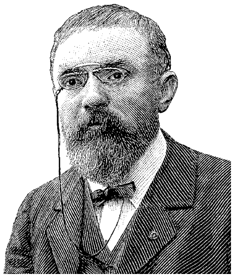

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Liouville quantum gravity surfaces with boundary as matings of trees

Morris Ang[a] and Ewain Gwynne[b]

[a]MIT, Cambridge, USA. E-mail: angm@mit.edu
[b]University of Cambridge, Cambridge, UK. E-mail: eg558@cam.ac.uk

Abstract. For $\gamma \in (0,2)$, the quantum disk and γ-quantum wedge are two of the most natural types of Liouville quantum gravity (LQG) surfaces with boundary. These surfaces arise as scaling limits of finite and infinite random planar maps with boundary, respectively. We show that the left/right quantum boundary length process of a space-filling $\mathrm{SLE}_{16/\gamma^2}$ curve on a quantum disk or on a γ-quantum wedge is a certain explicit conditioned two-dimensional Brownian motion with correlation $-\cos(\pi\gamma^2/4)$. This extends the mating of trees theorem of Duplantier, Miller, and Sheffield (2014) to the case of quantum surfaces with boundary (the disk case for $\gamma \in (\sqrt{2}, 2)$ was previously treated by Duplantier, Miller, Sheffield using different methods). As an application, we give an explicit formula for the conditional law of the LQG area of a quantum disk given its boundary length by computing the law of the corresponding functional of the correlated Brownian motion.

Résumé. Pour $\gamma \in (0,2)$, le disque quantique et le γ-secteur angulaire quantique sont deux types, parmi les plus naturels, de surfaces avec frontières pour la gravité quantique de Liouville (LQG). Ces surfaces apparaissent comme limites d'échelle des cartes planaires, respectivement finies et infinies, avec frontières. Nous montons que les processus des longueurs de la frontière quantique à gauche/droite d'une courbe $\mathrm{SLE}_{16/\gamma^2}$ sur un disque quantique ou un γ-secteur angulaire quantique est un mouvement Brownien 2-dimensionnel, sous un conditionnement explicite, avec corrélation $-\cos(\pi\gamma^2/4)$. Ceci étend le théorème d'accouplement d'arbres de Duplantier, Miller, et Sheffield (2014) au cas des surfaces quantiques avec frontières (le cas du disque pour $\gamma \in (\sqrt{2}, 2)$ avait été traité par Duplantier, Miller, Sheffield en utilisant des méthodes différentes). Comme application, nous donnons une formule explicite pour la loi conditionnelle de l'aire de la LQG d'un disque quantique étant donnée la longueur de sa frontière en calculant la loi de la fonctionnelle correspondante du mouvement Brownien corrélé.

MSC2020 subject classifications: 60J67; 60D05; 60J65

Keywords: Schramm–Loewner evolution; Liouville quantum gravity; Mating of trees; Quantum disk; Quantum wedge; Peanosphere

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Equidistribution of random walks on compact groups

Bence Borda

Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13–15, 1053 Budapest, Hungary. E-mail: borda.bence@renyi.hu

Abstract. Let $X_1, X_2, \dots$ be independent, identically distributed random variables taking values from a compact metrizable group G. We prove that the random walk $S_k = X_1 X_2 \cdots X_k$, $k = 1, 2, \dots$ equidistributes in any given Borel subset of G with probability 1 if and only if X_1 is not supported on any proper closed subgroup of G, and S_k has an absolutely continuous component for some $k \geq 1$. More generally, the sum $\sum_{k=1}^N f(S_k)$, where $f : G \to \mathbb{R}$ is Borel measurable, is shown to satisfy the strong law of large numbers and the law of the iterated logarithm. We also prove the central limit theorem with remainder term for the same sum, and construct an almost sure approximation of the process $\sum_{k \leq t} f(S_k)$ by a Wiener process provided S_k converges to the Haar measure in the total variation metric.

Résumé. Soient $X_1, X_2, \dots$ des variables aléatoires indépendantes et identiquement distribuées à valeurs dans un groupe compact métrisable G. Nous prouvons que la marche aléatoire $S_k = X_1 X_2 \cdots X_k$, $k = 1, 2, \dots$ est équidistribuée dans tout sous-ensemble borélien avec probabilité 1 si et seulement si le support de X_1 n'est pas inclus dans un sous-groupe fermé propre de G, et il existe $k \geq 1$ tel que S_k possède une composante absolument continue. Plus généralement, nous montrons que la somme $\sum_{k=1}^N f(S_k)$, où $f : G \to \mathbb{R}$ est mesurable au sens de Borel, vérifie la loi forte des grands nombres et la loi du logarithme itéré. Nous démontrons aussi le théorème central limite avec un terme d'erreur pour la même somme, et contruisons une approximation presque sûre du processus $\sum_{k \leq t} f(S_k)$ par un processus de Wiener à condition que S_k converge vers la mesure de Haar en variation totale.

MSC2020 subject classifications: 60G50; 60B15

Keywords: Empirical distribution; Ergodic theorem; Strong law of large numbers; Law of the iterated logarithm; Central limit theorem; Wiener process

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On the critical branching random walk III: The critical dimension

Qingsan Zhu

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel. E-mail: qingsanz@mail.tau.ac.il

Abstract. In this paper, we study the critical branching random walk in the critical dimension, four. We provide the asymptotics of the probability of visiting a fixed finite set and the range of the critical branching random walk conditioned on the total number of offspring. We also prove that conditioned on visiting a finite set, the first visiting point converges in distribution, when the starting point tends to infinity.

Résumé. Dans cet article, nous étudions la marche aléatoire de branchement critique en dimension critique (quatre). Nous déterminons les probabilités asymptotiques de visite d'un ensemble fini fixé, et l'image de la marche aléatoire de branchement critique conditionnée par le nombre total d'individus. Nous montrons également que, conditionnellement à l'événement de visite d'un ensemble fini, le premier point visité converge en loi, lorsque le point de départ tend vers l'infini.

MSC2020 subject classifications: 60G50; 60J80

Keywords: Critical branching random walk; Tree-indexed random walk; Visiting probability; Harmonic measure; Range

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Global observables for RW: Law of large numbers

Dmitry Dolgopyat[a], Marco Lenci[b,c] and Péter Nándori[d]

[a]Department of Mathematics, University of Maryland, College Park, MD 20741, USA. E-mail: dmitry@math.umd.edu
[b]Dipartimento di Matematica, Università di Bologna, 40126 Bologna, Italy. E-mail: marco.lenci@unibo.it
[c]Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, 40126 Bologna, Italy
[d]Department of Mathematical Sciences, Yeshiva University, New York, NY 10016, USA. E-mail: peter.nandori@yu.edu

Abstract. We consider the sums $T_N = \sum_{n=1}^{N} F(S_n)$ where S_n is a random walk on $\mathbb{Z}^d$ and $F : \mathbb{Z}^d \to \mathbb{R}$ is a global observable, that is, a bounded function which admits an average value when averaged over large cubes. We show that T_N always satisfies the weak Law of Large Numbers but the strong law fails in general except for one dimensional walks with drift. Under additional regularity assumptions on F, we obtain the Strong Law of Large Numbers and estimate the rate of convergence. The growth exponents which we obtain turn out to be optimal in two special cases: for quasiperiodic observables and for random walks in random scenery.

Résumé. Nous considérons la somme $T_N = \sum_{n=1}^{N} F(S_n)$, où S_n est une marche aléatoire à valeurs dans $\mathbb{Z}^d$ et $F : \mathbb{Z}^d \to \mathbb{R}$ est une observable globale, c'est-à-dire une fonction bornée ayant une valeur moyenne sur de grands cubes. Nous montrons que T_N satisfait toujours la loi faible des grands nombres mais la loi forte échoue en général, sauf dans le cas de la marche aléatoire unidimensionnelle avec dérive. Sous certaines hypothèses de régularité supplémentaires, nous obtenons la loi forte des grands nombres et nous estimons la vitesse de convergence. Les exposants que nous obtenons sont optimaux dans deux cas particuliers: pour les observables quasi-périodiques et pour les marches aléatoires en paysage aléatoire.

MSC2020 subject classifications: Primary 60F15; 60G50; secondary 37A40; 60K37

Keywords: Random walks; Law of large numbers; Arcsine law; Random walks in random scenery; Global observables

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Some random paths with angle constraints

Clément Berenfeld[a] and Ery Arias-Castro[b]

[a]*Université Paris-Dauphine & PSL, CNRS, CEREMADE, 75016 Paris, France. E-mail: berenfeld@ceremade.dauphine.fr*
[b]*University of California, San Diego, USA. E-mail: eariascastro@ucsd.edu*

Abstract. We propose a simple, geometrically-motivated construction of smooth random paths in the plane. The construction is such that, with probability one, the paths have finite curvature everywhere. Our construction is Markovian of order 2. We show that a simpler construction which is Markovian of order 1 fails to exhibit the desired finite curvature property.

Résumé. Nous étudions une manière élémentaire de construire des marches aléatoires du plan à l'aide d'angles aléatoires. Cette construction, issue de considérations géométriques, est telle que le processus limite possède presque sûrement des trajectoires dont la courbure est partout finie. Les marches aléatoires que nous exhibons sont markoviennes d'ordre 2, et nous montrons qu'une approche plus simple, avec des processus d'ordre 1, ne permet pas d'obtenir, à la limite, les propriétés désirées de courbure finie.

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Keywords: Random walk; Random angles; Central limit theorem for dependent variables; Brownian motion; Curvature

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Skorohod and rough integration for stochastic differential equations driven by Volterra processes

Thomas Cass[a] and Nengli Lim[b]

[a]*Department of Mathematics, Imperial College London, Huxley Building, London, SW7 2AZ, United Kingdom. E-mail: thomas.cass@imperial.ac.uk*
[b]*Singapore University of Technology and Design, 8 Somapah Rd., Singapore 487372. E-mail: nengli_lim@sutd.edu.sg*

Abstract. Given a solution Y to a rough differential equation (RDE), a recent result (*Ann. Probab.* **47** (2019) 1–60) extends the classical Itô-Stratonovich formula and provides a closed-form expression for $\int Y \circ \mathrm{d}\mathbf{X} - \int Y \,\mathrm{d}X$, i.e. the difference between the rough and Skorohod integrals of Y with respect to X, where X is a Gaussian process with finite p-variation less than 3. In this paper, we extend this result to Gaussian processes with finite p-variation such that $3 \le p < 4$. The constraint this time is that we restrict ourselves to Volterra Gaussian processes with kernels satisfying a natural condition, which however still allows the result to encompass many standard examples, including fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. As an application we recover Itô formulas in the case where the vector fields of the RDE governing Y are commutative.

Résumé. Étant donnée Y une solution d'une équation différentielle rugueuse (RDE), un résultat récent (*Ann. Probab.* **47** (2019) 1–60) étend la formule d'Itô-Stratonovich et propose une expression explicite pour $\int Y \circ \mathrm{d}\mathbf{X} - \int Y \,\mathrm{d}X$, c'est-à-dire pour la différence entre l'intégrale rugueuse et l'intégrale de Skorohod de Y par rapport à X, où X est un processus Gaussien avec p-variation plus petite que 3. Dans cet article, nous étendons ce résultat au cas de processus Gaussiens avec p-variation telle que $3 \le p < 4$. La contrainte ici est que nous nous restreignons au cas de processus Gaussiens de type Volterra avec des noyaux satisfaisant une condition naturelle, ce qui permet néanmoins de traiter beaucoup d'exemples classiques incluant le cas du mouvement Brownien fractionnaire avec paramètre de Hurst $H > \frac{1}{4}$. Comme application, nous retrouvons la formule d'Itô dans le cas où les champs de vecteurs de la RDE gouvernant Y sont commutatifs.

MSC2020 subject classifications: 60H07; 60L20

Keywords: Rough path theory; Volterra processes; Malliavin calculus

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Derivation of viscous Burgers equations from weakly asymmetric exclusion processes

M. Jara[a,*], C. Landim[a,b,†] and K. Tsunoda[c]

[a]*IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brasil. E-mail:* [*]*mjara@impa.br;* [†]*landim@impa.br*
[b]*CNRS UMR 6085, Université de Rouen, Avenue de l'Université, BP.12, Technopôle du Madrillet, F76801 Saint-Étienne-du-Rouvray, France*
[c]*Department of Mathematics, Osaka University, Osaka, 560-0043, Japan. E-mail: k-tsunoda@math.sci.osaka-u.ac.jp*

Abstract. We consider weakly asymmetric exclusion processes whose initial density profile is a small perturbation of a constant. We show that in the diffusive time-scale, in all dimensions, the density defect evolves as the solution of a viscous Burgers equation.

Résumé. Nous examinons le processus d'exclusion simple faiblement asymétrique partant d'une perturbation d'un profil de densité constant. Nous montrons qu'à l'échelle diffusive, en toute dimension, la perturbation évolue selon la solution d'une équation de Burgers visqueuse.

MSC2020 subject classifications: Primary 60K35; secondary 82C22

Keywords: Viscous Burgers equations; Weakly asymmetric exclusion processes; Incompressible limits

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Estimating a density, a hazard rate, and a transition intensity via the ρ-estimation method

Mathieu Sart

Univ Lyon, Université Jean Monnet Saint-Étienne, CNRS UMR 5208, Institut Camille Jordan, F-42023 Saint-Etienne, France.
E-mail: mathieu.sart@univ-st-etienne.fr

Abstract. We propose a unified study of three statistical settings by widening the ρ-estimation method developed in Baraud, Birgé and Sart (*Invent. Math.* **207** (2017) 425–517). More specifically, we aim at estimating a density, a hazard rate (from censored data), and a transition intensity of a time inhomogeneous Markov process. We show non-asymptotic risk bounds for an Hellinger-type loss when the models consist, for instance, of piecewise polynomial functions, multimodal functions, or functions whose square root is piecewise convex-concave. Under convex-type assumptions on the models, maximum likelihood estimators coincide with ρ-estimators, and satisfy therefore our risk bounds. However, our results also apply to some models where the maximum likelihood method does not work. Subsequently, we present an alternative way, based on estimator selection, to define a piecewise polynomial estimator. We control the risk of the estimator and carry out some numerical simulations to compare our approach with a more classical one based on maximum likelihood only.

Résumé. Nous proposons une étude unifiée de trois cadres statistiques par la méthode de la ρ-estimation développée dans Baraud, Birgé and Sart (*Invent. Math.* **207** (2017) 425–517). Plus précisément, nous proposons d'estimer une densité, un taux de risque (pour des données censurées) et une intensité de transition pour un processus de Markov à temps inhomogène. Nous montrons des bornes de risque non-asymptotiques pour une perte de type Hellinger lorsque les modèles sont, par exemple, constitués de fonctions polynomiales par morceaux, de fonctions multimodales, ou de fonctions dont la racine carrée est convexe-concave par morceaux. Sous des hypothèses de convexité sur les modèles, les estimateurs du maximum de vraisemblance coincident avec les ρ-estimateurs et vérifient donc nos bornes de risque. Cependant, nos résultats s'appliquent également à certains modèles où la méthode du maximum de vraisemblance ne fonctionne pas. Dans la suite, nous présentons une autre méthode, basée sur la sélection d'estimateurs, pour définir un estimateur polynomial par morceaux. Nous contrôlons le risque de l'estimateur et présentons quelques simulations numériques pour comparer notre approche à une plus classique basée uniquement sur la vraisemblance.

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Keywords: ρ-Estimator; Maximum likelihood; Qualitative assumptions; Piecewise polynomial estimation

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Existence of densities for stochastic differential equations driven by Lévy processes with anisotropic jumps

Martin Friesen[a,*], Peng Jin[b] and Barbara Rüdiger[a,†]

[a]*School of Mathematics and Natural Sciences, University of Wuppertal, Germany.*
E-mail: [*]*friesen@math.uni-wuppertal.de;* [†]*ruediger@uni-wuppertal.de*
[b]*Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China. E-mail:* pjin@stu.edu.cn

Abstract. We study existence and Besov regularity of densities for solutions to stochastic differential equations with Hölder continuous coefficients driven by a d-dimensional Lévy process $Z = (Z(t))_{t\geq 0}$, where, for $t > 0$, the density function f_t of $Z(t)$ exists and satisfies, for some $(\alpha_i)_{i=1,\dots,d} \subset (0, 2)$ and $C > 0$,

$$\limsup_{t\to 0} t^{1/\alpha_i} \int_{\mathbb{R}^d} \left| f_t(z + e_i h) - f_t(z)\right| dz \leq C|h|, \quad h \in \mathbb{R}, i = 1, \dots, d.$$

Here $e_1, \dots, e_d$ denote the canonical basis vectors in $\mathbb{R}^d$. The latter condition covers anisotropic $(\alpha_1, \dots, \alpha_d)$-stable laws but also particular cases of subordinate Brownian motion. To prove our result we use some ideas taken from (J. Funct. Anal. **264** (2013), 1757–1778).

Résumé. Nous étudions le problème de l'existence et de l'appartenance à un espace de Besov pour les densités de solutions d'équations différentielles stochastiques à coefficients hölderiens, conduites par un processus de Lévy d-dimensionnel $Z = (Z(t))_{t\geq 0}$, où, pour $t > 0$, la densité f_t de la loi de $Z(t)$ existe et vérifie, pour un certain $(\alpha_i)_{i=1,\dots,d} \subset (0, 2)$ et $C > 0$,

$$\limsup_{t\to 0} t^{1/\alpha_i} \int_{\mathbb{R}^d} \left| f_t(z + e_i h) - f_t(z)\right| dz \leq C|h|, \quad h \in \mathbb{R}, i = 1, \dots, d.$$

Ici, $e_1, \dots, e_d$ désignent les vecteurs de la base canonique de R^d. La précédente condition s'applique au cas de lois anisotropiques $(\alpha_1, \dots, \alpha_d)$-stables, mais aussi à des cas particuliers de mouvements browniens subordonnés. Pour démontrer ces résultats, nous utilisons certaines idées de (J. Funct. Anal. **264** (2013), 1757–1778).

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Keywords: Stochastic differential equation with jumps; Anisotropic Lévy process; Anisotropic Besov space; Transition density

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Erratum: *Central limit theorems for eigenvalues in a spiked population model* [Annales de l'Institut Henri Poincaré – Probabilités et Statistiques 2008, Vol. 44, No. 3, 447–474]

Zhidong Bai[a] and Jianfeng Yao[b]

[a]*KLASMOE and School of Mathematics and Statistics, Northeast Normal University, China. E-mail: baizd@nenu.edu.cn*
[b]*The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: jeffyao@hku.hk*

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Phase transition for the interchange and quantum Heisenberg models on the Hamming graph

Radosław Adamczak[a], Michał Kotowski[b,*] and Piotr Miłoś[b,†]

[a]*Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland. E-mail: r.adamczak@mimuw.edu.pl*
[b]*Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland.*
*E-mail: *michal.kotowski@mimuw.edu.pl; †pmilos@mimuw.edu.pl*

Abstract. We study a family of random permutation models on the Hamming graph $H(2, n)$ (i.e., the 2-fold Cartesian product of complete graphs), containing the interchange process and the cycle-weighted interchange process with parameter $\theta > 0$. This family contains the random walk representation of the quantum Heisenberg ferromagnet. We show that in these models the cycle structure of permutations undergoes a *phase transition* – when the number of transpositions defining the permutation is $\leq cn^2$, for small enough $c > 0$, all cycles are microscopic, while for more than $\geq Cn^2$ transpositions, for large enough $C > 0$, macroscopic cycles emerge with high probability.

We provide bounds on values C, c depending on the parameter θ of the model, in particular for the interchange process we pinpoint exactly the critical time of the phase transition. Our results imply also the existence of a phase transition in the quantum Heisenberg ferromagnet on $H(2, n)$, namely for low enough temperatures spontaneous magnetization occurs, while it is not the case for high temperatures.

At the core of our approach is a novel application of the cyclic random walk, which might be of independent interest. By analyzing explorations of the cyclic random walk, we show that sufficiently long cycles of a random permutation are uniformly spread on the graph, which makes it possible to compare our models to the mean-field case, i.e., the interchange process on the complete graph, extending the approach used earlier by Schramm.

Résumé. Nous étudions une famille de modèles de permutations aléatoires sur le graphe de Hamming $H(2, n)$ (c'est-à-dire le produit cartésien de deux graphes complets), incluant le processus d'échange et le processus d'échange pondéré par les cycles avec paramètre $\theta > 0$. Cette famille comprend la représentation par marches aléatoires du modèle de Heisenberg quantique ferromagnétique. Nous montrons que dans ces modèles, la structure des cycles des permutations satisfait une *transition de phase* – lorsque le nombre de transpositions définissant la permutation est $\leq cn^2$, pour $c > 0$ assez petit, tous les cycles sont microscopiques, tandis que lorsque ce nombre est $\geq Cn^2$ avec $C > 0$ assez grand, des cycles macroscopiques apparaissent avec grande probabilité.

Nous déterminons des bornes sur les constantes C, c dépendant du paramètre θ du modèle, en particulier, pour le processus d'échange, nous déterminons exactement le temps critique de la transition de phase. Nos résultats impliquent également l'existence d'une transition de phase pour le modèle de Heisenberg quantique ferromagnétique sur $H(2, n)$, stipulant que pour des températeurs assez basses, une magnétisation spontanée apparaît alors que cela n'est pas le cas aux hautes températures.

Le cœur de notre approche consiste en une nouvelle application de la marche aléatoire cyclique, qui pourrait être intéressante en elle-même. En analysant les explorations de la marche aléatoire cyclique, nous montrons que des cycles suffisamment longs d'une permutation aléatoire sont répartis uniformément sur le graphe, ce qui rend possible une comparaison entre nos modèles au cas du champ moyen, c'est-à-dire le processus d'échange sur le graphe complet, étendant ainsi l'approche antérieure utilisée par Schramm.

MSC2020 subject classifications: 60K35; 60J27

Keywords: Interchange process; Quantum Heisenberg model; Hamming graph

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Poisson statistics for Gibbs measures at high temperature

Gaultier Lambert

University of Zurich, Winterthurerstrasse 190, 8057 Zürich, Switzerland. E-mail: gaultier.lambert@math.uzh.ch

Abstract. We consider a gas of N particles subject to a two-body interaction and confined by an external potential in the mean field or high temperature regime, that is when the inverse temperature $\beta > 0$ satisfies $\beta N \to \gamma \geq 0$ as $N \to +\infty$. We show that under general conditions on the interaction and the potential, the local fluctuations are described by a Poisson point process in the large N limit. We present applications to Coulomb and Riesz gases on $\mathbb{R}^n$ for any $n \geq 1$, as well as to the edge behavior of β-ensembles on $\mathbb{R}$.

Résumé. On étudie le comportement asymptotique d'un gaz à N particules à l'équilibre modélisé par une interaction de type champ moyen, c'est-à-dire dans le régime à *haute température* où la constante de couplage satisfait $\beta N \to \gamma \geq 0$ quand $N \to +\infty$. On démontre que sous des hypothèses générales sur l'interaction à deux corps et le potentiel confinant, les fluctuations locales du gaz sont régies par un processus de Poisson. On discute des applications au gaz de Coulomb et de Riesz sur $\mathbb{R}^n$ pour $n \geq 1$ quelconque, ainsi que du comportement au bord des β-ensembles.

MSC2020 subject classifications: 60B20; 60G55; 82B31; 60G70; 60F10

Keywords: Statistical mechanics of particle systems; β-ensembles and Coulomb gas; Poisson statistics; Large deviations; Thermal equilibrium measure

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Efficient estimation of smooth functionals in Gaussian shift models

Vladimir Koltchinskii[1,*] and Mayya Zhilova[2,†]

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.
E-mail: [*]*vlad@math.gatech.edu;* [†]*mzhilova@math.gatech.edu*

Abstract. We study a problem of estimation of smooth functionals of parameter θ of Gaussian shift model

$$X = \theta + \xi, \quad \theta \in E,$$

where E is a separable Banach space and X is an observation of unknown vector θ in Gaussian noise ξ with zero mean and known covariance operator Σ. In particular, we develop estimators $T(X)$ of $f(\theta)$ for functionals $f : E \mapsto \mathbb{R}$ of Hölder smoothness $s > 0$ such that

$$\sup_{\|\theta\|\le 1} \mathbb{E}_\theta \big(T(X) - f(\theta)\big)^2 \lesssim \big(\|\Sigma\| \vee \big(\mathbb{E}\|\xi\|^2\big)^s\big) \wedge 1,$$

where $\|\Sigma\|$ is the operator norm of Σ, and show that this mean squared error rate is minimax optimal at least in the case of standard finite-dimensional Gaussian shift model ($E = \mathbb{R}^d$ equipped with the canonical Euclidean norm, $\xi = \sigma Z$, $Z \sim \mathcal{N}(0; I_d)$). Moreover, we determine a sharp threshold on the smoothness s of functional f such that, for all s above the threshold, $f(\theta)$ can be estimated efficiently with a mean squared error rate of the order $\|\Sigma\|$ in a "small noise" setting (that is, when $\mathbb{E}\|\xi\|^2$ is small). The construction of efficient estimators is crucially based on a "bootstrap chain" method of bias reduction. The results could be applied to a variety of special high-dimensional and infinite-dimensional Gaussian models (for vector, matrix and functional data).

Résumé. Dans cet article, nous étudions le problème d'estimation de fonctionnelles lisses d'un paramètre θ dans le modèle gaussien suivant:

$$X = \theta + \xi, \quad \theta \in E,$$

où E est un espace de Banach séparable, X est une observation du vecteur θ inconnu et le bruit ξ est gaussien de moyenne nulle et d'opérateur de covariance Σ connu. En particulier, nous développons des estimateurs $T(X)$ de $f(\theta)$ pour les fonctionnelles $f : E \mapsto \mathbb{R}$ de paramètre de régularité Hölderienne $s > 0$ tels que

$$\sup_{\|\theta\|\le 1} \mathbb{E}_\theta \big(T(X) - f(\theta)\big)^2 \lesssim \big(\|\Sigma\| \vee \big(\mathbb{E}\|\xi\|^2\big)^s\big) \wedge 1,$$

où $\|\Sigma\|$ est la norme d'opérateur de Σ, et nous montrons que cette estimation de l'erreur quadratique moyenne est minimax optimale au moins dans le cas du modèle gaussien de dimension finie avec une matrice de covariance identité ($E = \mathbb{R}^d$ est muni de la norme euclidienne canonique $\xi = \sigma Z$, $Z \sim \mathcal{N}(0; I_d)$). De plus, nous déterminons le seuil exact sur la régularité s de la fonctionnelle f tel que, pour tout s au-dessus de ce seuil, $f(\theta)$ peut-être estimé efficacement avec une erreur quadratique moyenne de l'ordre $\|\Sigma\|$ dans le régime de « bruit petit » (i.e. $\mathbb{E}\|\xi\|^2$ est petit). La construction des estimateurs efficaces est basée essentiellement sur une méthode de « chaîne bootstrap » pour la réduction du biais. Les résultats peuvent être appliqués à une grande variété de modèles gaussiens de dimension grande, voire infinie (pour les données vectorielles, matricielles et fonctionnelles).

MSC2020 subject classifications: Primary 62H12; secondary 62G20; 62H25; 60B20

Keywords: Efficiency; Smooth functionals; Gaussian shift model; Bootstrap; Effective rank; Concentration inequalities; Normal approximation

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Sharp phase transition for the continuum Widom–Rowlinson model

David Dereudre[a] and Pierre Houdebert[b]

[a]*Laboratoire de Mathématiques Paul Painlevé*
University of Lille, France. E-mail: david.dereudre@math.univ-lille.fr
[b]*Department of Mathematics, University of Potsdam. E-mail: pierre.houdebert@gmail.com*

Abstract. The Widom–Rowlinson model (or the Area-interaction model) is a Gibbs point process in $\mathbb{R}^d$ with the formal Hamiltonian defined as the volume of $\cup_{x\in\omega} B_1(x)$, where ω is a locally finite configuration of points and $B_1(x)$ denotes the unit closed ball centred at x. The model is also tuned by two other parameters: the activity $z > 0$ related to the intensity of the process and the inverse temperature $\beta \geq 0$ related to the strength of the interaction. In the present paper we investigate the phase transition of the model in the point of view of percolation theory and the liquid-gas transition. First, considering the graph connecting points with distance smaller than $2r > 0$, we show that for any $\beta \geq 0$, there exists $0 < \tilde{z}_c^a(\beta, r) < +\infty$ such that an exponential decay of connectivity at distance n occurs in the subcritical phase (i.e. $z < \tilde{z}_c^a(\beta, r)$) and a linear lower bound of the connection at infinity holds in the supercritical case (i.e. $z > \tilde{z}_c^a(\beta, r)$). These results are in the spirit of recent works using the theory of randomised tree algorithms (*Probab. Theory Related Fields* **173** (2019) 479–490, *Ann. of Math.* **189** (2019) 75–99, Duminil-Copin, Raoufi and Tassion (2018)). Secondly we study a standard liquid-gas phase transition related to the uniqueness/non-uniqueness of Gibbs states depending on the parameters z, β. Old results (*Phys. Rev. Lett.* **27** (1971) 1040–1041, *J. Chem. Phys.* **52** (1970) 1670–1684) claim that a non-uniqueness regime occurs for $z = \beta$ large enough and it is conjectured that the uniqueness should hold outside such an half line ($z = \beta \geq \beta_c > 0$). We solve partially this conjecture in any dimension by showing that for β large enough the non-uniqueness holds if and only if $z = \beta$. We show also that this critical value $z = \beta$ corresponds to the percolation threshold $\tilde{z}_c^a(\beta, r) = \beta$ for β large enough, providing a straight connection between these two notions of phase transition.

Résumé. Le modèle de Widom–Rowlinson (appelé aussi Area-interaction model) est un processus ponctuel de Gibbs dans $\mathbb{R}^d$ d'Hamiltonien le volume de $\cup_{x\in\omega} B_1(x)$, où ω est une configuration localement finie de points, et $B_1(x)$ la boule unité fermée centrée en x. Le modèle a deux paramètres : l'activité $z > 0$ liée à l'intensité du processus, et la température inverse $\beta \geq 0$ liée à la force de l'interaction. Dans cet article nous étudions la transition de phase du modèle du point de vue de la théorie de la percolation, et du point de vue de la transition liquide-gaz. Premièrement, en considérant le graphe connectant les points à distance au plus $2r > 0$, nous montrons que pour chaque $\beta \geq 0$, il existe $0 < \tilde{z}_c^a(\beta, r) < +\infty$ tel qu'il y ait décroissance exponentielle de la connectivité dans le régime sous-critique (i.e. $z < \tilde{z}_c^a(\beta, r)$) et une minoration linéaire de la connectivité à l'infini dans le régime sur-critique (i.e. $z > \tilde{z}_c^a(\beta, r)$). Ces résultats sont inspirés de travaux récents utilisant la théorie des algorithmes aléatoires (*Probab. Theory Related Fields* **173** (2019) 479–490, *Ann. of Math.* **189** (2019) 75–99, Duminil-Copin, Raoufi and Tassion (2018)). Deuxièmement nous étudions la transition de phase liquide-gaz, liée à l'unicité/non-unicité de la mesure de Gibbs en fonction des paramètres z, β. Des résultats anciens (*Phys. Rev. Lett.* **27** (1971) 1040–1041, *J. Chem. Phys.* **52** (1970) 1670–1684) montrent qu'il y a non-unicité lorsque $z = \beta$ sont assez grands, et il est conjecturé qu'il y a unicité en dehors de cette demi-droite ($z = \beta \geq \beta_c > 0$). Nous résolvons partiellement cette conjecture en toute dimension, en démontrant que pour chaque β assez grand, il y a non-unicité si et seulement si $z = \beta$. Nous démontrons également que la valeur critique $z = \beta$ correspond au seuil de percolation $\tilde{z}_c^a(\beta, r) = \beta$ pour β assez grand, donnant ainsi un lien étroit entre les deux notions de transition de phase développées dans le papier.

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The geometry of random walk isomorphism theorems

Roland Bauerschmidt[a,*], Tyler Helmuth[b,c] and Andrew Swan[a,†]

[a]*University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK.*
E-mail: [*]*rb812@cam.ac.uk;* [†]*acks2@cam.ac.uk*
[b]*University of Bristol, School of Mathematics, Fry Building, Woodland Road, Bristol, BS8 1UG, UK. E-mail: th17948@bristol.ac.uk*
[c]*Present address: Department of Mathematical Sciences, Durham University, Lower Mountjoy, DH1 3LE Durham, UK.*
E-mail: tyler.helmuth@durham.ac.uk

Abstract. The classical random walk isomorphism theorems relate the local times of a continuous-time random walk to the square of a Gaussian free field. A Gaussian free field is a spin system that takes values in Euclidean space, and this article generalises the classical isomorphism theorems to spin systems taking values in hyperbolic and spherical geometries. The corresponding random walks are no longer Markovian: they are the vertex-reinforced and vertex-diminished jump processes. We also investigate supersymmetric versions of these formulas.

Our proofs are based on exploiting the continuous symmetries of the corresponding spin systems. The classical isomorphism theorems use the translation symmetry of Euclidean space, while in hyperbolic and spherical geometries the relevant symmetries are Lorentz boosts and rotations, respectively. These very short proofs are new even in the Euclidean case.

Isomorphism theorems are useful tools, and to illustrate this we present several applications. These include simple proofs of exponential decay for spin system correlations, exact formulas for the resolvents of the joint processes of random walks together with their local times, and a new derivation of the Sabot–Tarrès formula for the limiting local time of the vertex-reinforced jump process.

Résumé. Les théorèmes classiques d'isomorphisme de marche aléatoire relient les heures locales d'une marche aléatoire en temps continu au carré d'un champ libre gaussien. Un champ libre gaussien est un système de spin qui prend des valeurs dans l'espace euclidien, et cet article généralise les théorèmes d'isomorphisme classiques aux systèmes de spin prenant des valeurs de géométries hyperboliques et sphériques. Les marche aléatoires correspondantes ne sont plus markoviennes : elles sont les processus de saut renforcé par sommet et de saut réduit par sommet. Nous étudions également les versions supersymétriques de ces formules.

Nos preuves sont basées sur l'exploitation des symétries continues des systèmes de spin correspondants. Les théorèmes d'isomorphisme classiques utilisent la symétrie de traduction de l'espace euclidien, tandis qu'en géométries hyperboliques et sphériques les symétries pertinentes sont respectivement des amplificateurs de Lorentz et des rotations. Ces très courtes preuves sont nouvelles même dans le cas euclidien.

Les théorèmes d'isomorphisme sont des outils utiles, et pour illustrer cela, nous présentons plusieurs applications. Celles-ci incluent notamment de simples preuves de décroissance exponentielle pour des corrélations du système de spin, des formules exactes pour les résolvants des processus conjoints de marches aléatoires combinés à leur heure locale, et une nouvelle dérivation de la formule de Sabot–Tarrès pour l'heure locale limitée du processus de saut renforcé par sommet.

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Continuity in κ in SLE$_\kappa$ theory using a constructive method and Rough Path Theory

Dmitry Beliaev[a,*], Terry J. Lyons[a,†] and Vlad Margarint[b]

[a]*University of Oxford, Mathematical Institute, Radcliffe Observatory, Andrew Wiles Building, Woodstock Rd, Oxford OX2 6GG, United Kingdom. E-mail: *belyaev@maths.ox.ac.uk; †terry.lyons@maths.ox.ac.uk*
[b]*NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 1555 Century Ave, Pudong, Shanghai, 200122, China. E-mail:* vdm2@nyu.edu

Abstract. Questions regarding the continuity in κ of the SLE$_\kappa$ traces and maps appear very naturally in the study of SLE. In order to study the first question, we consider a natural coupling of SLE traces: for different values of κ we use the same Brownian motion. It is very natural to assume that with probability one, SLE$_\kappa$ depends continuously on κ. It is rather easy to show that SLE is continuous in the Carathéodory sense, but showing that SLE traces are continuous in the uniform sense is much harder. In this note we show that for a given sequence $\kappa_j \to \kappa \in (0, 8/3)$, for almost every Brownian motion SLE$_\kappa$ traces converge locally uniformly. This result was also recently obtained by Friz, Tran and Yuan using different methods. In our analysis, we provide a constructive way to study the SLE$_\kappa$ traces for varying parameter $\kappa \in (0, 8/3)$. The argument is based on a new dynamical view on the approximation of SLE curves by curves driven by a piecewise square root approximation of the Brownian motion.

The second question can be answered naturally in the framework of Rough Path Theory. Using this theory, we prove that the solutions of the backward Loewner Differential Equation driven by $\sqrt{\kappa} B_t$ when started away from the origin are continuous in the p-variation topology in the parameter κ, for all $\kappa \in \mathbb{R}_+$.

Résumé. Des questions touchant à la continuité en κ des traces et des applications conformes du SLE$_\kappa$ apparaissent très naturellement dans l'étude des SLE. Afin d'étudier la première de ces questions, nous considérons un couplage naturel des traces des SLE: pour différentes valeurs de κ nous utilisons le même mouvement brownien. Il est très naturel de supposer qu'avec probabilité 1, SLE$_\kappa$ dépend continument de κ. Il est assez facile de montrer la continuité dans le sens de Carathéodory, mais montrer une telle continuité uniforme est bien plus ardu. Dans cette note, nous montrons que pour une suite donnée $\kappa_j \to \kappa \in (0, 8/3)$, et pour presque toute trajectoire du mouvement brownien, les traces de SLE$_\kappa$ convergent localement uniformément. Ce résultat a été également obtenu récemment par Friz, Tran et Yuan par d'autres méthodes. Dans notre analyse, nous donnons une façon constructive d'étudier les traces de SLE$_\kappa$ pour un paramètre variable $\kappa \in (0, 8/3)$. L'argument se base sur un nouveau point de vue dynamique sur les approximations des courbes SLE par des courbes conduites par des approximations par morceaux de fonctions racine carrée du mouvement brownien.

La seconde question peut être résolue naturellement dans le cadre de la théorie des chemins rugueux. À l'aide de cette théorie, nous montrons que les solutions de l'équation différentielle de Loewner rétrograde conduite par $\sqrt{\kappa} B_t$, partant loin de l'origine, sont continues dans la topologie de la p-variation en le paramètre κ, pour tout $\kappa \in \mathbb{R}_+$.

MSC2020 subject classifications: 60-XX; 30-XX

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Edgeworth expansions for weakly dependent random variables

Kasun Fernando[a] and Carlangelo Liverani[b]

[a]*Department of Mathematics, University of Maryland, 4176 Campus Drive, College Park, MD 20742-4015, USA. E-mail: abkf@math.umd.edu*
[b]*Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy. E-mail: liverani@mat.uniroma2.it*

Abstract. We discuss sufficient conditions that guarantee the existence of asymptotic expansions for the CLT for *weakly dependent* random variables including observations arising from sufficiently chaotic dynamical systems like piece-wise expanding maps, and strongly ergodic Markov chains. As a corollary we obtain refinements of the Local Limit Theorem and moderate deviation results. We primarily use spectral techniques to obtain the results.

Résumé. Nous discutons des conditions suffisantes garantissant l'existence d'expansions asymptotiques du théorème central limite pour des variables aléatoires *faiblement dépendantes*, dont des observations provenant de systèmes dynamiques suffisamment chaotiques comme des applications dilatantes par morceaux, et des chaînes de Markov fortement ergodiques. Comme corollaire, nous obtenons des raffinements du théorème local limite et de résultats de déviations modérées. Nos méthodes sont principalement des techniques spectrales.

MSC2020 subject classifications: 60F05; 37A50

Keywords: Edgeworth expansions; Weak dependence; Markov chains; Expanding maps

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Central limit theorem for mesoscopic eigenvalue statistics of deformed Wigner matrices and sample covariance matrices

Yiting Li*, Kevin Schnelli[1,†] and Yuanyuan Xu[2,‡]

*KTH Royal Institute of Technology, Stockholm, Sweden. E-mail: *yitingl@kth.se; †schnelli@kth.se; ‡yuax@kth.se*

Abstract. We consider N by N deformed Wigner random matrices of the form $X_N = H_N + A_N$, where H_N is a real symmetric or complex Hermitian Wigner matrix and A_N is a deterministic real bounded diagonal matrix. We prove a universal Central Limit Theorem for the linear eigenvalue statistics of X_N for all mesoscopic scales both in the spectral bulk and at regular edges where the global eigenvalue density vanishes as a square root. The method relies on studying the characteristic function of the linear statistics (Landon and Sosoe (2018)) by using the cumulant expansion method, along with local laws for the Green function of X_N (*Ann. Probab.* **48** (2020) 963–1001; *Probab. Theory Related Fields* **169** (2017) 257–352; *J. Math. Phys.* **54** (2013) 103504) and analytic subordination properties of the free additive convolution (Dallaporta and Fevrier (2019); *Random Matrices Theory Appl.* **9** (2020) 2050011). We also prove the analogous results for high-dimensional sample covariance matrices.

Résumé. Nous considérons des matrices aléatoires de Wigner déformées de taille N de la forme $X_N = H_N + A_N$, où H_N est une matrice hermitienne de Wigner symétrique ou complexe réelle, et A_N est une matrice diagonale déterministe avec des entrées réelles et bornées. Nous prouvons un théorème de limite centrale universel pour les statistiques linéaires des valeurs propres de X_N pour toutes les échelles mésoscopiques à la fois dans le centre de spectre et aux bords réguliers où la densité globale des valeurs propres disparait sous forme de racine carrée. La méthode repose sur l'étude de la fonction caractéristique des statistiques linéaires (Landon and Sosoe (2018)) en utilisant la méthode des cumulants, ainsi que les lois locales pour la fonction de Green de X_N (*Ann. Probab.* **48** (2020) 963–1001; *Probab. Theory Related Fields* **169** (2017) 257–352; *J. Math. Phys.* **54** (2013) 103504) et les propriétés de subordination analytique de la convolution libre additive (Dallaporta and Fevrier (2019); *Random Matrices Theory Appl.* **9** (2020) 2050011). Nous prouvons également les résultats analogues pour des matrices de corrélation empirique de haute dimension.

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Strong convergence order for slow–fast McKean–Vlasov stochastic differential equations

Michael Röckner[a], Xiaobin Sun[b,*] and Yingchao Xie[b,†]

[a]*Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany, and Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS), Beijing, 100190, China. E-mail: roeckner@math.uni-bielefeld.de*
[b]*School of Mathematics and Statistics and Research Institute of Mathematical Science, Jiangsu Normal University, Xuzhou, 221116, China. E-mail: *xbsun@jsnu.edu.cn; †ycxie@jsnu.edu.cn*

Abstract. In this paper, we consider the averaging principle for a class of McKean–Vlasov stochastic differential equations with slow and fast time-scales. Under some proper assumptions on the coefficients, we first prove that the slow component strongly converges to the solution of the corresponding averaged equation with convergence order $1/3$ using the approach of time discretization. Furthermore, under stronger regularity conditions on the coefficients, we use the technique of Poisson equation to improve the order to $1/2$, which is the optimal order of strong convergence in general.

Résumé. Dans cet article, nous considérons le principe de moyennisation pour une classe d'équations différentielles stochastiques de type McKean–Vlasov avec une échelle de temps lente et une échelle de temps rapide. Sous des hypothèses adéquates sur les coefficients, nous montrons d'abord que la composante lente converge vers la solution de l'équation moyennée correspondante avec un ordre de convergence $1/3$ en utilisant une approche par discrétisation en temps. D'autre part, sous des hypothèses de régularité plus fortes sur les coefficients, nous utilisons la technique de l'équation de Poisson pour améliorer l'ordre à $1/2$, ce qui est l'ordre optimal de la convergence forte en général.

MSC2020 subject classifications: Primary 60H10; secondary 34F05

Keywords: Averaging principle; McKean–Vlasov stochastic differential equations; Slow–fast; Poisson equation; Strong convergence order

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Global martingale solutions for quasilinear SPDEs via the boundedness-by-entropy method

Gaurav Dhariwal[a,*], Florian Huber[a,†], Ansgar Jüngel[a,‡], Christian Kuehn[b,§] and Alexandra Neamţu[b,¶]

[a] *Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, 1040 Wien, Austria.*
E-mail: [*] *gaurav.dhariwal@tuwien.ac.at;* [†] *florian.huber@asc.tuwien.ac.at;* [‡] *juengel@tuwien.ac.at*
[b] *Department of Mathematics, Technical University of Munich, Boltzmannstr. 3, 85748 Garching bei München, Germany.*
E-mail: [§] *ckuehn@ma.tum.de;* [¶] *alexandra.neamtu@tum.de*

Abstract. The existence of global-in-time bounded martingale solutions to a general class of cross-diffusion systems with multiplicative Stratonovich noise is proved. The equations describe multicomponent systems from physics or biology with volume-filling effects and possess a formal gradient-flow or entropy structure. This structure allows for the derivation of almost surely positive lower and upper bounds for the stochastic processes. The existence result holds under some assumptions on the interplay between the entropy density and the multiplicative noise terms. The proof is based on a stochastic Galerkin method, a Wong–Zakai type approximation of the Wiener process, the boundedness-by-entropy method, and the tightness criterion of Brzeźniak and coworkers. Three-species Maxwell–Stefan systems and n-species biofilm models are examples that satisfy the general assumptions.

Résumé. Nous prouvons l'existence des solutions globales et bornées pour une classe générale de systèmes de diffusion croisées avec bruit de Stratonovich multiplicatif. Les équations décrivent des systèmes à plusieurs composantes issus de la physique ou de la biologie, avec des effets de remplissage de volume (« volume-filling »), et possèdent une structure formelle de flot de gradient ou d'entropie. Cette structure permet d'obtenir des limites presque-sures inférieures et supérieures positives pour les processus stochastiques. Le résultat d'existence a lieu sous certaines hypothèses sur l'interaction entre la densité d'entropie et les termes de bruit multiplicatif. La preuve est basée sur une méthode stochastique de Galerkin, une approximation de type Wong–Zakai du processus de Wiener, la méthode « boundedness-by-entropy » et le critère de tension de Brzeźniak et de ses collaborateurs. Les systèmes de Maxwell–Stefan à trois espèces et les modèles de biofilms à n espèces sont des exemples qui satisfont les hypothèses générales.

MSC2020 subject classifications: 60H15; 35R60; 35Q35; 35Q92

Keywords: Cross diffusion; Martingale solutions; Entropy method; Tightness; Skorokhod–Jakubowski theorem; Maxwell–Stefan systems; Biofilm model

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