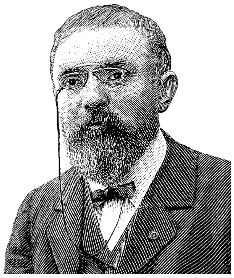

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Path dependent Feynman–Kac formula for forward backward stochastic Volterra integral equations

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Abstract. This paper is concerned with the relationship between forward–backward stochastic Volterra integral equations (FBSVIEs, for short) and a system of (nonlocal in time) path dependent partial differential equations (PPDEs, for short). Due to the nature of Volterra type equations, the usual flow property (or semigroup property) does not hold. Inspired by Viens–Zhang (*Ann. Appl. Probab.* **29** (2019) 3489–3540) and Wang–Yong (*Stochastic Process. Appl.* **129** (2019) 4926–4964), auxiliary processes are introduced so that the flow property of adapted solutions to the FBSVIEs is recovered in a suitable sense, and thus the functional Itô formula is applicable. Having achieved this stage, a natural PPDE is found so that the adapted solution of the backward SVIEs admits a representation in terms of the solution to the forward SVIE via the solution to a PPDE. On the other hand, the solution of the PPDE admits a representation in terms of adapted solution to the (path dependent) FBSVIE, which is referred to as a Feynman–Kac formula. This leads to the existence and uniqueness of a classical solution to the PPDE, under smoothness conditions on the coefficients of the FBSVIEs. Further, when the smoothness conditions are relaxed with the backward component of FBSVIE being one-dimensional, a new (and suitable) notion of viscosity solution is introduced for the PPDE, for which a comparison principle of the viscosity solutions is established, leading to the uniqueness of the viscosity solution. Finally, some results have been extended to coupled FBSVIEs and type-II BSVIEs, and a representation formula for the path derivatives of PPDE solution is obtained by a closer investigation of linear FBSVIEs.

Résumé. Cet article étudie les relations entre les équations intégrales de Volterra forward-backward stochastiques (FBSVIE) et un système d'équations aux dérivées partielles, non locales en temps, dépendant des trajectoires (PPDE). En raison de la nature des équations du type Volterra, la propriété habituelle de flot, ou de semigroupe, n'est pas vérifiée. Inspirés par les travaux de Viens–Zhang (*Ann. Appl. Probab.* **29** (2019) 3489–3540) et Wang–Yong (*Stochastic Process. Appl.* **129** (2019) 4926–4964), nous introduisons des processus auxiliaires de sorte que la propriété de flot de solutions adaptées aux FBSVIE soit retrouvée dans un sens approprié, et que la formule d'Itô fonctionnelle soit applicable. Puis, nous exhibons une PPDE naturelle telle que la solution adaptée de la SVIE backward admet une représentation en termes de la solution de la SVIE forward via la solution de cette PPDE. Par ailleurs, la solution de la PPDE admet une représentation en termes de la solution à la FBSVIE (dépendant de la trajectoire), ce que nous interprétons comme une formule de Feynman–Kac. Ceci conduit à l'existence et l'unicité d'une solution classique de la PPDE, sous des conditions de régularité pour les coefficients de la FBSVIE. De plus, sous l'hypothèse que la composante backward de la FBSVIE est de dimension 1, on peut affaiblir ces conditions de régularité en introduisant une nouvelle notion de solution de viscosité pour la PPDE, et établir un principe de comparaison de ces solutions qui implique leur unicité. Enfin, certains résultats ont été étendus aux FBSVIE couplées et aux BSVIE de type II, et une formule de représentation des dérivées des trajectoires des solutions de la PPDE est obtenue par une étude plus approfondie des FBSVIE linéaires.

MSC2020 subject classifications: 60H20; 45D05; 35K10; 35D40; 60G22

Keywords: Forward backward stochastic Volterra integral equation; Path dependent partial differential equation; Feynman–Kac formula; Viscosity solution; Comparison principle

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Nonlinear predictable representation and $\mathbb{L}^1$-solutions of backward SDEs and second-order backward SDEs

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Abstract. The theory of backward SDEs extends the predictable representation property of Brownian motion to the nonlinear framework, thus providing a path-dependent analog of fully nonlinear parabolic PDEs. In this paper, we consider backward SDEs, their reflected version, and their second-order extension, in the context where the final data and the generator satisfy $\mathbb{L}^1$-integrability condition. Our main objective is to provide the corresponding existence and uniqueness results for general Lipschitz generators. The uniqueness holds in the so-called Doob class of processes, simultaneously under an appropriate class of measures. We emphasize that the previous literature only deals with backward SDEs, and requires either that the generator is separable in (y, z), see Peng (In *Backward Stochastic Differential Equations* (1997) 141–159 Longman), or strictly sublinear in the gradient variable z, see Briand, Delyon, Hu, Pardoux and Stoica (*Stochastic Process. Appl.* **108** (1) (2003) 109–129), or that the final data satisfies an $L \ln L$-integrability condition, see Hu and Tang (*Electron. Commun. Probab.* **23** (2018) 27). We bypass these conditions by defining $\mathbb{L}^1$-integrability under the nonlinear expectation operator induced by the previously mentioned class of measures.

Résumé. La théorie des équations différentielles stochastiques rétrogrades étend la propriété de représentation prévisible du mouvement brownien au cadre non linéaire, offrant ainsi un analogue non-markovien aux équations aux dérivées partielles paraboliques complètement non linéaires. Dans ce papier, nous considérons les EDS rétrogrades, leur version réfléchies et son extension au second ordre, dans le contexte d'une donnée terminale et d'un générateur $\mathbb{L}^1$-intégrables. Notre objectif est d'établir un résultat d'existence et d'unicité pour un générateur Lipschitzien. Nous montrons que l'unicité a lieu dans la classe des processus de Doob, simultanément sous une classe appropriée de mesures de probabilité sur l'espace des trajectoires. Notons que ce résultat est nouveau, même dans le cas particulier des EDS rétrogrades, où la littérature précédente établit l'unicité pour des générateurs, soit séparables en (y, z) (Peng (In *Backward Stochastic Differential Equations* (1997) 141–159 Longman)), soit strictement sous-linéaires en la variable de gradient z, (Briand, Delyon, Hu, Pardoux and Stoica (*Stochastic Process. Appl.* **108** (1) (2003) 109–129)), ou alors sous une condition d'intégrabilité $L \ln L$ (Hu and Tang (*Electron. Commun. Probab.* **23** (2018) 27)). Nous évitons de recourir à de telles conditions en introduisant l'intégrabilité $\mathbb{L}^1$ sous l'opérateur d'espérance non linéaire induit par la classe ci-dessus de mesures de probabilité.

MSC2020 subject classifications: 60H10

Keywords: Backward SDE; Second-order backward SDE; Nonlinear expectation; Nondominated probability measures

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A central limit theorem for descents of a Mallows permutation and its inverse

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Abstract. This paper studies the asymptotic distribution of descents $\mathrm{des}(w)$ in a permutation w, and its inverse, distributed according to the Mallows measure. The Mallows measure is a non-uniform probability measure on permutations introduced to study ranked data. Under this measure, permutations are weighted according to the number of inversions they contain, with the weighting controlled by a parameter q. The main results are a Berry–Esseen theorem for $\mathrm{des}(w) + \mathrm{des}(w^{-1})$ as well as a joint central limit theorem for $(\mathrm{des}(w), \mathrm{des}(w^{-1}))$ to a bivariate normal with a non-trivial correlation depending on q. The proof uses Stein's method with size-bias coupling along with a regenerative process associated to the Mallows measure.

Résumé. Cet article étudie la distribution asymptotique des descentes $\mathrm{des}(w)$ dans une permutation w, et son inverse, distribuée suivant la mesure de Mallows. La mesure de Mallows est une probabilité non-uniforme sur les permutations introduite pour étudier les données ordonnées. Sous cette mesure, les permutations sont pondérées suivant le nombre d'inversions qu'elles contiennent, avec des poids contrôlés par un paramètre q. Les résultats principaux consistent en un théorème de Berry–Esseen pour $\mathrm{des}(w) + \mathrm{des}(w^{-1})$ ainsi qu'un théorème central limite joint pour $(\mathrm{des}(w), \mathrm{des}(w^{-1}))$ ayant comme limite une loi normale bi-variée avec une corrélation non-triviale dépendant de q. La preuve utilise la méthode de Stein avec des couplages biaisés par la taille, et un processus de renouvellement associé à la mesure de Mallows.

MSC2020 subject classifications: 60F05; 60B15; 05E16; 20F55

Keywords: Mallows permutations; Descents; Central limit theorem; Stein's method

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Topological expansion in isomorphism theorems between matrix-valued fields and random walks

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Abstract. We consider Gaussian fields of real symmetric, complex Hermitian or quaternionic Hermitian matrices over an electrical network, and describe how the isomorphisms between these fields and random walks give rise to topological expansions encoded by ribbon graphs. We further consider matrix-valued Gaussian fields twisted by an orthogonal, unitary or symplectic connection. In this case the isomorphisms involve traces of holonomies of the connection along random walk loops parametrized by boundary cycles of ribbon graphs.

Résumé. On considère des champs gaussiens de matrices symétriques réelles, hermitiennes complexes ou hermitiennes quaternioniques au dessus un réseau électrique, et on décrit comment l'isomorphisme entre ces champs et les marches aléatoires fait apparaître des développements topologiques représentées par des graphes à rubans. De plus, on considère des champs gaussiens matriciels tordus par une connexion orthogonale, unitaire ou symplectique. Dans ce cas les isomorphismes font intervenir des traces d'holonomies de la connexion le long des boucles de marche aléatoire paramétrées par les cycles de bord des graphes à rubans.

MSC2020 subject classifications: Primary 60G15; 81T18; 81T25; secondary 15B52; 60J55

Keywords: Discrete gauge theory; Gaussian free field; Gaussian orthogonal ensemble; Gaussian symplectic ensemble; Gaussian unitary ensemble; Isomorphism theorems; Holonomy; Matrix integrals; Matrix models; Random matrices; Random walks; Ribbon graphs; Topological expansion; Wilson loops

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Universality classes for general random matrix flows

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Abstract. We consider matrix-valued processes described as solutions to stochastic differential equations of very general form. We study the family of the empirical measure-valued processes constructed from the corresponding eigenvalues. We show that the family indexed by the size of the matrix is tight under very mild assumptions on the coefficients of the initial SDE. We characterize the limiting distributions of its subsequences as solutions to an integral equation. We use this result to study some universality classes of random matrix flows. These generalize the classical results related to Dyson Brownian motion and squared Bessel particle systems. We study some new phenomena as the existence of the generalized Marchenko–Pastur distributions supported on the real line. We also introduce universality classes related to generalized geometric matrix Brownian motions and Jacobi processes. Finally we study, under some conditions, the convergence of the empirical measure-valued process of eigenvalues associated to matrix flows to the law of a free diffusion.

Résumé. Nous considérons des processus à valeurs matricielles décrits comme des solutions d'équations différentielles stochastiques d'une forme très générale. Nous étudions la famille des processus empiriques à valeurs mesures construits à partir des valeurs propres correspondantes. Nous montrons que la famille indexée par la taille de la matrice est tendue sous des hypothèses faibles sur les coefficients de l'EDS initiale. Nous caractérisons les distributions limites des sous-suites comme solutions d'une équation intégrale. Nous utilisons ce résultat pour étudier certaines classes d'universalité de flots de matrices aléatoires. Ceci généralise le résultat classique lié au mouvement brownien de Dyson et aux systèmes de particules de carré de Bessel. Nous étudions de nouveaux phénomènes comme l'existence de la distribution généralisée de Marchenko–Pastur sur la droite réelle. Nous introduisons aussi une classe d'universalité reliée aux processus matriciels browniens géométriques et aux processus de Jacobi. Enfin, nous étudions, sous certaines conditions, la convergence du processus des mesures empiriques des valeurs propres associées aux flots de matrices, vers la loi d'une diffusion libre.

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Hamilton–Jacobi equations for inference of matrix tensor products

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Abstract. We study the high-dimensional limit of the free energy associated with the inference problem of finite-rank matrix tensor products. In general, we bound the limit from above by the unique solution to a certain Hamilton–Jacobi equation. Under additional assumptions on the nonlinearity in the equation which is determined explicitly by the model, we identify the limit with the solution. Two notions of solutions, weak solutions and viscosity solutions, are considered, each of which has its own advantages and requires different treatments. For concreteness, we apply our results to a model with i.i.d. entries and symmetric interactions. In particular, for the first order and even order tensor products, we identify the limit and obtain estimates on convergence rates; for other odd orders, upper bounds are obtained.

Résumé. Nous étudions la limite en grande dimension de l'énergie libre associée au problème d'inférence de produits tensoriels de matrices de rang fini. En toute généralité, nous montrons que la limite est bornée supérieurement par la solution unique d'une certaine équation de Hamilton–Jacobi. Sous des hypothèses supplémentaires sur la non-linéarité de l'équation qui est déterminée explicitement en fonction du modèle, nous identifions la limite avec cette solution. Deux notions de solutions sont considérées, les solutions faibles et les solutions de viscosité, chacune ayant ses propres avantages et requérant une approche différente. Comme exemple concret, nous appliquons nos résultats à un modèle à coefficients i.i.d. et à interactions symétriques. En particulier, pour les produits tensoriels de premier ordre et d'ordre pair, nous identifions la limite et obtenons des estimées des vitesses de convergence. Pour les autres ordres impairs, nous obtenons des bornes supérieures.

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Keywords: Inference problem; Hamilton–Jacobi equation; Tensor

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Spectral gap and cutoff phenomenon for the Gibbs sampler of $\nabla\varphi$ interfaces with convex potential

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Abstract. We consider the Gibbs sampler, or heat bath dynamics associated to log-concave measures on $\mathbb{R}^N$ describing $\nabla\varphi$ interfaces with convex potentials. Under minimal assumptions on the potential, we find that the spectral gap of the process is always given by $\text{gap}_N = 1 - \cos(\pi/N)$, and that for all $\epsilon \in (0,1)$, its ϵ-mixing time satisfies $T_N(\epsilon) \sim \frac{\log N}{2\text{gap}_N}$ as $N \to \infty$, thus establishing the cutoff phenomenon. The results reveal a universal behavior in that they do not depend on the choice of the potential.

Résumé. Nous considérons l'échantillonneur de Gibbs, aussi appelé dynamique "heat bath", associé à des mesures log-concaves sur $\mathbb{R}^N$ et décrivant des interfaces $\nabla\varphi$ avec potentiels convexes. Sous des hypothèses minimales sur le potentiel, nous montrons que le trou spectral du processus est toujours donné par $\text{gap}_N = 1 - \cos(\pi/N)$, et que pour tout $\epsilon \in (0,1)$, le temps de mélange de seuil ϵ satisfait $T_N(\epsilon) \sim \frac{\log N}{2\text{gap}_N}$ quand $N \to \infty$, ce qui établit un phénomène de cutoff. Ces résultats exhibent un comportement universel, en ce qu'ils ne dépendent pas du potentiel choisi.

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Keywords: Spectral gap; Mixing time; Cutoff

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Spectral analysis of the zigzag process

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Abstract. The zigzag process is a variant of the telegraph process with position dependent switching intensities. A characterization of the L^2-spectrum for the generator of the one-dimensional zigzag process is obtained in the case where the marginal stationary distribution on $\mathbb{R}$ is unimodal and the refreshment intensity is zero. Sufficient conditions are obtained for a spectral mapping theorem, mapping the spectrum of the generator to the spectrum of the corresponding Markov semigroup. Furthermore results are obtained for symmetric stationary distributions and for perturbations of the spectrum, in particular for the case of a non-zero refreshment intensity. In the examples we consider (including a Gaussian target distribution) a slight increase of the refreshment intensity above zero results in a larger L^2-spectral gap, corresponding to an improved convergence in L^2.

Résumé. Le processus de ZigZag est une variante du processus du télégraphe avec des intensités de retournement dépendant de la position. Nous obtenons une caractérisation du spectre en norme L^2 du générateur du processus de zigzag unidimensionnel, dans le cas où la distribution marginale stationnaire sur $\mathbb{R}$ est unimodale, avec une fréquence de rééchantillonnage nulle. Nous obtenons des conditions suffisantes pour un théorème d'isomorphisme spectral, identifiant le spectre du générateur à celui du semi-groupe de Markov correspondant. Par ailleurs, nous obtenons des résultats pour les distributions stationnaires symétriques ainsi que pour les perturbations du spectre dans le cas particulier d'une fréquence de rééchantillonnage non nulle. Enfin, nous considérons dans les exemples (avec une distribution cible gaussienne) un faible taux de rafraichissement positif par rapport aux résultats du taux nul, ce qui induit une plus grande bande dans le spectre L^2, correspondant à une convergence améliorée en norme L^2.

MSC2020 subject classifications: Primary 47A10; secondary 37A30; 60J25

Keywords: Spectral theory; Non-reversible Markov process; Markov semigroup; Zigzag process; Exponential ergodicity; Perturbation theory; Telegraph process; Piecewise deterministic Markov process

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Zero kinetic undercooling limit in the supercooled Stefan problem

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Abstract. We study the solutions of the one-phase supercooled Stefan problem with kinetic undercooling, which describes the freezing of a supercooled liquid, in one spatial dimension. Assuming that the initial temperature lies between the equilibrium freezing point and the characteristic invariant temperature throughout the liquid our main theorem shows that, as the kinetic undercooling parameter tends to zero, the free boundary converges to the (possibly irregular) free boundary in the supercooled Stefan problem without kinetic undercooling, whose uniqueness has been recently established in (Delarue, Nadtochiy and Shkolnikov (2019), Ledger and Søjmark (2018)). The key tools in the proof are a Feynman–Kac formula, which expresses the free boundary in the problem with kinetic undercooling through a local time of a reflected process, and a resulting comparison principle for the free boundaries with different kinetic undercooling parameters.

Résumé. Nous étudions les solutions de la phase unique de surfusion du problème de Stefan, avec surfusion cinétique, qui décrit le gel d'un liquide en surfusion en une dimension spatiale. En supposant que la température initiale est entre le point d'équilibre de gel et la température invariante caractéristique dans le liquide, notre théorème principal montre que, lorsque le paramètre de surfusion cinétique tend vers 0, la frontière libre converge vers la frontière libre (éventuellement irrégulière) dans le problème de surfusion de Stefan sans surfusion cinétique. L'unicité de cette dernière a été montrée récemment dans (Delarue, Nadtochiy and Shkolnikov (2019), Ledger and Søjmark (2018)). Les outils clés de la preuve sont une formule de Feynman–Kac, qui exprime la frontière libre dans le problème avec surfusion cinétique à travers le temps local d'un processus réfléchi, et un principe de comparaison pour les frontières libres avec différents paramètres de surfusion cinétique.

MSC2020 subject classifications: Primary 60J70; secondary 35R35; 60J55; 80A22

Keywords: Kinetic undercooling; Supercooled Stefan problem; Free boundary problems; Feynman–Kac formula; Local time; Reflected processes

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Geometric convergence bounds for Markov chains in Wasserstein distance based on generalized drift and contraction conditions

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Abstract. Let $(X_n)_{n=0}^{\infty}$ denote a Markov chain on a Polish space that has a stationary distribution ϖ. This article concerns upper bounds on the Wasserstein distance between the distribution of X_n and ϖ. In particular, an explicit geometric bound on the distance to stationarity is derived using generalized drift and contraction conditions whose parameters vary across the state space. These new types of drift and contraction allow for sharper convergence bounds than the standard versions, whose parameters are constant. Application of the result is illustrated in the context of a non-linear autoregressive process and a Gibbs algorithm for a random effects model.

Résumé. Soit $(X_n)_{n=0}^{\infty}$ une chaîne de Markov définie sur un espace polonais qui a une distribution stationnaire ϖ. Cet article s'intéresse aux bornes supérieures pour la distance de Wasserstein entre les distributions X_n et ϖ. En particulier, une borne géométrique explicite est obtenue sur la distance à l'équilibre en utilisant des conditions de dérive et de contraction dont les paramètres varient dans l'espace d'états. Ces nouveaux types de dérive et contraction permettent d'obtenir des bornes de convergence plus précises que les versions standard où les paramètres sont constants. Des applications de ce résultat sont données dans le contexte des processus auto-régressifs non-linéaires et dans le contexte d'un algorithme de Gibbs pour le modèle à effets aléatoires.

MSC2020 subject classifications: 60J05

Keywords: Convergence analysis; Exponential convergence; Kantorovich–Rubinstein distance; Lyapunov drift function; Polish space; Quantitative bound

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A microscopic derivation of coupled SPDE's with a KPZ flavor

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Abstract. We consider an interacting particle system driven by a Hamiltonian dynamics and perturbed by a conservative stochastic noise so that the full system conserves two quantities: energy and volume. The Hamiltonian part is regulated by a scaling parameter vanishing in the limit. We study the form of the fluctuations of these quantities at equilibrium and derive coupled stochastic partial differential equations with a KPZ flavor.

Résumé. Nous considérons un système de particules en interaction régi par une dynamique hamiltonienne perturbée par un bruit stochastique conservatif de sorte que le système complet conserve deux quantités : l'énergie et le volume. La partie hamiltonienne est régulée par un paramètre d'échelle disparaissant à la limite. Nous étudions la forme des fluctuations de ces quantités à l'équilibre et dérivons des équations aux dérivées partielles stochastiques couplées qui ont une fragrance de KPZ.

MSC2020 subject classifications: 60K35; 35R60; 60H15; 60H40; 82C22

Keywords: Equilibrium fluctuations; Stochastic Burgers Equation; Fluctuating hydrodynamics

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Couplings for Andersen dynamics

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Abstract. Andersen dynamics is a standard method for molecular simulations, and a precursor of the Hamiltonian Monte Carlo algorithm used in MCMC inference. The stochastic process corresponding to Andersen dynamics is a PDMP (piecewise deterministic Markov process) that iterates between Hamiltonian flows and velocity randomizations of randomly selected particles. Both from the viewpoint of molecular dynamics and MCMC inference, a basic question is to understand the convergence to equilibrium of this PDMP particularly in high dimension. Here we introduce a coupling approach to derive explicit convergence bounds in a Wasserstein sense. The bounds are dimension free for not necessarily convex potentials with weakly interacting components on a high dimensional torus, and for strongly convex and gradient Lipschitz potentials on a Euclidean product space.

Résumé. La dynamique d'Andersen est une méthode standard pour les simulations moléculaires et un précurseur de l'algorithme de Monte Carlo Hamiltonien utilisé dans l'inférence MCMC. Le processus stochastique correspondant à la dynamique d'Andersen est un PDMP (processus de Markov déterministe par morceaux) qui itère entre les écoulements hamiltoniens et les randomisations de vitesse de particules sélectionnées au hasard. Tant du point de vue de la dynamique moléculaire que de l'inférence MCMC, une question fondamentale est de comprendre la convergence vers l'équilibre de ce PDMP, surtout en dimension supérieure. Nous présentons ici des couplages pour obtenir des bornes de convergence au sens de Wasserstein qui ne nécessitent pas de convexité globale de l'énergie potentielle sous-jacente.

MSC2020 subject classifications: Primary 60J25; secondary 65C05

Keywords: Molecular dynamics; Markov Chain Monte Carlo; Hamiltonian Monte Carlo; Couplings

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Rates of convergence in the central limit theorem for martingales in the non stationary setting

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Abstract. In this paper, we give rates of convergence, for minimal distances and for the uniform distance, between the law of partial sums of martingale differences and the limiting Gaussian distribution. More precisely, denoting by P_X the law of a random variable X and by G_a the normal distribution $\mathcal{N}(0,a)$, we are interested by giving quantitative estimates for the convergence of $P_{S_n/\sqrt{V_n}}$ to G_1, where S_n is the partial sum associated with either martingale differences sequences or more general dependent sequences, and $V_n = \operatorname{Var}(S_n)$. Applications to linear statistics, non stationary ρ-mixing sequences and sequential dynamical systems are given.

Résumé. Dans cet article nous donnons des vitesses de convergence, pour des distances minimales ainsi que pour la distance uniforme, entre la loi des sommes partielles de différences de martingales et la loi Gaussienne limite. Plus précisément, en notant P_X la loi d'une variable aléatoire X et G_a la loi normale $\mathcal{N}(0,a)$, nous donnons des estimées quantitatives de la vitesse de convergence de $P_{S_n/\sqrt{V_n}}$ vers G_1, où S_n est la somme partielle formée à partir de différences de martingales ou d'une suite de variables dépendantes, et $V_n = \operatorname{Var}(S_n)$. Nous présentons également des applications du résultat principal à certaines statistiques linéaires, à des suites ρ-mélangeantes non stationnaires, ainsi qu'à une classe de systèmes dynamiques séquentiels.

MSC2020 subject classifications: 60F05; 60G42; 60G48

Keywords: Minimal distances; Ideal distances; Gaussian approximation; Berry–Esseen type inequalities; Martingales; ρ-mixing sequences; Sequential dynamical systems

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Random walk in cooling random environment: Recurrence versus transience and mixed fluctuations

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Abstract. This is the third in a series of papers in which we consider one-dimensional Random Walk in Cooling Random Environment (RWCRE). The latter is obtained by starting from one-dimensional Random Walk in Random Environment (RWRE) and resampling the environment along a sequence of deterministic times, called *refreshing times*. In the present paper we explore two questions for general refreshing times. First, we investigate how the recurrence versus transience criterion known for RWRE changes for RWCRE. Second, we explore the fluctuations for RWCRE when RWRE is either recurrent or satisfies a classical central limit theorem. We show that the answer depends in a delicate way on the choice of the refreshing times. An overarching goal of our paper is to investigate how the behaviour of a random process with a rich correlation structure can be affected by resettings.

Résumé. Ceci est le troisième d'une série d'articles dans lesquels nous considérons une marche aléatoire unidimensionnelle dans un milieu aléatoire refroidissant (MAMAR). Ce processus est obtenu en partant d'une marche aléatoire unidimensionnelle dans un milieu aléatoire (MAMA) et en rafraîchissant l'environnement le long d'une séquence de temps déterministes, appelée *temps de rafraîchissement*. Dans le présent article, nous explorons deux questions pour des moments de rafraîchissement généraux. Tout d'abord, nous examinons comment le critère de récurrence connu pour MAMA change pour MAMAR. Deuxièmement, nous explorons les fluctuations de MAMAR lorsque MAMA est récurrent ou satisfait un théorème central limite classique. Nous montrons que la réponse dépend de manière subtile du choix des moments de rafraîchissement. Un objectif primordial de notre article est d'étudier comment le comportement d'un processus aléatoire avec une riche structure de corrélation peut être affecté par des rafraîchissements.

MSC2020 subject classifications: Primary 60F05; secondary 60G50; 60K37

Keywords: Random walk; Dynamic random environment; Refreshing times; Cooling regimes; Recurrence versus transience; Mixed fluctuations

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A shape theorem and a variational formula for the quenched Lyapunov exponent of random walk in a random potential

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Abstract. We prove a shape theorem and derive a variational formula for the limiting quenched Lyapunov exponent and the Green's function of random walk in a random potential on a square lattice of arbitrary dimension and with an arbitrary finite set of steps. The potential is a function of a stationary environment and the step of the walk. This potential is subject to a moment assumption which has strictness tied to the mixing rate of the environment. Our setting includes directed and undirected polymers, random walk in static and dynamic random environment, and, in the zero-temperature case, our results also give a shape theorem and a variational formula for the time constant of both site and edge directed last-passage percolation and standard first-passage percolation.

Résumé. Nous prouvons un théorème de forme et déduisons une formule variationnelle pour l'exposant de Lyapunov et la fonction de Green de la marche aléatoire dans un potentiel aléatoire sur un réseau carré de dimension arbitraire et avec un ensemble fini arbitraire des pas possibles. Le potentiel est une fonction d'un environnement stationnaire et du pas de la marche. Ce potentiel est soumis à une hypothèse sur les moments qui est liée à la vitesse de mélange du milieu. Notre cadre comprend les modèles de polymères dirigés et non dirigés, les marche aléatoire dans un environnement aléatoire statique et dynamique, et, dans le cas de température nulle, nos résultats donnent également un théorème de forme et une formule variationnelle pour la constante de temps de la percolation du dernier passage dirigée par site et par arête et de la percolation du premier passage standard.

MSC2020 subject classifications: 60K35; 60K37

Keywords: Cocycle; First-passage percolation; FPP; Green's function; Last-passage percolation; LPP; Lyapunov exponent; Random polymer measure; Random walk; Random environment; Random potential; RWRE; RWRP; Shape theorem; Variational formula

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Transience and recurrence of sets for branching random walk via non-standard stochastic orders

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Abstract. We study how the recurrence and transience of space-time sets for a branching random walk on a graph depends on the offspring distribution. Here, we say that a space-time set A is *recurrent* if it is visited infinitely often almost surely on the event that the branching random walk survives forever, and say that A is *transient* if it is visited at most finitely often almost surely. We prove that if μ and ν are supercritical offspring distributions with means $\bar{\mu} < \bar{\nu}$ then every space-time set that is recurrent with respect to the offspring distribution μ is also recurrent with respect to the offspring distribution ν and similarly that every space-time set that is transient with respect to the offspring distribution ν is also transient with respect to the offspring distribution μ. To prove this, we introduce a new order on probability measures that we call the *germ order* and prove more generally that the same result holds whenever μ is smaller than ν in the germ order. Our work is inspired by the work of Johnson and Junge (AIHP 2018), who used related stochastic orders to study the frog model.

Résumé. Nous étudions comment la récurrence et la transience d'ensembles d'espace-temps pour une marche aléatoire branchante sur un graphe dépendent de la distribution de la progéniture. Un ensemble d'espace-temps A est appelé récurrent s'il est visité infiniment souvent sur l'événement de survie de la marche aléatoire branchante ; il est appelé transient si le nombre de visites est fini presque sûrement. Nous prouvons que si μ et ν sont deux distributions surcritiques de moyennes $\bar{\mu} < \bar{\nu}$ alors tout ensemble d'espace-temps qui est récurrent pour μ l'est aussi pour ν, et pareillement tout ensemble transient pour ν l'est aussi pour μ. Afin de prouver ce résultat nous introduisons un nouvel ordre partiel sur les mesures de probabilités que nous appelons germinal. Le résultat ci-dessus est ainsi vrai plus généralement pour deux mesures μ et ν telles que μ est plus petite que ν dans l'ordre germinal. Cette approche est inspirée par les travaux de Johnson et Junge (AIHP 2018) qui ont utilisé des notions reliées d'ordres stochastiques pour étudier le "frog model".

MSC2020 subject classifications: 60J80; 60J10

Keywords: Branching random walk; Recurrence; Transience; Stochastic ordering; Maximum displacement; Minkowski dimension

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Central limit theorems for parabolic stochastic partial differential equations

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Abstract. Let $\{u(t,x)\}_{t\geq 0, x\in\mathbb{R}^d}$ denote the solution of a d-dimensional nonlinear stochastic heat equation that is driven by a Gaussian noise, white in time with a homogeneous spatial covariance that is a finite Borel measure f and satisfies Dalang's condition. We prove two general functional central limit theorems for occupation fields of the form $N^{-d}\int_{\mathbb{R}^d} g(u(t,x))\psi(x/N)\,\mathrm{d}x$ as $N\to\infty$, where g runs over the class of Lipschitz functions on $\mathbb{R}^d$ and $\psi\in L^2(\mathbb{R}^d)$. The proof uses Poincaré-type inequalities, Malliavin calculus, compactness arguments, and Paul Lévy's classical characterization of Brownian motion as the only mean zero, continuous Lévy process. Our result generalizes central limit theorems of Huang et al. (*Stochastic Process. Appl.* **131** (2020) 7170–7184; *Stoch. Partial Differ. Equ., Anal. Computat.* **8** (2020) 402–421) valid when $g(u)=u$ and $\psi=\mathbf{1}_{[0,1]^d}$.

Résumé. Soit $\{u(t,x)\}_{t\geq 0, x\in\mathbb{R}^d}$ la solution d'une équation de la chaleur stochastique non-linéaire d-dimensionnelle, perturbée par un bruit gaussien, blanc en temps et avec une covariance homogène en espace donnée par une mesure de Borel finie qui satisfait la condition de Dalang. Nous démontrons deux théorèmes de la limite centrale fonctionnels pour des champs d'occupation de la forme $N^{-d}\int_{\mathbb{R}^d} g(u(t,x))\psi(x/N)\,\mathrm{d}x$ quand $N\to\infty$, où g est une function lipschitzienne sur $\mathbb{R}^d$ et $\psi\in L^2(\mathbb{R}^d)$. La preuve utilise des inegalités de type Poincaré, le calcul de Malliavin, des arguments de compacité et la caractérisation du mouvement brownien comme le seul processus de Lévy continu de moyenne nulle. Notre résultat généralise les théorèmes de la limite centrale de Huang et al (*Stochastic Process. Appl.* **131** (2020) 7170–7184 ; *Stoch. Partial Differ. Equ., Anal. Computat.* **8** (2020) 402–421) qui sont valables lorsque $g(u)=u$ et $\psi=\mathbf{1}_{[0,1]^d}$.

MSC2020 subject classifications: 60H15; 60F17; 60H07

Keywords: Stochastic heat equation; Central limit theorem; Poincaré inequalities; Malliavin calculus; Metric entropy

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Renormalisation from non-geometric to geometric rough paths

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Abstract. The Hairer–Kelly map has been introduced for establishing a correspondence between geometric and non-geometric rough paths. Recently, a new renormalisation on rough paths has been proposed in (*Proc. Lond. Math. Soc.* **121**(2) (2020) 220–251), built on this map and the Lyons–Victoir extension theorem. In this work, we compare this renormalisation with the existing ones such as BPHZ and the local products renormalisations. We prove that they commute in a certain sense with the Hairer–Kelly map and exhibit an explicit formula in the framework of (*Proc. Lond. Math. Soc.* **121**(2) (2020) 220–251). We also see how the renormalisation behaves in the alternative approach in (*Ann. Inst. Henri Poincaré Probab. Stat.* **55**(2) (2019) 1131–1148) for moving from non-geometric to geometric rough paths.

Résumé. L'application de Hairer–Kelly a été introduite pour établir une correspondance entre les chemins rugueux géométriques et non-géométriques. Récemment, une nouvelle renormalisation sur les chemins rugueux a été proposée dans (*Proc. Lond. Math. Soc.* **121**(2) (2020) 220–251), construite d'après cette application et le théorème d'extension de Lyons–Victoir. Dans ce travail, on compare cette renormalisation avec celles déjà existantes comme la renormalisation BPHZ et la renormalisation des produits locaux. On montre qu'elles commutent dans un certain sens avec l'application de Hairer–Kelly et on dévoile une formule explicite dans le contexte de (*Proc. Lond. Math. Soc.* **121**(2) (2020) 220–251). On considère aussi le comportement de ces renormalisations dans l'approche alternative (*Ann. Inst. Henri Poincaré Probab. Stat.* **55**(2) (2019) 1131–1148) qui permet de passer d'un chemin rugueux non-géométrique à un chemin rugueux géométrique.

MSC2020 subject classifications: Primary 60H10; secondary 16T05

Keywords: Rough paths; Renormalisation; Hopf algebra

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The Brownian disk viewed from a boundary point

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Abstract. We provide a new construction of Brownian disks in terms of forests of continuous random trees equipped with nonnegative labels corresponding to distances from a distinguished point uniformly distributed on the boundary of the disk. This construction shows in particular that distances from the distinguished point evolve along the boundary as a five-dimensional Bessel bridge. As an important ingredient of our proofs, we show that the uniform measure on the boundary, as defined in the earlier work of Bettinelli and Miermont, is the limit of the suitably normalized volume measure on a small tubular neighborhood of the boundary. Our construction also yields a simple proof of the equivalence between the definition of the Brownian half-plane given by Gwynne and Miller as the scaling limit of the uniform infinite half-plane quadrangulation and the alternative definition proposed by Caraceni and Curien.

Résumé. Nous donnons une nouvelle construction du disque brownien à partir d'une forêt d'arbres aléatoires continus munis de labels positifs correspondant aux distances depuis un point distingué uniformément distribué sur la frontière du disque. Cette construction montre en particulier que les distances depuis le point distingué évoluent le long de la frontière selon un pont de Bessel de dimension 5. Un ingrédient important de nos preuves consiste à montrer que la mesure uniforme sur la frontière, définie dans le travail précédent de Bettinelli et Miermont, est limite de la mesure de volume convenablement normalisée sur un petit voisinage tubulaire de la frontière. Notre construction fournit aussi une preuve simple de l'équivalence entre la définition du demi-plan brownien donnée par Gwynne et Miller comme limite d'échelle de la quadrangulation infinie uniforme du demi-plan et la définition alternative proposée par Caraceni et Curien.

MSC2020 subject classifications: 60D05

Keywords: Brownian disk; Continuous random tree; Bessel bridge; Boundary measure; Brownian half-plane

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The density of the (α, d, β)-superprocess and singular solutions to a fractional non-linear PDE

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Abstract. We consider the density X_t of the critical (α, d, β)-superprocess with $\alpha \in (0, 2)$, $1+\beta \in (1, 2]$ and $\beta < \frac{\alpha}{d}$. A recent result on partial differential equations (*Nonlinear Anal.* **137** (2016) 306–337) implies the following dichotomy when the process is conditioned on survival: if $x \in \mathbb{R}^d$ is fixed and $\beta \leq \beta^* = \frac{\alpha}{d+\alpha}$, then $X_t(x) > 0$ a.s.; otherwise, the probability that $X_t(x)$ is positive has power law decay. We strengthen this by proving that if $\beta < \beta^*$ and the density is continuous, then the density function is strictly positive almost surely.

We then consider the event that X_t charges a measure μ, that is, $\int X_t(x)\mu(dx)$ is positive. We give close to sharp conditions on μ under which X_t charges μ almost surely when conditioned on survival. Our characterization is based on the size of the support of μ, in the sense of Hausdorff measure and dimension. There is a critical dimension d_c such that all measures with support of dimension greater than d_c are charged almost surely. If μ has support of dimension less than d_c and satisfies a uniform density condition, we estimate the decay of the probability that X_t charges μ.

Our methods also give new results for the non-linear fractional partial differential equation

$$\partial_t u(t, x) = \Delta_\alpha u(t, x) - u(t, x)^{1+\beta} \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

where $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian. The initial trace of a positive solution consists of a Radon measure and a closed set $\mathcal{S}$, called the singular set, where the solution has non-integrable singularities as $t \to 0$. For $\beta < \frac{\alpha}{d}$, the existence of solutions with singular set $\mathcal{S}$ is characterized via the critical dimension d_c. Solutions do not exist if the Hausdorff dimension of $\mathcal{S}$ exceeds (and in some cases equals) d_c. If the dimension of $\mathcal{S}$ is less than d_c and $\mathcal{S}$ satisfies a technical condition, we prove that a solution with singular set $\mathcal{S}$ exists.

Résumé. Nous nous intéressons à la densité X_t du (α, d, β)-superprocessus critique avec $\alpha \in (0, 2)$, $1+\beta \in (1, 2]$ et $\beta < \frac{\alpha}{d}$. Un résultat récent en théorie des équations aux dérivées partielles (*Nonlinear Anal.* **137** (2016) 306–337) implique la dichotomie suivante lorsque l'on conditionne le processus à la survie : si $x \in \mathbb{R}^d$ est fixé et $\beta \leq \beta^* = \frac{\alpha}{d+\alpha}$, alors $X_t(x) > 0$ p.s. ; dans le cas contraire, la probabilité que $X_t(x)$ soit strictement positif a une décroissance polynomiale. Nous renforçons cela en montrant que si $\beta < \beta^*$ et si la densité est continue, alors la fonction de densité est strictement positive presque surement.

Nous considérons alors l'événement que X_t charge une mesure μ, c'est-à-dire que $\int X_t(x)\mu(dx)$ est strictement positif. Nous donnons des conditions presque optimales sur μ pour que X_t charge μ presque surement conditionnellement à l'événement de survie. Notre caractérisation est formulée en termes de la taille du support de μ, au sens des mesures et dimensions de Hausdorff. Il existe une dimension critique d_c telle que toutes les mesures ayant un support de dimension supérieure à d_c sont chargées presque surement. Si μ a un support de dimension inférieure à d_c et satisfait une condition de densité uniforme, nous estimons la probabilité que X_t charge μ.

Nos méthodes donnent également des résultats nouveaux sur l'équation aux dérivées partielles non linéaire fractionnaire

$$\partial_t u(t, x) = \Delta_\alpha u(t, x) - u(t, x)^{1+\beta} \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

où $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ est le laplacien fractionnaire. La trace initiale d'une solution positive consiste en une mesure de Radon et un ensemble fermé $\mathcal{S}$, appelé l'ensemble singulier, où la solution a des singularités non intégrables lorsque $t \to 0$. Quand $\beta < \frac{\alpha}{d}$, l'existence de solutions avec ensemble singulier $\mathcal{S}$ est caractérisée en termes de la dimension critique d_c. Aucune solution n'existe si la dimension de Hausdorff de $\mathcal{S}$ dépasse (et dans certains cas, est égale à) d_c. Si la dimension de $\mathcal{S}$ est inférieure à d_c et $\mathcal{S}$ satisfait une condition technique, nous montrons qu'il existe une solution d'ensemble singulier $\mathcal{S}$.

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Keywords: Superprocess densities; Stable branching; Fractional semilinear pde; Initial trace

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Intrinsic area near the origin for self-similar growth-fragmentations and related random surfaces

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Abstract. We study the behaviour of a natural measure defined on the leaves of the genealogical tree of some branching processes, namely self-similar growth-fragmentation processes. Each particle, or cell, is attributed a positive mass that evolves in continuous time according to a positive self-similar Markov process and gives birth to children at negative jumps events. We are interested in the asymptotics of the mass of the ball centered at the root, as its radius decreases to 0. We obtain the almost sure behaviour of this mass when the Eve cell starts with a strictly positive size. This differs from the situation where the Eve cell grows indefinitely from size 0. In this case, we show that, when properly rescaled, the mass of the ball converges in distribution towards a non-degenerate random variable. We then derive bounds describing the almost sure behaviour of the rescaled mass. Those results are applied to certain random surfaces, exploiting the connection between growth-fragmentations and random planar maps obtained in (*Probab. Theory Related Fields* **172** (2018) 663–724). This allows us to extend a result of Le Gall (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 237–313) on the volume of a free Brownian disk close to its boundary, to a larger family of stable disks. The upper bound of the mass of a typical ball in the Brownian map is refined, and we obtain a lower bound as well.

Résumé. Nous étudions le comportement d'une mesure naturellement définie sur les feuilles de l'arbre généalogique de certains processus de branchement, à savoir les processus de croissance-fragmentation autosimilaires. Chaque particule, ou cellule, possède une masse positive qui évolue dans le temps suivant un processus de Markov autosimilaire positif et donne naissance à des enfants à chaque saut négatif. Nous nous intéressons aux asymptotiques de la masse de la boule centrée sur la racine, lorsque son rayon décroit vers 0. Nous obtenons le comportement presque sûr de cette masse lorsque la cellule Eve démarre avec une taille strictement positive. Cela diffère de la situation ou la cellule Eve croît indéfiniment depuis une taille nulle. Dans ce cas, nous montrons que, proprement renormalisée, la masse de la boule converge en loi vers une variable aléatoire non-dégénérée. Nous déduisons alors des bornes décrivant le comportement presque sûr de la masse remise à l'échelle. Ces résultats sont appliqués à certaines surfaces aléatoires, en exploitant la connexion entre processus de croissance-fragmentation et cartes planaires aléatoires obtenue dans (*Probab. Theory Related Fields* **172** (2018) 663–724). Cela nous permet d'étendre des résultats de Le Gall (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 237–313) sur le volume du disque Brownien libre proche de sa frontière, à une famille plus grande de disques stables. La borne supérieure de la masse d'une boule typique dans la carte Brownienne est améliorée et nous obtenons une borne inférieure également.

MSC2020 subject classifications: 60J25; 60G18; 60G57

Keywords: Self-similar growth-fragmentations; Intrinsic area; Rate of growth

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On the explosion of the number of fragments in simple exchangeable fragmentation-coagulation processes

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Abstract. We consider the exchangeable fragmentation-coagulation (EFC) processes, where the coagulations are multiple and not simultaneous, as in a Λ-coalescent, and the fragmentations dislocate at finite rate an individual block into sub-blocks of infinite size. Sufficient conditions are found for the block counting process to explode (i.e. to reach ∞) or not and for ∞ to be either an exit boundary or an entrance boundary. In a case of regularly varying fragmentation and coagulation mechanisms, we find regimes where the boundary ∞ can be either an exit, an entrance or a regular boundary. In the latter regular case, the EFC process leaves instantaneously the set of partitions with an infinite number of blocks and returns to it immediately. Our proofs are based on a new sufficient condition of explosion for positive continuous-time Markov chains, which is of independent interest.

Résumé. Nous considérons les processus de fragmentation-coagulation échangeables (EFC), où les coagulations sont multiples et non simultanées, comme dans un Λ-coalescent, et où les fragmentations disloquent à taux fini un bloc individuel en sous-blocs de taille infinie. Des conditions suffisantes sont trouvées pour que le processus du nombre de blocs explose (c'est-à-dire atteigne l'infini) ou non et pour que l'infini soit une frontière de sortie ou une frontière d'entrée. Lorsque les mécanismes de fragmentation et de coagulation satisfont une hypothèse de variation régulière, nous trouvons des régimes où la frontière ∞ peut être une sortie, une entrée ou un point régulier. Dans ce dernier cas régulier, le processus EFC quitte instantanément l'ensemble des partitions avec un nombre infini de blocs et y revient immédiatement. Nos preuves sont basées sur une nouvelle condition suffisante d'explosion pour les chaînes de Markov positives en temps continu, laquelle peut avoir un intérêt indépendant.

MSC2020 subject classifications: 60J80; 60J70; 60J90; 92D25

Keywords: Λ-coalescent; Fragmentation; Branching process; Explosion; Coming down from infinity; Entrance boundary; Regular boundary; Continuous-time Markov chains

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Weak convergence of the intersection point process of Poisson hyperplanes

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Abstract. This paper deals with the intersection point process of a stationary and isotropic Poisson hyperplane process in $\mathbb{R}^d$ of intensity $t > 0$, where only hyperplanes that intersect a centred ball of radius $R > 0$ are considered. Taking $R = t^{-\frac{d}{d+1}}$ it is shown that this point process converges in distribution, as $t \to \infty$, to a Poisson point process on $\mathbb{R}^d \setminus \{0\}$ whose intensity measure has power-law density proportional to $\|x\|^{-(d+1)}$ with respect to the Lebesgue measure. A bound on the speed of convergence in terms of the Kantorovich–Rubinstein distance is provided as well. In the background is a general functional Poisson approximation theorem on abstract Poisson spaces. Implications on the weak convergence of the convex hull of the intersection point process and the convergence of its f-vector are also discussed, disproving and correcting thereby a conjecture of Devroye and Toussaint (*J. Algorithms* **14** (1993) 381–394) in computational geometry.

Résumé. Cet article traite du processus ponctuel d'intersection d'un processus d'hyperplans de Poisson stationnaire et isotrope dans $\mathbb{R}^d$ d'intensité $t > 0$, où seuls les hyperplans qui intersectent une boule centrée de rayon $R > 0$ sont considérés. En prenant $R = t^{-\frac{d}{d+1}}$, on montre que ce processus ponctuel converge en distribution, lorsque $t \to \infty$, vers un processus ponctuel de Poisson dans $\mathbb{R}^d \setminus \{0\}$ dont la mesure d'intensité a une densité proportionnelle à $\|x\|^{-(d+1)}$ par rapport à la mesure de Lebesgue. Une borne sur la vitesse de convergence en termes de distance de Kantorovich–Rubinstein est également fournie. En arrière-plan se trouve un théorème général d'approximation de Poisson fonctionnel sur les espaces de Poisson abstraits. Les implications sur la convergence faible de l'enveloppe convexe du processus de point d'intersection et la convergence de son vecteur f sont également discutées, réfutant et corrigeant ainsi une conjecture de Devroye et Toussaint (*J. Algorithms* **14** (1993) 381–394) en géométrie computationelle.

MSC2020 subject classifications: Primary 60D05; 60F05; secondary 52A22; 53C65

Keywords: Convex hull; Integral geometry; Intersection point process; Poisson hyperplane process; Poisson point process approximation; Rate of convergence; Weak convergence

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Two-dimensional Lorentz process for magnetotransport: Boltzmann-Grad limit

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Abstract. We study a system of charged, noninteracting classical particles moving in a Poisson distribution of hard-disk scatterers in two dimensions, under the effect of a magnetic field perpendicular to the plane. We prove that, in the low-density (Boltzmann-Grad) limit, the particle distribution evolves according to a generalized linear Boltzmann equation, previously derived and solved by Bobylev et al. (*Phys. Rev. Lett.* **75** (1995) 2, *J. Stat. Phys.* **87** (1997) 1205–1228, *J. Stat. Phys.* **102** (2001) 1133–1150). In this model, Boltzmann's chaos fails, and the kinetic equation includes non-Markovian terms. The ideas of (*Phys. Rev.* **185** (1969) 308–322) can be however adapted to prove convergence of the process with memory.

Résumé. Nous étudions un système de particules classiques chargées, sans interaction, se déplaçant dans une distribution de Poisson de disques durs en deux dimensions, sous l'effet d'un champ magnétique perpendiculaire au plan. Nous montrons que, dans la limite de faible densité (Boltzmann-Grad), la distribution des particules évolue selon une équation de Boltzmann linéaire généralisée, déjà obtenue et résolue par Bobylev et al. (*Phys. Rev. Lett.* **75** (1995) 2, *J. Stat. Phys.* **87** (1997) 1205–1228, *J. Stat. Phys.* **102** (2001) 1133–1150). Dans ce modèle, le chaos de Boltzmann n'est pas vérifié et l'équation cinétique inclut des termes non markoviens. Les idées de (*Phys. Rev.* **185** (1969) 308–322) peuvent cependant être adaptées pour prouver la convergence du processus avec mémoire.

MSC2020 subject classifications: 82C05; 82C40; 35Q70; 60J99

Keywords: Lorentz gas; Magnetic field; Generalized Boltzmann equation; Low-density limit; Non-Markovian process; Memory terms

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Asymptotic expansion of correlation functions for $\mathbb{Z}^d$ covers of hyperbolic flows

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Abstract. We establish expansion of an arbitrary order for the correlation function of sufficiently regular observables of $\mathbb{Z}^d$ extensions of some hyperbolic flows. Our examples include the $\mathbb{Z}^2$ periodic Lorentz gas and geodesic flows on abelian covers of compact manifolds with negative curvature.

Résumé. Nous établissons des développements asymptotiques de tous ordres pour la fonction corrélation d'observables suffisamment régulières de $\mathbb{Z}^d$-extensions de flots hyperboliques. Nos résultats s'appliquent au gaz de Lorentz $\mathbb{Z}^2$-periodique et au flot géodésique sur des revêtements abéliens de variétés compactes de courbure négative.

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Iterated invariance principle for slowly mixing dynamical systems

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Abstract. We give sufficient Gordin-type criteria for the iterated (enhanced) weak invariance principle to hold for deterministic dynamical systems. Such an invariance principle is intrinsically related to the interpretation of stochastic integrals. We illustrate this with examples of deterministic fast-slow systems where our iterated invariance principle yields convergence to a stochastic differential equation.

Résumé. Nous donnons des critères suffisants de type Gordin assurant qu'un principe d'invariance itéré (renforcé) faible est satisfait pour des systèmes dynamiques déterministes. Un tel principe d'invariance est intrinsèquement relié à l'interprétation des intégrales stochastiques. Nous illustrons ceci par des exemples de systèmes lents-rapides déterministes où notre principe d'invariance itéré implique la convergence vers une équation différentielle stochastique.

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