le théorème de la limite centrale. Le régime critique a beaucoup plus de variantes possibles. Nous donnons alors des conditions nécessaires et suffisantes pour la recurrence/transience de la marche et une caractérisation complète des lois limites possibles sous la mesure moyennisée dans le cas transient.

Le cadre de ce papier généralise le modèle étudié précédemment où les piles de cookies étaient périodiques [Excited random walk with periodic cookies (2014) Preprint]. Nos résultats sur la ballisticité et les théorèmes limites sont nouveaux même pour le modèle périodique.

*MSC:* Primary 60K37; secondary 60F05; 60J10; 60J15; 60K35

*Keywords:* Excited random walk; Diffusion approximation; Stable limit laws; Random environment; Branching-like processes

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Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 3, 1498–1511
DOI: 10.1214/16-AIHP767


# Pairing of zeros and critical points for random polynomials[1]

Boris Hanin

*Department of Mathematics, MIT, Cambridge, MA 02142, USA. E-mail: bhanin@mit.edu*

**Abstract.** Let $p_N$ be a random degree $N$ polynomial in one complex variable whose zeros are chosen independently from a fixed probability measure $\mu$ on the Riemann sphere $S^2$. This article proves that if we condition $p_N$ to have a zero at some fixed point $\xi \in S^2$, then, with high probability, there will be a critical point $w_\xi$ at a distance $N^{-1}$ away from $\xi$. This $N^{-1}$ distance is much smaller than the $N^{-1/2}$ typical spacing between nearest neighbors for $N$ i.i.d. points on $S^2$. Moreover, with the same high probability, the argument of $w_\xi$ relative to $\xi$ is a deterministic function of $\mu$ plus fluctuations on the order of $N^{-1}$.

**Résumé.** Soit $p_N$ un polynôme aléatoire de degré $N$ en une variable complexe tel que ses zéros sont distribués indépendamment suivant une mesure de probabilité $\mu$ fixée et définie sur la sphère de Riemann $S^2$. Cet article prouve que si nous conditionnons $p_N$ pour avoir un zéro en un point fixé $\xi \in S^2$, alors, avec grande probabilité, il y aura un point critique $w_\xi$ à une distance $N^{-1}$ de $\xi$. Cette distance $N^{-1}$ est beaucoup plus petite que l'espacement typique entre deux points voisins pour $N$ points i.i.d. sur $S^2$, qui lui est d'ordre $N^{-1/2}$. De plus, avec la méme grande probabilité, l'argument de $w_\xi$ relativement à $\xi$ est une fonction déterministe de $\mu$, plus des fluctuations d'ordre $N^{-1}$.

*MSC:* 30C10; 30C15; 60G60

*Keywords:* Zeros; Critical points; Random polynomials; Gauss–Lucas

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