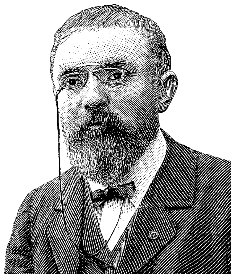

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On mean estimation for heteroscedastic random variables

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Abstract. We study the problem of estimating the common mean μ of n independent symmetric random variables with different and unknown standard deviations $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$. We show that, under some mild regularity assumptions on the distribution, there is an adaptive estimator $\widehat{\mu}$ such that it is invariant to permutations of the elements of the sample and satisfies that, up to logarithmic factors, with high probability,

$$|\widehat{\mu} - \mu| \lesssim \min\left\{\sigma_{m^*}, \frac{\sqrt{n}}{\sum_{i=\sqrt{n}}^{n} \sigma_i^{-1}}\right\},$$

where the index $m^* \lesssim \sqrt{n}$ satisfies $m^* \approx \sqrt{\sigma_{m^*} \sum_{i=m^*}^{n} \sigma_i^{-1}}$.

Résumé. Nous étudions le problème de l'estimation de la moyenne commune μ de n variables aléatoires symétriques indépendantes avec des écarts types différents et inconnus $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$. Nous montrons que, sous faibles hypothèses de régularité sur la distribution, il existe un estimateur adaptatif $\widehat{\mu}$ invariant par rapport aux permutations des éléments de l'échantillon qui satisfait à facteurs logarithmiques près et avec une grande probabilité

$$|\widehat{\mu} - \mu| \lesssim \min\left\{\sigma_{m^*}, \frac{\sqrt{n}}{\sum_{i=\sqrt{n}}^{n} \sigma_i^{-1}}\right\},$$

où l'indice $m^* \lesssim \sqrt{n}$ satisfait $m^* \approx \sqrt{\sigma_{m^*} \sum_{i=m^*}^{n} \sigma_i^{-1}}$.

MSC2020 subject classifications: Primary 62G30; 62F25; secondary 62F35

Keywords: Mean estimation; Heteroscedastic observations; Order statistic; Robustness; Adaptivity

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Trees, forests, and impurity-based variable importance in regression

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Abstract. Tree ensemble methods such as random forests (*Mach. Learn.* **45** (2001) 5–32) are very popular to handle high-dimensional tabular data sets, notably because of their ability to detect sparse signals and their resulting good predictive accuracy. However, when machine learning is used for decision-making problems, settling for the best predictive procedures may not be reasonable since enlightened decisions require to understand the phenomena underlying the data, which is accessible only with an in-depth comprehension of the algorithm prediction process. Unfortunately, random forests are not intrinsically interpretable since their prediction results from averaging several hundreds of decision trees. A classic approach to gain knowledge on this so-called black-box algorithm is to compute variable importances, that are employed to assess the predictive impact of each input variable. Variable importances are then used to rank or select variables and thus play a great role in data analysis. Mean Decrease Impurity (MDI) is one of the two variable importance measures in random forests. However, there is no theoretical justification to use MDI: we do not even know what this indicator estimates. In this paper, we analyze MDI and prove that if input variables are independent and in absence of interactions, MDI provides a variance decomposition of the output, where the contribution of each variable is clearly identified. We also study models exhibiting dependence between input variables or interaction, for which the variable importance is intrinsically ill-defined.

Résumé. Les méthodes d'ensemble basées sur les arbres de décision comme les forêts aléatoires (*Mach. Learn.* **45** (2001) 5–32) sont très prisées pour traiter des jeux de données tabulaires de grande dimension, notamment de par leur capacité à détecter des signaux parcimonieux et les bonnes performances prédictives qui en découlent. Cependant, lorsque l'apprentissage automatique est utilisé pour des problèmes d'aide à la décision, choisir la procédure à utiliser uniquement au regard de ses capacités prédictives n'est pas souhaitable. En effet, prendre une décision éclairée requiert de comprendre les phénomènes régissant le comportement des données, ce qui n'est possible qu'en ayant une compréhension précise du processus de prédiction de l'algorithme. Malheureusement, les forêts aléatoires ne sont pas intrinsèquement interprétables puisque leur prédiction résulte de l'agrégation de plusieurs centaines d'arbres de décision. Une approche classique pour améliorer la compréhension de ces "boîtes noires" est de calculer les indices d'importance de variables, qui sont employés pour quantifier l'influence de chaque variable d'entrée sur la sortie. Les mesures d'importance de variables sont ensuite utilisées pour classer ou sélectionner les variables et jouent ainsi un rôle prépondérant dans les analyses de données. Le MDI (Mean Decrease Impurity) est l'une des deux mesures d'importance calculées par les forêts aléatoires. Cependant, il n'y a aucune justification théorique à l'utilisation du MDI : nous ne savons toujours pas vers quelle quantité cet indicateur converge. Dans cet article, nous analysons le MDI et prouvons qu'en l'absence d'interactions entre les variables d'entrées, si celles-ci sont de plus indépendantes, alors le MDI fournit une décomposition de la variance de la sortie, où la contribution de chaque variable d'entrée est clairement identifiée. Nous étudions également des modèles contenant des interactions ou de la dépendance (entre les variables d'entrées), et montrons que le MDI est intrinsèquement mal défini pour ces modèles.

MSC2020 subject classifications: 62G08; 62G20

Keywords: Random forests; Variable importance; Mean Decrease Impurity

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Risk bounds when learning infinitely many response functions by ordinary linear regression

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Abstract. Consider the problem of learning a large number of response functions simultaneously based on the same input variables. The training data consist of a single independent random sample of the input variables drawn from a common distribution together with the associated responses. The input variables are mapped into a high-dimensional linear space, called the feature space, and the response functions are modelled as linear functionals of the mapped features, with coefficients calibrated via ordinary least squares. We provide convergence guarantees on the worst-case excess prediction risk by controlling the convergence rate of the excess risk uniformly in the response function. The dimension of the feature map is allowed to tend to infinity with the sample size. The collection of response functions, although potentially infinite, is supposed to have a finite Vapnik–Chervonenkis dimension. The bound derived can be applied when building multiple surrogate models in a reasonable computing time.

Résumé. Nous considérons le problème de l'apprentissage simultané d'un grand nombre de fonctions de réponse en s'appuyant sur un échantillon d'entrées commun composé de variables aléatoires supposées indépendantes et identiquement distribuées. Ces données sont envoyées dans un espace de haute dimension, appelé espace caractéristique. Les fonctions de réponse sont approchées par des combinaisons linéaires de coordonnées de l'espace caractéristique dont les coefficients sont calibrés par la méthode des moindres carrés ordinaires. Nous fournissons des garanties de convergence sur l'excès de risque de prédiction en contrôlant le taux de convergence de l'excès de risque uniformément sur la classe de fonctions de réponse. La dimension de l'espace caractéristique peut tendre vers l'infini avec la taille de l'échantillon. Cependant, la collection de fonctions de réponse, bien que potentiellement infinie, est supposée avoir une dimension de Vapnik–Chervonenkis finie. Les bornes obtenues garantissent le bon fonctionnement de modèles de substitution pour remplacer une famille de modèles complexes dans l'optique de réduire les coûts numériques.

MSC2020 subject classifications: 62G20; 62J05

Keywords: Response surface model; Multitask learning; Ordinary least squares; Monte Carlo integration; Control variates

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Semiparametric estimation of McKean–Vlasov SDEs

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Abstract. In this paper we study the problem of semiparametric estimation for a class of McKean–Vlasov stochastic differential equations. Our aim is to estimate the drift coefficient of a MV-SDE based on observations of the corresponding particle system. We propose a semiparametric estimation procedure and derive the rates of convergence for the resulting estimator. We further prove that the obtained rates are essentially optimal in the minimax sense.

Résumé. Dans cet article, nous étudions le problème d'estimation semi-paramétrique pour une classe d'équations différentielles stochastiques de type McKean–Vlasov. Notre but est d'estimer le coefficient de dérive d'une EDS de type MV à partir d'observations du système de particules associé. Nous proposons une méthode d'estimation semi-paramétrique et obtenons les vitesses de convergence pour les estimateurs correspondants. Nous démontrons également que les vitesses de convergence sont quasi-optimales au sens minimax.

MSC2020 subject classifications: Primary 62G20; 62M05; secondary 60G07; 60H10

Keywords: Deconvolution; McKean–Vlasov SDEs; Mean field models; Multi-agent learning; Minimax bounds; Semiparametric estimation

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Extreme order statistics of random walks

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Abstract. This paper is concerned with the limit theory of the extreme order statistics derived from random walks. We establish the joint convergence of the order statistics near the minimum of a random walk in terms of the Feller chains. Detailed descriptions of the limit process are given in the case of simple symmetric walks and Gaussian walks.

Résumé. Cet article traite de la théorie limite des statistiques d'ordre extrêmes provenant des marches aléatoires. Nous établissons la convergence conjointe des statistiques d'ordre près du minimum d'une marche aléatoire en termes des chaînes de Feller. Des descriptions détaillées du processus limite sont données dans le cas de marches simples symétriques et des marches gaussiennes.

MSC2020 subject classifications: 60G50; 60F17; 60J05

Keywords: Bessel processes; Brownian embedding; Fluctuation theory; Limit theorems; Order statistics; Path decomposition; Random walk

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Busemann process and semi-infinite geodesics in Brownian last-passage percolation

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Abstract. We prove the existence of semi-infinite geodesics for Brownian last-passage percolation (BLPP). Specifically, on a single event of probability one, there exist semi-infinite geodesics started from every space-time point and traveling in every asymptotic direction. Properties of these geodesics include uniqueness for a fixed initial point and direction, non-uniqueness for fixed direction but random initial points, and coalescence of all geodesics traveling in a common, fixed direction. Along the way, we prove that for fixed northeast and southwest directions, there almost surely exist no bi-infinite geodesics in the given directions. The semi-infinite geodesics are constructed from Busemann functions. Our starting point is a result of Alberts, Rassoul-Agha and Simper that established Busemann functions for fixed points and directions. Out of this, we construct the global process of Busemann functions simultaneously for all initial points and directions, and then the family of semi-infinite Busemann geodesics. The uncountable space of the semi-discrete setting requires extra consideration and leads to new phenomena, compared to discrete models.

Résumé. Nous prouvons l'existence de géodésiques semi-infinies pour la percolation brownienne de dernier passage (BLPP). Plus précisément, sur un seul événement de probabilité 1, il existe des géodésiques semi-infinies partant de chaque point d'espace-temps et voyageant dans toutes les directions asymptotiques. Les propriétés de ces géodésiques comprennent l'unicité pour un point et une direction initiaux fixes, la non-unicité pour une direction fixe mais des points initiaux aléatoires, et la coalescence de toutes les géodésiques se déplaçant dans une direction fixe commune. En cours de route, nous prouvons que pour les directions fixes nord-est et sud-ouest, il n'existe presque sûrement pas de géodésiques bi-infinies dans les directions données. Les géodésiques semi-infinies sont construites à partir des fonctions de Busemann. Notre point de départ est le résultat d'Alberts, Rassoul-Agha et Simper qui ont établi des fonctions Busemann pour des points et directions fixes. À partir de là, nous construisons le processus global des fonctions de Busemann simultanément pour tous les points et directions initiaux, puis la famille des géodésiques de Busemann semi-infinies. L'espace indénombrable du cadre semi-discret nécessite une considération supplémentaire et conduit à de nouveaux phénomènes, par rapport aux modèles discrets.

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Keywords: Brownian motion; Busemann function; Semi-infinite geodesic; Coalescence; Last-passage percolation; Midpoint problem

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Scaling limit for random walk on the range of random walk in four dimensions

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Abstract. We establish scaling limits for the random walk whose state space is the range of a simple random walk on the four-dimensional integer lattice. These concern the asymptotic behaviour of the graph distance from the origin and the spatial location of the random walk in question. The limiting processes are the analogues of those for higher-dimensional versions of the model, but additional logarithmic terms in the scaling factors are needed to see these. The proof applies recently developed machinery relating the scaling of resistance metric spaces and stochastic processes, with key inputs being natural scaling statements for the random walk's invariant measure, the associated effective resistance metric, the graph distance, and the cut times for the underlying simple random walk.

Résumé. Nous établissons les limites d'échelle pour une marche aléatoire dont l'espace d'états est l'image d'une marche aléatoire simple sur le réseau des entiers de dimension 4. Celles-ci concernent le comportement asymptotique de la distance de graphe à partir de l'origine et la position spatiale de la marche aléatoire en question. Les processus limites sont des analogues de ceux pour le modèle en dimension supérieure, mais un terme logarithmique supplémentaire dans les facteurs de renormalisation est nécessaire. La preuve utilise la machinerie récemment développée liant limites d'échelle d'espaces avec métrique de résistance et processus stochastiques, avec comme éléments clés l'échelle naturelle pour la mesure invariante de la marche aléatoire, la métrique de résistance effective associée, la distance de graphe, et les temps de coupure pour la marche aléatoire simple sous-jacente.

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Keywords: Random walk; Scaling limit; Range of random walk; Random environment

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A functional stable limit theorem for Gibbs–Markov maps

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Abstract. For a class of locally (but not necessarily uniformly) Lipschitz continuous d-dimensional observables over a Gibbs–Markov system, we show that convergence of (suitably normalized and centered) ergodic sums to a non-Gaussian stable vector is equivalent to the distribution belonging to the classical domain of attraction, and that it implies a weak invariance principle in the (strong) Skorohod $\mathcal{J}_1$-topology on $\mathcal{D}([0,\infty),\mathbb{R}^d)$. The argument uses the classical approach via finite-dimensional marginals and $\mathcal{J}_1$-tightness. As applications, we record a Spitzer-type arcsine law for certain $\mathbb{Z}$-extensions of Gibbs–Markov systems, and prove an asymptotic independence property of excursion processes of intermittent interval maps.

Résumé. Pour une classe d'observables d-dimensionnelles localement (mais pas nécessairement uniformément) Lipschitz sur un système Gibbs–Markov, nous montrons que la convergence des sommes ergodiques (convenablement centrées et normalisées) vers un vecteur aléatoire de loi stable non gaussienne est équivalente au fait que la distribution appartient au domaine d'attraction classique. Dans ce cas nous montrons aussi un principe d'invariance faible dans la topologie forte de Skorohod $\mathcal{J}_1$ sur $\mathcal{D}([0,\infty),\mathbb{R}^d)$. L'argument utilise l'approche classique via les lois marginales de dimension finie et la tension de la suite dans le bon espace. Comme applications, nous démontrons une loi arcsinus à la Spitzer pour certaines $\mathbb{Z}$-extensions des systèmes Gibbs–Markov, ainsi qu'une propriété d'indépendance asymptotique des processus d'excursion de certaines applications intermittentes de l'intervalle.

MSC2020 subject classifications: 28D05; 37A25; 37C30; 11K50

Keywords: Weak invariance principle; Stable laws; Stable Lévy processes; Weakly dependent processes; Stationary sequences

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Diffusivity of a walk on fractures of a hypertorus

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Abstract. This article studies discrete height functions on the discrete hypertorus. These are functions on the vertices of this hypertorus graph for which the derivative satisfies a specific condition on each edge. We then perform a random walk on the set of such height functions, in the spirit of *Diffusivity of a random walk on random walks*, a work of Boissard, Cohen, Espinasse, and Norris. The goal is to estimate the diffusivity of this random walk in the mesh limit. It turns out that each height functions is characterised by a number of so-called *fractures* of the hypertorus. These fractures are then studied in isolation; we are able to understand their asymptotic behaviour in the mesh limit due to the recent understanding of the associated random surfaces. This allows for an asymptotic reduction to a one-dimensional continuous system consisting of $\gcd \mathbf{n}$ parts where $\mathbf{n} \in \mathbb{N}^d$ is the fundamental parameter of the original model. We then prove that the diffusivity of the random walk tends to $1/(1 + 2\gcd \mathbf{n})$ in this mesh limit.

Résumé. Cet article étudie des fonctions de hauteur discrètes sur l'hypertore discret. Il s'agit de fonctions définies sur les sommets de l'hypertore dont la dérivée satisfait une certaine condition en chaque arête. Nous considérons une marche aléatoire sur cet ensemble de fonctions, à l'instar des travaux de Boissard, Cohen, Espinasse et Norris dans leur article *Diffusivity of a random walk on random walks*. L'objectif est d'estimer la diffusivité de cette marche aléatoire dans la limite d'échelle. Nous montrons que toute fonction de hauteur est caractérisée par le nombre de *fractures* qu'elle induit sur l'hypertore. Nous étudions ensuite ces fractures ; il est possible de comprendre leur comportement asymptotique dans la limite d'échelle grâce à de récents travaux sur les surfaces aléatoires qui leur sont associées. Cela permet de réduire notre étude asymptotique à un système continu à une dimension, constitué de $\operatorname{pgcd} \mathbf{n}$ parties, où $\mathbf{n} \in \mathbb{N}^d$ est un paramètre fondamental du modèle initial. Nous montrons alors que la diffusivité de la marche aléatoire converge vers $1/(1 + 2\operatorname{pgcd} \mathbf{n})$ dans cette limite d'échelle.

MSC2020 subject classifications: Primary 60J10; secondary 60F05

Keywords: Random walk; Markov chain; Central limit theorem; Martingale approximation

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Cutoff for permuted Markov chains

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Abstract. Let P be a bistochastic matrix of size n, and Π be a permutation matrix of size n. In this paper, we are interested in the mixing time of the Markov chain whose transition matrix is given by $Q = P\Pi$. In other words, the chain alternates between random steps governed by P and deterministic steps governed by Π. We show that if the permutation Π is chosen uniformly at random, then under mild assumptions on P, with high probability, the chain Q exhibits cutoff at time $\frac{\log n}{\mathbf{h}}$, where $\mathbf{h}$ is the entropic rate of P. Moreover, for deterministic permutations, we improve the upper bound on the mixing time obtained by Chatterjee and Diaconis (*Probab. Theory Related Fields* **178** (2020) 1193–1214).

Résumé. Soit P une matrice bistochastique de taille n, et Π une matrice de permutation de taille n. Dans cet article, nous nous intéressons au temps de mélange de la chaîne de Markov dont la matrice de transition est donnée par $Q = P\Pi$. En d'autres termes, la chaîne alterne entre des sauts aléatoires gouvernés par P et des sauts déterministes gouvernés par Π. Nous montrons que si la permutation Π est choisie uniformément au hasard, alors, sous de légères hypothèses sur P, avec grande probabilité, la chaîne Q présente un cutoff au temps $\frac{\log n}{\mathbf{h}}$, où $\mathbf{h}$ est le taux entropique de P. De plus, pour des permutations déterministes, nous améliorons la borne supérieure sur le temps de mélange obtenue par Chatterjee and Diaconis (*Probab. Theory Related Fields* **178** (2020) 1193–1214).

MSC2020 subject classifications: 60J10

Keywords: Markov chains; Mixing times; Random permutations

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The diameter of the directed configuration model

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Abstract. We show that the diameter of the directed configuration model with n vertices rescaled by $\log n$ converges in probability to a constant. Our assumptions are the convergence of the in- and out-degree of a uniform random vertex in distribution, first and second moment. Our result extends previous results on the diameter of the model and applies to many other random directed graphs.

Résumé. Nous montrons que le diamètre du modèle de configuration dirigé avec n sommets renormalisé par $\log n$ converge en probabilité vers une constante. Nos hypothèses sont la convergence en loi des degrés entrant et sortant d'un sommet uniformément choisi, et celles en moments d'ordre 1 et d'ordre 2. Notre résultat étend les résultats précédents sur le diamètre du modèle et s'applique à de nombreux autres graphes orientés aléatoires.

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Keywords: Directed configuration model; Diameter; Distances

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A simplified second-order Gaussian Poincaré inequality in discrete setting with applications

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Abstract. In this paper, a simplified second-order Gaussian Poincaré inequality for normal approximation of functionals over infinitely many Rademacher random variables is derived. It is based on a new bound for the Kolmogorov distance between a general Rademacher functional and a Gaussian random variable, which is established by means of the discrete Malliavin–Stein method and is of independent interest. As an application, the number of vertices with prescribed degree and the subgraph counting statistic in the Erdős–Rényi random graph are discussed. The number of vertices of fixed degree is also studied for percolation on the Hamming hypercube. Moreover, the number of isolated faces in the Linial–Meshulam–Wallach random κ-complex and infinite weighted 2-runs are treated.

Résumé. Dans cet article, une inégalité de Poincaré gaussienne simplifiée du second ordre pour l'approximation normale de fonctionnelles sur une infinité de variables aléatoires de Rademacher est obtenue. Elle est basée sur une nouvelle limite pour la distance de Kolmogorov entre une fonction générale de Rademacher et une variable gaussienne, qui est établie au moyen de la méthode discrète de Malliavin–Stein et qui présente un intérêt indépendant. Comme application, le nombre de sommets de degré prescrit et la statistique de comptage des sous-graphes dans le graphe aléatoire d'Erdős–Rényi sont discutés. Le nombre de sommets de degré fixé est également étudié pour la percolation sur l'hypercube de Hamming. De plus, le nombre de faces isolées dans le κ-complexe aléatoire de Linial–Meshulam–Wallach et les succès consécutifs pondérés infinis sont examinés.

MSC2020 subject classifications: 05C80; 60F05; 60H07

Keywords: Berry–Esseen bound; Discrete stochastic analysis; Erdős–Rényi random graph; Infinite weighted 2-run; Isolated face; Hypercube percolation; Malliavin–Stein method; Rademacher functional; Random simplicial complex; Second-order Poincaré inequality; Subgraph count; Vertex of given degree

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Diffusive limits of two-parameter ordered Chinese Restaurant Process up-down chains

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Abstract. We construct a two-parameter family of Feller diffusions on the set of open subsets of (0, 1) that arise as diffusive limits of two-parameter ordered Chinese Restaurant Process up-down chains. The diffusions we construct are natural ordered analogues of Petrov's two-parameter extension of Ethier and Kurtz's infinitely-many-neutral-alleles diffusion model. Recently, there has been significant interest in ordered analogues of the diffusions Petrov constructed. Existing methods for constructing such processes have been based on pathwise methods using marked Lévy processes and an outstanding conjecture about these processes is that they are, in fact, the diffusive limit of the ordered Chinese Restaurant Process up-down chains that we consider here. We make progress on this conjecture by showing that the diffusive limit of the ordered Chinese Restaurant Process up-down chains exists. Moreover, our methods yield a simple, explicit description of the generator of the limiting processes on a core described in terms of quasisymmetric functions.

Résumé. Nous construisons une famille de diffusions de Feller à deux paramètres sur l'ensemble des sous-ensembles ouverts de (0, 1). Ces diffusions apparaissent comme les limites diffusives des chaînes ascendantes et descendantes du processus du restaurant chinois à deux paramètres. Les diffusions que nous construisons sont des analogues ordonnés naturels de l'extension à deux paramètres, introduite par Petrov, du modèle de diffusion avec un nombre infini d'allèles neutres considéré par Ethier et Kurtz. Récemment, il y a eu un intérêt significatif pour les analogues ordonnés des diffusions construites par Petrov. Les méthodes existantes pour construire de tels processus utilisent des processus de Lévy marqués. Il a été conjecturé que ces processus sont la limite diffusive des chaînes ascendantes et descendantes du processus de restauration chinois ordonné que nous considérons ici. Nous progressons sur cette conjecture en montrant que la limite diffusive de ces chaînes ordonnées du processus du restaurant chinois existe. De plus, nos méthodes donnent une description simple et explicite du générateur des processus limites sur un noyau décrit en termes de fonctions quasi-symétriques.

MSC2020 subject classifications: Primary 60J60; 60J45; 60C05; secondary 60J10; 05E05

Keywords: Chinese Restaurant Process; Up-down Markov chains; Diffusive limits

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Dynamical Gibbs–non-Gibbs transitions in Widom–Rowlinson models on trees

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Abstract. We consider the soft-core Widom–Rowlinson model for particles with spins and holes, on a Cayley tree of order d (which has $d+1$ nearest neighbours), depending on repulsion strength β between particles of different signs and on an activity parameter λ for particles. We analyse Gibbsian properties of the time-evolved intermediate Gibbs measure of the static model, under a spin-flip time evolution, in a regime of large repulsion strength β.

We first show that there is a dynamical transition, in which the measure becomes non-Gibbsian at large times, independently of the particle activity, for any $d \geq 2$. In our second and main result, we also show that for large β and at large times, the measure of the set of bad configurations (discontinuity points) changes from zero to one as the particle activity λ increases, assuming that $d \geq 4$. Our proof relies on a general zero-one law for bad configurations on the tree, and the introduction of a set of uniformly bad configurations given in terms of subtree percolation, which we show to become typical at high particle activity.

Résumé. Nous considérons le modèle de Widom–Rowlinson à contraintes molles constitué des particules de spins et de trous, sur un arbre de Cayley d'ordre d (à $d+1$ plus proches voisins), dépendant d'une force répulsive β entre particules de signes différents et d'un paramètre d'activité λ sur les particules. Nous analysons les propriétés gibbsiennes de l'évolution temporelle de la mesure de Gibbs intermédiaire du modèle statique, au cours d'une dynamique de renversement des spins, dans le régime de forte répulsion β.

Nous mettons d'abord en évidence l'existence d'une transition dynamique, au cours de laquelle la mesure devient non-gibbsienne en temps long, indépendamment de l'activité particulaire, pour tout $d \geq 2$. Dans notre second et principal résultat, nous montrons également que pour les grandes valeurs de β, en temps long, la mesure de l'ensemble des mauvaises configurations (points de discontinuités) passe de zéro à un lorsque l'activité des particules λ croît, en supposant $d \geq 4$. Notre preuve repose sur une loi du 0-1 générale pour les mauvaises configurations sur l'arbre, et sur l'introduction d'un ensemble de configurations uniformément mauvaises exprimé en terme de percolation sur un sous-arbre, que nous montrons être typique à haute activité particulaire.

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Keywords: Widom–Rowlinson model; Gibbs measures; Non-Gibbsianness; Stochastic dynamics; Dynamical Gibbs–non-Gibbs transitions; Phase transitions; Cayley tree; Percolation; Zero-one law

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Large deviation principle for the intersection measure of Brownian motions on unbounded domains

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Abstract. Consider the intersection measure ℓ_t^{IS} of p independent Brownian motions on $\mathbb{R}^d$. We prove the large deviation principle for the normalized intersection measure $t^{-p}\ell_t^{\mathrm{IS}}$ as $t \to \infty$, before exiting a (possibly unbounded) domain $D \subset \mathbb{R}^d$ with smooth boundary. This is an extension of the result of König and Mukherjee [*Comm. Pure Appl. Math.* **66** (2013) 263–306] which deals with the case D is bounded. The essential contribution of this paper is to prove the so-called super-exponential estimate for the intersection measure of killed Brownian motions on such D by an application of the Chapman–Kolmogorov relation. As a consequence, the new argument in this paper gives not only an extension to unbounded domains but also a simpler proof even for bounded domains.

Résumé. Nous considérons la mesure d'intersection ℓ_t^{IS} de p mouvements browniens indépendants sur $\mathbb{R}^d$. Nous prouvons un principe de grande déviation pour la mesure d'intersection normalisée $t^{-p}\ell_t^{\mathrm{IS}}$ lorsque t tend vers l'infini, avant de sortir d'un domaine $D \subset \mathbb{R}^d$ (qui peut être non borné) avec une frontière lisse. Ce travail généralise [*Comm. Pure Appl. Math.* **66** (2013) 263–306] dans lequel D est borné. La contribution essentielle de cet article est de prouver, par une application de la relation de Chapman–Kolmogorov, une estimation sur-exponentielle pour la mesure d'intersection des mouvements browniens tués sur un tel D. Ce nouvel argument apporte aussi une preuve plus simple dans le cas des domaines bornés.

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Keywords: Intersection measure; Large deviations

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A characterization of transportation-information inequalities for Markov processes in terms of dimension-free concentration

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Abstract. Inequalities between transportation costs and Fisher information are known to characterize certain concentration properties of Markov processes around their invariant measures. This note provides a new characterization of the quadratic transportation-information inequality W_2I in terms of a dimension-free concentration property for i.i.d. (conditionally on the initial positions) copies of the underlying Markov process. This parallels Gozlan's characterization of the quadratic transportation-entropy inequality W_2H. The proof is based on a new Laplace-type principle for the operator norms of Feynman-Kac semigroups, which is of independent interest. Lastly, we illustrate how both our theorem and (a form of) Gozlan's are instances of a general convex-analytic tensorization principle.

Résumé. Il est connu que les inégalités entre les coûts de transport et l'information de Fisher caractérisent certaines propriétés de concentration des processus de Markov autour de leurs mesures invariantes. Cette note apporte une nouvelle caractérisation de l'inégalité quadratique transport-information W_2I, en termes d'une propriété de concentration indépendante de la dimension pour les copies i.i.d. (conditionnellement aux positions initiales) du processus de Markov sous-jacent. Ceci est comparable à la caractérisation de Gozlan de l'inégalité quadratique transport-entropie W_2H. La preuve est basée sur un nouveau principe de type Laplace pour les normes d'opérateurs des semigroupes de Feynman-Kac, qui peut avoir un intérêt indépendant. Enfin, nous expliquons comment notre théorème et (une forme de) celui de Gozlan sont des exemples d'un principe général de tensorisation convexe-analytique.

MSC2020 subject classifications: Primary 26D10; 60E15; secondary 60J25

Keywords: Transportation-information inequalities; Dimension-free concentration

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Noise effects in some stochastic evolution equations: Global existence and dependence on initial data

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Abstract. In this paper, we consider the noise effects on a class of stochastic evolution equations including the stochastic Camassa–Holm equations with or without rotation. We first obtain the existence, uniqueness and a blow-up criterion of pathwise solutions in Sobolev space H^s with $s > 3/2$. Then we prove that strong enough noise can prevent blow-up with probability 1, which justifies the regularization effect of strong nonlinear noise in preventing singularities. Besides, such strengths of noise are estimated in different examples. Finally, for the interplay between regularization effect induced by the noise and the dependence on initial conditions, we introduce and investigate the stability of the exiting time and construct an example to show that the multiplicative noise cannot improve both the stability of the exiting time and the continuity of the dependence on initial data simultaneously.

Résumé. Dans cet article, nous considérons les effets du bruit sur une classe d'équations d'évolution stochastiques y compris les équations stochastiques de Camassa–Holm avec ou sans rotation. Nous obtenons d'abord l'existence, l'unicité et un critère d'explosion de solutions *pathwise* dans l'espace de Sobolev H^s avec $s > 3/2$. Ensuite, nous prouvons qu'un bruit suffisamment fort peut empêcher l'explosion avec probabilité 1, ce qui justifie l'effet régularisant du bruit non linéaire fort dans la prévention des singularités. De plus, de telles forces de bruit sont estimées dans les différents exemples. Enfin, pour l'interaction entre l'effet de régularisation induit par le bruit et la dépendance par rapport aux conditions initiales, nous introduisons et étudions la stabilité du temps de sortie et construisons un exemple pour montrer que le bruit multiplicatif ne peut pas améliorer simultanément la stabilité du temps de sortie et la continuité de la dépendance par rapport aux données initiales.

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Keywords: Stochastic evolution equations; Pathwise solution; Blow-up criterion; Regularization effect of noise; Weak instability

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Boundary traces of shift-invariant diffusions in half-plane

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Abstract. We study boundary traces of shift-invariant diffusions: two-dimensional diffusions in the upper half-plane $\mathbb{R} \times [0, \infty)$ (or in $\mathbb{R} \times [0, R)$) invariant under horizontal translations. We prove that the corresponding trace processes are Lévy processes with completely monotone jumps, and, conversely, every Lévy process with completely monotone jumps is a boundary trace of some shift-invariant diffusion. Up to some natural transformations of space and time, this correspondence is bijective. We also reformulate this result in the language of additive functionals of the Brownian motion in $[0, \infty)$ (or in $[0, R)$), and Brownian excursions. Our main tool is the recent extension of Krein's spectral theory of strings, due to Eckhardt and Kostenko.

Résumé. Nous étudions les traces marginales de diffusions invariantes par translation : des diffusions bidimensionnelles dans le demi-plan supérieur $\mathbb{R} \times [0, \infty)$ (ou dans $\mathbb{R} \times [0, R)$) invariantes par translation horizontale. Nous prouvons que les processus de trace correspondants sont des processus de Lévy avec des sauts complètement monotones et, réciproquement, tout processus de Lévy avec des sauts complètement monotones est une trace marginale d'une diffusion invariante par translation. Moyennant certaines transformations naturelles de l'espace et du temps, cette correspondance est bijective. Nous reformulons également ce résultat dans le langage des fonctionnelles additives du mouvement brownien dans $[0, \infty)$ (ou dans $[0, R)$), et des excursions browniennes. Notre outil principal est la récente extension de la théorie spectrale des cordes de Krein, grâce à Eckhardt et Kostenko.

MSC2020 subject classifications: 60J60; 60J75; 35J25; 35J70; 35R11; 47G20

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Limit theorems in Wasserstein distance for empirical measures of diffusion processes on Riemannian manifolds

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Abstract. Let (M,ρ) be a connected compact Riemannian manifold without boundary or with a convex boundary ∂M, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\,\mathrm{d}x$ is a probability measure, where $\mathrm{d}x$ is the volume measure. Let $\{\lambda_i\}_{i\ge 1}$ be all non-trivial eigenvalues of $-L$ with Neumann boundary condition if $\partial M \ne \varnothing$, where $L := \Delta + \nabla V$ for Δ being the Laplace–Beltrami operator on M. Then the empirical measures $\{\mu_t\}_{t>0}$ of the diffusion process generated by L (with reflecting boundary if $\partial M \ne \varnothing$) satisfy

$$\lim_{t\to\infty}\{t\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2]\} = \sum_{i=1}^{\infty}\frac{2}{\lambda_i^2} \quad \text{uniformly in } x \in M,$$

where $\mathbb{E}^x$ is the expectation for the diffusion process starting at point x, and $\mathbb{W}_2$ is the L^2-Wasserstein distance induced by the Riemannian metric. The limit is finite if and only if $d \le 3$, and in this case we derive

$$\lim_{t\to\infty}\sup_{x\in M}\left|\mathbb{P}^x\big(t\mathbb{W}_2(\mu_t,\mu)^2 < a\big) - \mathbb{P}\left(\sum_{k=1}^{\infty}\frac{2\xi_k^2}{\lambda_k^2} < a\right)\right| = 0, \quad a \ge 0,$$

where $\{\xi_k\}_{k\ge 1}$ are i.i.d. standard Gaussian random variables. Moreover, $\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2] \sim t^{-\frac{2}{d-2}}$ for $d \ge 5$, and when $d = 4$ we have $\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2] \le ct^{-1}\log t$ for some constant $c > 0$ and large t while the same type lower bound estimate holds for $M = \mathbb{T}^4$. Finally, we establish the long-time large deviation principle for $\{\mathbb{W}_2(\mu_t,\mu)^2\}_{t\ge 0}$ with a good rate function given by the information with respect to μ.

Résumé. Soit (M,ρ) une variété riemannienne compacte connexe sans bord, ou à bord convexe ∂M. Soit $V \in C^2(M)$ tel que $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\,\mathrm{d}x$ est une mesure de probabilité, où $\mathrm{d}x$ est la mesure de volume. Soient $\{\lambda_i\}_{i\ge 1}$ les valeurs propres non triviales de $-L$ avec condition aux limites de Neumann si $\partial M \ne \varnothing$, où $L := \Delta + \nabla V$ et Δ est l'opérateur de Laplace–Beltrami sur M. Alors les measures empiriques $\{\mu_t\}_{t>0}$ du processus engendré par L (avec réflexion au bord si $\partial M \ne \varnothing$) vérifient

$$\lim_{t\to\infty}\{t\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2]\} = \sum_{i=1}^{\infty}\frac{2}{\lambda_i^2} \quad \text{uniformément en } x \in M,$$

où $\mathbb{E}^x$ est l'espérance par rapport au processus avec condition initiale x et $\mathbb{W}_2$ est la distance L^2-Wasserstein associée à la métrique riemannienne de l'espace. La limite est finie si et seulement si $d \le 3$ et dans ce cas on a

$$\lim_{t\to\infty}\sup_{x\in M}\left|\mathbb{P}^x\big(t\mathbb{W}_2(\mu_t,\mu)^2 < a\big) - \mathbb{P}\left(\sum_{k=1}^{\infty}\frac{2\xi_k^2}{\lambda_k^2} < a\right)\right| = 0, \quad a \ge 0,$$

avec $\{\xi_k\}_{k\ge 1}$ une suite de variables i.i.d. gaussiennes standard. De plus, $\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2] \sim t^{-\frac{2}{d-2}}$ si $d \ge 5$ et $\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2] \le ct^{-1}\log t$ si $d = 4$, avec $c > 0$ et t suffisamment grand. Des estimations inférieures similaires sont établies si $M = \mathbb{T}^4$. Pour conclure, nous démontrons un principe de grandes déviations en temps long pour $\{\mathbb{W}_2(\mu_t,\mu)^2\}_{t\ge 0}$ avec une bonne fonction de taux donnée par l'information par rapport à μ.

MSC2020 subject classifications: 60D05; 58J65

Keywords: Empirical measure; Diffusion process; Riemannian manifold; Wasserstein distance; Eigenvalues

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Lyapunov exponents of the SHE under general initial data

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Abstract. We consider the $(1+1)$-dimensional stochastic heat equation (SHE) with multiplicative white noise and the Cole-Hopf solution of the Kardar–Parisi–Zhang (KPZ) equation. We show an exact way of computing the Lyapunov exponents of the SHE for a large class of initial data which includes any bounded deterministic positive initial data and the stationary initial data. As a consequence, we derive exact formulas for the upper tail large deviation rate functions of the KPZ equation for general initial data.

Résumé. Nous considérons l'équation de chaleur stochastique (SHE) de dimension $(1+1)$ avec bruit blanc multiplicatif et la solution de Cole-Hopf de l'équation de Kardar–Parisi–Zhang. Nous montrons une manière exacte de calculer les exposants de Lyapunov du SHE pour une grande classe de données initiales qui inclut toutes les données initiales positives déterministes bornées et les données initiales stationnaires. En conséquence, nous déduisons des formules exactes pour les fonctions de taux de grande déviation de la queue supérieure de l'équation KPZ pour les données initiales générales.

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Keywords: Stochastic heat equation; KPZ equation; Lyapunov exponents; Large deviation

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Operator-valued matrices with free or exchangeable entries

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Abstract. We study matrices whose entries are free or exchangeable noncommutative elements in some tracial W^*-probability space. More precisely, we consider operator-valued Wigner and Wishart matrices and prove quantitative convergence to operator-valued semicircular elements over some subalgebra in terms of Cauchy transforms and the Kolmogorov distance. As direct applications, we obtain explicit rates of convergence for a large class of random block matrices with independent or correlated blocks. Our approach relies on a noncommutative extension of the Lindeberg method and operator-valued Gaussian interpolation techniques.

Résumé. Nous étudions des matrices dont les entrées sont des variables libres ou échangeables d'un W*-espace de probabilités tracial. Plus précisément, nous considérons des matrices de Wigner et de Wishart avec des entrées à valeurs opérateurs et nous montrons la convergence quantitative, vers des variables semi-circulaires à valeurs opérateurs sur une certaine sous-algèbre, en termes de transformées de Cauchy et de distance de Kolmogorov. Comme applications directes, nous obtenons des taux de convergence explicites pour une large classe de matrices aléatoires par blocs avec des blocs indépendants ou corrélés. Notre approche repose sur une extension non-commutative de la méthode de Lindeberg et de techniques d'interpolation gaussiennes à valeurs opérateurs.

MSC2020 subject classifications: 46L54; 60B10; 60B20

Keywords: Random block matrices; Noncommutative Lindeberg method; Matrices with noncommutative exchangeable entries; Matrices with free entries; Operator-valued free probability; Random operators

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