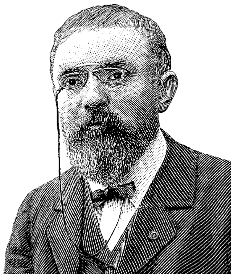

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# The maximum of a branching random walk with stretched exponential tails

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**Abstract.** We study a one-dimensional branching random walk in the case when the step size distribution has a stretched exponential tail, and, in particular, no finite exponential moments. The tail of the step size $X$ decays as $\mathbb{P}[X > t] \sim a \exp\{-\lambda t^r\}$ for some constants $a, \lambda > 0$ where $r \in (0, 1)$. We give a detailed description of the asymptotic behaviour of the position of the rightmost particle, proving almost sure limit theorems, convergence in law and a growth condition dichotomy. The limit theorems reveal interesting differences betweens the two regimes $r \in (0, 2/3)$ and $r \in (2/3, 1)$, with yet different limits in the boundary case $r = 2/3$.

**Résumé.** Nous étudions une marche aléatoire branchante uni-dimensionelle quand les déplacements n'ont pas des moments exponentiels. Plus précisement, la queue d'un déplacement $X$ se comporte comme suit : $\mathbb{P}[X > t] \sim a \exp\{-\lambda t^r\}$, pour des constantes $a, \lambda > 0$ et $r \in (0, 1)$. Nous donnons une description détaillée du comportement asymptotique du maximum, en montrant des lois limites presque sûres, des theorèmes de convergence en loi et une dichotomie basée sur une condition de croissance. Ces lois limites diverses font apparaître des différences interéssantes entre les deux régimes $r \in (0, 2/3)$ et $r \in (2/3, 1)$, et le cas critique $r = 2/3$ est encore différent.

*MSC2020 subject classifications:* 60F10; 60J80; 60G50

*Keywords:* Branching random walk; Stretched exponential random variables; Limit theorems; Point processes; Extreme values

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# Extinction times of multitype continuous-state branching processes

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**Abstract.** A multitype continuous-state branching process (MCSBP) $Z = (Z_t)_{t \geq 0}$, is a Markov process with values in $[0, \infty)^d$ that satisfies the branching property. Its distribution is characterised by its branching mechanism, that is the data of $d$ Laplace exponents of $\mathbb{R}^d$-valued spectrally positive Lévy processes, each one having $d - 1$ increasing components. We give an expression of the probability for a MCSBP to tend to 0 at infinity in term of its branching mechanism. Then we prove that this extinction holds at a finite time if and only if some condition bearing on the branching mechanism holds. This condition extends Grey's condition that is well known for $d = 1$. Our arguments bear on elements of fluctuation theory for spectrally positive additive Lévy fields recently obtained in (*Electron. J. Probab.* **25** (2020) 26) and an extension of the Lamperti representation in higher dimension proved in (*Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 1280–1304).

**Résumé.** Un processus de branchement multitype, continu (MCSBP) $Z = (Z_t)_{t \geq 0}$, est un processus de Markov à valeurs dans $[0, \infty)^d$ qui satisfait à la propriété de branchement. Sa loi est caractérisée par son mécanisme de branchement qui est donné par $d$ exposants de Laplace de processus de Lévy spectralement positifs, à valeurs dans $\mathbb{R}^d$, chacun d'entre eux possédant $d - 1$ coordonnées croissantes. Nous donnons une expression de la probabilité pour un MCSBP de tendre vers 0 à l'infini en terme de son mécanisme de branchement. Nous montrons que cette extinction a lieu en un temps fini si et seulement si une certaine condition portant sur le mécanisme de branchement est satisfaite. Cette condition étend la condition de Grey bien connue en dimension 1. Nos arguments portent sur des éléments de théorie des fluctuations pour les champs de Lévy additifs, spectrallement positifs récemment établis en (*Electron. J. Probab.* **25** (2020) 26) ainsi qu'une extension de la représentation de Lamperti en dimension supérieure obtenue en (*Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 1280–1304).

*MSC2020 subject classifications:* Primary 60J80; secondary 60G51

*Keywords:* Multitype continuous-state branching process; Extinction time; Spectrally positive additive Lévy field; Lamperti representation

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# Recurrence of horizontal–vertical walks

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**Abstract.** Consider a nearest neighbor random walk on the two-dimensional integer lattice, where each vertex is initially labeled either 'H' or 'V', uniformly and independently. At each discrete time step, the walker resamples the label at its current location (changing 'H' to 'V' and 'V' to 'H' with probability $q$). Then, it takes a mean zero horizontal step if the new label is 'H', and a mean zero vertical step if the new label is 'V'. This model is a randomized version of the deterministic rotor walk, for which its recurrence (i.e., visiting every vertex infinitely often with probability 1) in two dimensions is still an open problem. We answer the analogous question for the horizontal–vertical walk, by showing that the horizontal–vertical walk is recurrent for $q \in (\frac{1}{3}, 1]$.

**Résumé.** Considérons une marche aléatoire aux plus proches voisins sur le réseau entier bidimensionnel, où chaque sommet est initialement étiqueté soit "H" soit "V", uniformément et indépendamment. À chaque pas de temps discret, le marcheur ré-échantillonne l'étiquette à son emplacement actuel (en changeant "H" en "V" et "V" en "H" avec la probabilité $q$). Ensuite, il fait un pas horizontal de moyenne nulle si la nouvelle étiquette est "H", et un pas vertical de moyenne nulle si la nouvelle étiquette est "V". Ce modèle est une version randomisée de la marche déterministe du rotor, pour laquelle sa récurrence (c'est-à-dire visiter chaque sommet infiniment souvent avec une probabilité de 1) en deux dimensions est encore un problème ouvert. Nous répondons à la question analogue pour la marche horizontale-verticale, en montrant que la marche horizontale-verticale est récurrente pour $q \in ]\frac{1}{3}, 1]$.

*MSC2020 subject classifications:* Primary 60K35; secondary 60F20; 60J10; 82C41

*Keywords:* Recurrence; Transience; Random walk; Random environment; Rotor-router

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# Recurrence and transience of random difference equations in the critical case

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**Abstract.** For i.i.d. random vectors $(M_1, Q_1), (M_2, Q_2), \dots$ such that $M > 0$ a.s., $Q \geq 0$ a.s. and $\mathbb{P}(Q = 0) < 1$, the random difference equation $X_n = M_n X_{n-1} + Q_n$, $n = 1, 2, \dots$, is studied in the critical case when the random walk with increments $\log M_1$, $\log M_2, \dots$ is oscillating. We provide conditions for the null recurrence and transience of the Markov chain $(X_n)_{n\geq 0}$ by inter alia drawing on techniques developed in the related article (*J. Appl. Probab.* **54** (2017) 1089–1110) for another case exhibiting the null recurrence/transience dichotomy.

**Résumé.** Étant donnés des vecteurs aléatoires i.i.d. $(M_1, Q_1), (M_2, Q_2), \dots$ tels que $M > 0$ et $Q \geq 0$ p.s., et $\mathbb{P}(Q = 0) < 1$, nous étudions l'équation aux différences aléatoires $X_n = M_n X_{n-1} + Q_n$, $n = 1, 2, \dots$ dans le cas critique, lorsque la marche aléatoire avec incréments $\log M_1$, $\log M_2, \dots$ est oscillante. Nous obtenons des conditions pour la récurrence nulle et la transience de la chaîne de Markov $(X_n)_{n\geq 0}$, en utilisant notamment des techniques développées dans l'article lié (*J. Appl. Probab.* **54** (2017) 1089–1110), qui traite d'un autre cas présentant la dichotomie récurrence nulle/transience.

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*Keywords:* Invariant Radon measure; Null recurrence; Perpetuity; Random difference equation; Transience

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# Large fluctuations and transport properties of the Lévy–Lorentz gas

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**Abstract.** The Lévy–Lorentz gas describes the motion of a particle on the real line in the presence of a random array of scattering points, whose distances between neighboring points are heavy-tailed i.i.d. random variables with finite mean. The motion is a continuous-time, constant-speed interpolation of the simple symmetric random walk on the marked points. In this paper we study the large fluctuations of the continuous-time process and the resulting transport properties of the model, both annealed and quenched, confirming and extending previous work by physicists that pertain to the annealed framework. Specifically, focusing on the particle displacement, and under the assumption that the tail distribution of the interdistances between scatterers is regularly varying at infinity, we prove a precise large deviation principle for the annealed fluctuations and present the asymptotics of annealed moments, demonstrating annealed superdiffusion. Then, we provide an upper large deviation estimate for the quenched fluctuations and the asymptotics of quenched moments, showing that the asymptotic diffusive regime conditional on a typical arrangement of the scatterers is normal diffusion, and not superdiffusion. Although the Lévy–Lorentz gas seems to be accepted as a model for anomalous diffusion, our findings suggest that superdiffusion is a transient behavior which develops into normal diffusion on long timescales, and raise a new question about how the transition from the quenched normal diffusion to the annealed superdiffusion occurs.

**Résumé.** Le gaz de Lévy–Lorentz modélise le déplacement d'une particule sur l'axe des nombres réels en présence d'obstacles distribués de telle façon que les distances entre les obstables sont des variables aléatoires i.i.d. à queue lourde et de moyenne finie. La dynamique est donnée par l'interpolation, en temps continu et vitesse fixe, de la marche aléatoire symétrique sur les obstacles. Cet article étudie les grandes fluctuations du processus en temps continu et les propriétés de transport du modèle sous-jacent. Les résultats obtenus dans les cas de désordre "annealed" et "quenched" confirment et généralisent des résultats précédents issus de la physique dans le cas "annealed". En particulier, sous l'hypothèse que la queue de la loi des distances inter-obstacles est à variation régulière à l'infini, nous prouvons dans le cas "annealed" un principe de grandes déviations et obtenons l'expression asymptotique des moments, qui montrent l'existence d'un regime de sur-diffusion. Dans le cas quenched, nous obtenons l'expression asymptotique des moments et une borne supérieure sur les grandes deviations des fluctuations. Cela nous permet de montrer, pour des configurations d'obstacles typiques, que le régime asymptotique de diffusion est la diffusion normale, et non la sur-diffusion. Bien que le gaz de Lévy–Lorentz soit en général utilisé pour modéliser la diffusion anormale, notre résultat suggère que le regime sur-diffusif est seulement transitoire. Cela soulève également la question de la nature de la transition entre diffusion normale "quenched" et sur-diffusion "annealed".

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*Keywords:* Random walks on point processes; Random walks in random environment; Lévy–Lorentz gas; Regularly varying tails; Precise large deviation principles; Convergence of moments; Transport properties; Anomalous diffusion

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# Stochastic homogenization of random walks on point processes

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**Abstract.** We consider random walks on the support of a random purely atomic measure on $\mathbb{R}^d$ with random jump probability rates. The jump range can be unbounded. The purely atomic measure is reversible for the random walk and stationary for the action of the group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$. By combining two-scale convergence and Palm theory for $\mathbb{G}$-stationary random measures and by developing a cut-off procedure, under suitable second moment conditions we prove for almost all environments the homogenization for the massive Poisson equation of the associated Markov generator. In addition, we obtain the quenched convergence of the $L^2$-Markov semigroup and resolvent of the diffusively rescaled random walk to the corresponding ones of the Brownian motion with covariance matrix $2D$. For symmetric jump rates, the above convergence plays a crucial role in the derivation of hydrodynamic limits when considering multiple random walks with site-exclusion or zero range interaction. We do not require any ellipticity assumption, neither non-degeneracy of the homogenized matrix $D$. Our results cover a large family of models, including e.g. random conductance models on $\mathbb{Z}^d$ and on general lattices (possibly with long conductances), Mott variable range hopping, simple random walks on Delaunay triangulations, simple random walks on supercritical percolation clusters.

**Résumé.** Nous considérons des marches aléatoires sur le support d'une mesure aléatoire purement atomique sur $\mathbb{R}^d$ avec taux de sauts aléatoires. Les sauts peuvent être arbitrairement longs. La mesure purement atomique est réversible pour la marche aléatoire et stationnaire pour l'action du groupe $\mathbb{G} = \mathbb{R}^d$ ou $\mathbb{G} = \mathbb{Z}^d$. En combinant la convergence à deux échelles et la théorie de Palm pour les mesures aléatoires $\mathbb{G}$-stationnaires et en développant une procédure de troncation, sous des conditions de moment d'ordre deux appropriées, nous prouvons pour presque tous les environnements l'homogénéisation pour l'équation de Poisson massive du générateur de Markov associé. De plus, nous obtenons la convergence du semi-groupe de Markov $L^2$ et de la résolvante de la marche aléatoire, après renormalisation diffusive, vers leur équivalent pour le mouvement brownien de matrice de covariance $2D$. Pour des taux de sauts symétriques, cette convergence joue un rôle crucial dans l'obtention de la limite hydrodynamique pour des modèles de marches multiples avec exclusion ou à portée nulle. Aucune hypothèse d'ellipticité, ou de non-dégénérescence de la matrice homogénéisée D, n'est nécessaire. Nos résultats couvrent une large classe de modèles, qui inclue notamment les modèles de conductances aléatoires sur $\mathbb{Z}^d$ et sur réseaux généraux (éventuellement à conductances longues), les modèles de sauts à distance variable de Mott, les marches aléatoires simples sur les triangulations de Delaunay et les marches aléatoires simples sur des amas de percolation surcritiques.

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*Keywords:* Random measure; Point process; Palm distribution; Random walk in random environment; Stochastic homogenization; Two-scale convergence; Mott variable range hopping; Conductance model; Hydrodynamic limit

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# On the uniqueness of Gibbs distributions with a non-negative and subcritical pair potential

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**Abstract.** We prove that the distribution of a Gibbs process with non-negative pair potential is uniquely determined as soon as an associated Poisson-driven random connection model (RCM) does not percolate. Our proof combines disagreement coupling in continuum with a coupling of a Gibbs process and a RCM. The improvement over previous uniqueness results is illustrated both in theory and simulations.

**Résumé.** Nous prouvons que la loi d'un processus de Gibbs avec potentiel d'interaction positif est déterminée de façon unique dès qu'un modèle de connexion aléatoire (RCM) associé, dirigé par un processus de Poisson, ne percole pas. Notre preuve combine le couplage de désaccords dans le continuum avec le couplage d'un processus de Gibbs et d'un RCM. L'amélioration par rapport aux résultats d'unicité antérieurs est illustrée à la fois de manière théorique et avec des simulations.

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*Keywords:* Gibbs process; Uniqueness of Gibbs distributions; Pair potentials; Disagreement coupling; Poisson embedding; Random connection model; Percolation

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# Stationary states of the one-dimensional facilitated asymmetric exclusion process

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**Abstract.** We describe the translation invariant stationary states (TIS) of the one-dimensional facilitated asymmetric exclusion process in continuous time, in which a particle at site $i \in \mathbb{Z}$ jumps to site $i+1$ (respectively $i-1$) with rate $p$ (resp. $1-p$), provided that site $i-1$ (resp. $i+1$) is occupied and site $i+1$ (resp. $i-1$) is empty. All TIS states with density $\rho \le 1/2$ are supported on trapped configurations in which no two adjacent sites are occupied; we prove that if in this case the initial state is i.i.d. Bernoulli then the final state is independent of $p$. This independence also holds for the system on a finite ring. For $\rho > 1/2$ there is only one TIS. It is the infinite volume limit of the probability distribution that gives uniform weight to all configurations in which no two holes are adjacent, and is isomorphic to the Gibbs measure for hard core particles with nearest neighbor exclusion.

**Résumé.** Nous décrivons les états stationnaires invariants par translation (TIS) du processus d'exclusion asymétrique facilité unidimensionnel en temps continu, dans lequel une particule sur le site $i \in \mathbb{Z}$ saute vers le site $i+1$ (respectivement $i-1$) avec un taux $p$ (resp. $1-p$), à condition que le site $i-1$ (resp. $i+1$) soit occupé et que le site $i+1$ (resp. $i-1$) soit vide. Tous les états TIS avec une densité $\rho \le 1/2$ sont supportés par des configurations piégées dans lesquelles aucun des deux sites adjacents n'est occupé ; dans ce cas, nous prouvons que si l'état initial est i.i.d. Bernoulli alors l'état final est indépendant de $p$. Cette indépendance est également valable pour le système sur un anneau fini. Pour $\rho > 1/2$ il n'y a qu'un seul TIS. Il s'agit de la limite en volume infini de la mesure de probabilité qui donne un poids uniforme à toutes les configurations dans lesquelles deux trous ne sont pas adjacents, et isomorphe à la mesure de Gibbs pour les particules à noyau dur avec exclusion du plus proche voisin.

*MSC2020 subject classifications:* Primary 60K35; 82C22; secondary 82C23; 82C26

*Keywords:* Asymmetric facilitated exclusion processes; One dimensional conserved lattice gas; Facilitated jumps; Translation invariant steady states; Asymmetry independence; F-ASEP; F-TASEP

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# A Kac model with exclusion

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**Abstract.** We consider a one dimensional Kac model with conservation of energy and an exclusion rule. Fix a number of particles $n$, and an energy $E > 0$. Let each of the particles have an energy $x_j \geq 0$, with $\sum_{j=1}^n x_j = E$. For $\epsilon$ positive, the allowed configurations $(x_1, \dots, x_n)$ are those that satisfy $|x_i - x_j| \geq \epsilon$ for all $i \neq j$. At each step of the process, a pair $(i, j)$ of particles is selected uniformly at random, and then they "collide", and there is a repartition of their total energy $x_i + x_j$ between them producing new energies $x_i^*$ and $x_j^*$ with $x_i^* + x_j^* = x_i + x_j$, but with the restriction that exclusion rule is still observed for the new pair of energies. This process bears some resemblance to Kac models for Fermions in which the exclusion represents the effects of the Pauli exclusion principle. However, the "non-quantized" exclusion rule here, with only a lower bound on the gaps, introduces interesting novel features, and a detailed notion of Kac's chaos is required to derive an evolution equation for the rescaled empirical measures for the process, as we show here.

**Résumé.** Nous considerons un modèle de Kac unidimensionnel avec conservation de l'énergie et une règle d'exclusion. Pour un nombre de particules $n$, et une énergie $E > 0$ fixes, soit $x_j \geq 0$ l'énergie de la particule $j$ avec $\sum_{j=1}^n x_j = E$. Pour $\epsilon > 0$ les configurations admises de $(x_1, \dots, x_n)$ sont celles qui satisfont $|x_i - x_j| \geq \epsilon$, pour tout $i \neq j$. À chaque pas du processus, une paire $(i, j)$ de particules est sélectionnée uniformément au hasard, puis les particules « collisionnent ». Leur énergie totale $x_i + x_j$ est ensuite redistribuée produisant de nouvelles énergies $x_i^*$ et $x_j^*$ avec $x_i^* + x_j^* = x_i + x_j$, de telle sorte que la règle d'exclusion soit toujours observée pour la nouvelle paire. Ce processus présente des ressamblances avec modèles de Kac pour Fermions dans lesquels l'exclusion représente les effets du principe d'exclusion de Pauli. Cependant, la règle d'exclusion « non quantifié » ici, avec seulement une borne inférieure sur les écarts, introduit des nouveautées intéressantes, et une notion détaillée du chaos de Kac nécessaire pour dériver une équation d'évolution pour des mesures empiriques réchelonnée pour la processus, comme nous le montrons ici.

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*Keywords:* Jump process; Chaos

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# Weak convergence of directed polymers to deterministic KPZ at high temperature

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**Abstract.** It is shown that when $d \geq 3$, the growing random surface generated by the $(d+1)$-dimensional directed polymer model at sufficiently high temperature, after being smoothed by taking microscopic local averages, converges to a solution of the deterministic KPZ equation in a suitable scaling limit.

**Résumé.** On montre que quand $d \geq 3$, la surface aléatoire croissante engendrée par le modèle de polymère dirigé $(d+1)$-dimensionnel à une température suffisamment haute, après avoir été lissée en prenant des moyennes locales microscopiques, converge vers une solution de l'équation de KPZ déterministe dans une limite d'échelle appropriée.

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*Keywords:* Directed polymer; KPZ; Random surface; Scaling limit

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# Equivalence of Liouville measure and Gaussian free field

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**Abstract.** Given an instance $h$ of the Gaussian free field on a planar domain $D$ and a constant $\gamma \in (0,2)$, one can use various regularization procedures to make sense of the *Liouville quantum gravity area measure* $\mu := e^{\gamma h(z)}\,dz$. It is known that the field $h$ a.s. determines the measure $\mu_h$. We show that the converse is true: namely, $h$ is measurably determined by $\mu_h$. More generally, given a random closed fractal subset $\mathcal{A}$ endowed with a Frostman measure $\sigma$ whose support is $\mathcal{A}$ (independent of $h$), a Gaussian multiplicative chaos measure $\mu_{\sigma,h}$ can be constructed. We give a mild condition on $(\mathcal{A},\sigma)$ under which $\mu_{\sigma,h}$ determines $h$ restricted to $\mathcal{A}$, in the sense that it determines its harmonic extension off $\mathcal{A}$. Our condition is satisfied by the occupation measures of planar Brownian motion and SLE curves under natural parametrizations. Along the way we obtain general positive moment bounds for Gaussian multiplicative chaos. Contrary to previous results, this does not require any assumption on the underlying measure $\sigma$ such as scale invariance, and hence may be of independent interest.

**Résumé.** Etant donnée une réalisation $h$ d'un champ libre Gaussien dans un domaine $D$ du plan et une constante $\gamma \in (0,2)$ il est possible de donner un sens à la mesure aléatoire $\mu_h := e^{\gamma h(z)}\,dz$ dite de gravité quantique de Liouville, dont il est connu qu'elle est une fonction mesurable du champ $h$. Nous montrons la réciproque de ce résultat : c'est-à-dire, $h$ est entièrement déterminé de façon mesurable par la mesure $\mu_h$. Plus généralement, étant donné un ensemble fractal fermé $\mathcal{A}$ aléatoire équipé d'une mesure de Frostman de référence $\sigma$ (tous deux indépendants de $h$), il est possible de construire le chaos multiplicatif Gaussien $\mu_{\sigma,h}$ de $h$ par rapport à $\sigma$. Nous donnons une condition simple et générique sur $(\mathcal{A},\sigma)$ sous laquelle $\mu_{\sigma,h}$ détermine la restriction de $h$ à $\sigma$, ou plus précisément l'extension harmonique de $h$ en dehors de $\mathcal{A}$. Cette condition est satisfaite par la mesure d'occupation du mouvement Brownien plan et par des courbes SLE munies de paramétrisations naturelles. En cours de route nous obtenons des résultats généraux sur les moments positifs du chaos multiplicatif Gaussien. Contrairement à de précédents travaux, nous ne faisons pas d'hypothèse sur la mesure de référence $\sigma$ de type invariance par échelle. Les résultats ainsi obtenus peuvent donc être d'un intérêt indépendant.

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*Keywords:* Gaussian free field; Gaussian multiplicative chaos; Liouville measure

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# Finite-size scaling, phase coexistence, and algorithms for the random cluster model on random graphs

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**Abstract.** For $\Delta \geq 5$ and $q$ large as a function of $\Delta$, we give a detailed picture of the phase transition of the random cluster model on random $\Delta$-regular graphs. In particular, we determine the limiting distribution of the weights of the ordered and disordered phases at criticality and prove exponential decay of correlations and central limit theorems away from criticality. Our techniques are based on using polymer models and the cluster expansion to control deviations from the ordered and disordered ground states. These techniques also yield efficient approximate counting and sampling algorithms for the Potts and random cluster models on random $\Delta$-regular graphs at *all* temperatures when $q$ is large. This includes the critical temperature at which it is known the Glauber and Swendsen–Wang dynamics for the Potts model mix slowly. We further prove new slow-mixing results for Markov chains, most notably that the Swendsen–Wang dynamics mix exponentially slowly throughout an open interval containing the critical temperature. This was previously only known at the critical temperature.

Many of our results apply more generally to $\Delta$-regular graphs satisfying a small-set expansion condition.

**Résumé.** Pour $\Delta \geq 5$ et $q$ grand en fonction de $\Delta$, nous donnons une description détaillée de la transition de phase du modèle de composantes connexes aléatoires (i.e., le modèle FK) sur des graphes $\Delta$-réguliers aléatoires. En particulier, nous déterminons la distribution limite des poids des phases ordonnées et désordonnées au point critique et prouvons la décroissance exponentielle des corrélations et le comportement gaussien des fluctuations loin du point critique. Nos techniques sont basées sur l'utilisation de modèles de polymères et l'expansion en clusters pour contrôler les écarts par rapport aux états fondamentaux ordonnés et désordonnés. Ces techniques produisent également des algorithmes de comptage et d'échantillonnage efficaces pour les modèles de Potts et FK sur des graphes $\Delta$-réguliers aléatoires à toutes les températures lorsque $q$ est grand. Cela inclut la température critique à laquelle on sait que la dynamique de Glauber et de Swendsen–Wang pour le modèle de Potts mélangent lentement. Nous prouvons en outre de nouveaux résultats de mélange lent pour les chaînes de Markov, notamment que la dynamique de Swendsen–Wang mélange exponentiellement lentement tout au long d'un intervalle ouvert contenant la température critique. Ceci n'était auparavant connu qu'à la température critique.

Beaucoup de nos résultats s'appliquent plus généralement aux graphes $\Delta$-réguliers qui satisfont une borne inférieure sur le nombre d'arêtes quittant chaque « petit ensemble » de sommets dans le graphe.

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# Random nearest neighbor graphs: The translation invariant case

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**Abstract.** If $(\omega(e))$ is a family of random variables (weights) assigned to the edges of $\mathbb{Z}^d$, the nearest neighbor graph is the directed graph induced by all edges $\langle x, y\rangle$ such that $\omega(\{x, y\})$ is minimal among all neighbors $y$ of $x$. That is, each vertex points to its closest neighbor, if the weights are viewed as edge-lengths. Nanda–Newman introduced nearest neighbor graphs when the weights are i.i.d. and continuously distributed and proved that a.s., all components of the undirected version of the graph are finite. We study the case of translation invariant, distinct weights, and prove that nearest neighbor graphs do not contain doubly-infinite directed paths. In contrast to the i.i.d. case, we show that in this stationary case, the graphs can contain either one or two infinite components (but not more) in dimension two, and $k$ infinite components for any $k \in [1, \infty]$ in dimension $\geq 3$. The latter constructions use a general procedure to exhibit a certain class of directed graphs as nearest neighbor graphs with distinct weights, and thereby characterize all translation invariant nearest neighbor graphs. We also discuss relations to geodesic graphs from first-passage percolation and implications for the coalescing walk model of Chaika–Krishnan.

**Résumé.** Si $(\omega(e))$ est une famille de variables aléatoires (poids) affectées aux arêtes de $\mathbb{Z}^d$, le graphe à plus proches voisins est le graphe orienté induit par toutes les arêtes $\langle x, y\rangle$ tel que $\omega(\{x, y\})$ soit minimal parmi les voisins $y$ de $x$. Autrement dit, chaque sommet pointe vers son voisin le plus proche, si les longueurs des arêtes sont données par les poids. Nanda et Newman ont introduit ces graphes lorsque les poids sont i.i.d. avec distribution continue et ils ont prouvé que toutes les composantes de la version non orientée du graphe sont finies. Nous étudions le cas avec poids distincts et invariants par translation et nous prouvons que ces graphes ne contiennent pas de chemins orientés doublement infinis. Contrairement au cas i.i.d., nous montrons que dans ce cas stationnaire les graphes peuvent contenir une ou deux composantes infinies (mais pas plus) en dimension deux, et $k$ composantes infinies pour tout $k \in [1, \infty]$ en dimension trois ou plus. Dans la construction de ces graphes nous utilisons une procédure générale qui introduit une certaine classe de graphes orientés en tant que graphes à plus proches voisins avec des poids distincts, et nous caractérisont ainsi tous les graphes à plus proches voisins invariants par translation. Nous discutons également des relations avec les graphes géodésiques de la percolation de premier passage et des implications pour le modèle de marche coalescente de Chaika et Krishnan.

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*Keywords:* Nearest neighbor graphs; Stationary percolation; Mass transport

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# Scaling limit of small random perturbation of dynamical systems

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**Abstract.** In this article, we prove that a small random perturbation of a dynamical system with multiple stable equilibria converges to a Markov chain whose states are neighborhoods of the deepest stable equilibria, under a suitable time-rescaling, provided that the perturbed dynamics is reversible in time. Such a result has been anticipated from the 70s, when the foundation of the mathematical treatment for this problem has been established by Freidlin and Wentzell, but the process level convergence was still open. We solve this problem by reducing the entire analysis to an investigation of the solution of an associated Poisson equation, and furthermore provide a method to carry out this analysis by using well-known test functions in a novel manner.

**Résumé.** Dans cet article, nous prouvons qu'une petite perturbation aléatoire d'un système dynamique avec plusieurs équilibres stables converge vers une chaîne de Markov dont les états sont des voisinages des équilibres stables les plus profonds, sur une échelle temporelle adaptée, à condition que la dynamique perturbée soit réversible dans le temps. Un tel résultat a été anticipé dès les années 1970, lorsque les fondements du traitement mathématique de ce problème ont été établis par Freidlin et Wentzell, mais la convergence au niveau des processus restait ouverte jusqu'à aujourd'hui. Nous résolvons ce problème en réduisant l'analyse à l'étude de la solution d'une équation de Poisson associée. De plus, nous introduisons une méthode pour effectuer cette analyse en utilisant d'une manière inédite des fonctions de test bien connues.

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*Keywords:* Metastability; diffusion processes; generator; Poisson equation

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# Wasserstein perturbations of Markovian transition semigroups

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**Abstract.** In this paper, we deal with a class of time-homogeneous continuous-time Markov processes with transition probabilities bearing a nonparametric uncertainty. The uncertainty is modelled by considering perturbations of the transition probabilities within a proximity in Wasserstein distance. As a limit over progressively finer time periods, on which the level of uncertainty scales proportionally, we obtain a convex semigroup satisfying a nonlinear PDE in a viscosity sense. A remarkable observation is that, in standard situations, the nonlinear transition operators arising from nonparametric uncertainty coincide with the ones related to parametric drift uncertainty. On the level of the generator, the uncertainty is reflected as an additive perturbation in terms of a convex functional of first order derivatives. We additionally provide sensitivity bounds for the convex semigroup relative to the reference model. The results are illustrated with Wasserstein perturbations of Lévy processes, infinite-dimensional Ornstein–Uhlenbeck processes, geometric Brownian motions, and Koopman semigroups.

**Résumé.** Dans cet article, nous traitons d'une classe de processus de Markov à temps continu homogène dans le temps avec des probabilités de transition portant une incertitude non paramétrique. L'incertitude est modélisée en considérant des perturbations de probabilités de transition proches en distance de Wasserstein. Comme limite sur des périodes de temps de plus en plus fines, sur lesquelles le niveau d'incertitude s'étend proportionnellement, nous obtenons un semigroupe convexe satisfaisant une EDP non linéaire dans un sens de viscosité. Une observation remarquable est que, dans des situations standards, les opérateurs de transition non linéaires découlant de l'incertitude non paramétrique coïncident avec ceux liés à l'incertitude paramétrique de dérive. Au niveau du générateur, l'incertitude se traduit par une perturbation additive en termes d'une fonction convexe de dérivées de premier ordre. Nous fournissons en outre des bornes de sensibilité pour le semigroupe convexe par rapport au modèle de référence. Les résultats sont illustrés par les perturbations de Wasserstein des processus de Lévy, les processus d'Ornstein–Uhlenbeck de dimension infinie, les mouvements browniens géométriques et les semigroupes de Koopman.

*MSC2020 subject classifications:* Primary 60J35; 47H20; secondary 60G65; 90C31; 62G35

*Keywords:* Markov process; Wasserstein distance; Nonparametric uncertainty; Convex semigroup; Nonlinear PDE; Viscosity solution

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# Quantitative control of Wasserstein distance between Brownian motion and the Goldstein–Kac telegraph process

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**Abstract.** In this manuscript, we provide a non-asymptotic process level control between the telegraph process and the Brownian motion with suitable diffusivity constant via a Wasserstein distance with quadratic average cost. In addition, we derive non-asymptotic estimates for the corresponding time average $p$-th moments. The proof relies on coupling techniques such as coin-flip coupling, synchronous coupling and the Komlós–Major–Tusnády coupling.

**Résumé.** Dans cet article, nous fournissons un contrôle au niveau de processus et non asymptotique entre le processus télégraphique et le mouvement brownien avec une constante de diffusivité appropriée par rapport à la distance de Wasserstein et avec un coût moyen quadratique. De plus, nous dérivons des estimations non asymptotiques pour les $p$-ièmes moments moyens correspondants. La preuve repose sur des techniques de couplage telles que le couplage pile ou face, le couplage synchrone et le couplage Komlós–Major–Tusnády.

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*Keywords:* Brownian motion; Coin-flip coupling; Decoupling; Free velocity flip model; Komlós–Major–Tusnády coupling; Random evolutions; Synchronous coupling; Telegraph process; Wasserstein distance

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# Coalescing-fragmentating Wasserstein dynamics: Particle approach

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**Abstract.** We construct a family of semimartingales that describes the behavior of a particle system with sticky-reflecting interaction. The model is a physical improvement of the Howitt–Warren flow (*Ann. Probab.* **37** (2009) 1237–1272), an infinite system of diffusion particles on the real line that sticky-reflect from each other. But now particles have masses obeying the conservation law and the diffusion rate of each particle depends on its mass. The equation which describes the evolution of the particle system is a new type of equations in infinite-dimensional space and can be interpreted as an infinite-dimensional analog of the equation for sticky-reflected Brownian motion. The particle model appears as a particular solution to the corrected version of the Dean–Kawasaki equation.

**Résumé.** Nous construisons une famille de semimartingales décrivant le comportement d'un système de particules avec interactions à effet réflectif et adhésif. Ce modèle est un amélioration plus physique du flot de Howitt–Warren (*Ann. Probab.* **37** (2009) 1237–1272), un système infini de particules diffusives sur la droite réelle interagissant avec effet réflectif et adhésif. Dans cet article, les particules ont désormais des masses qui satisfont à la loi de conservation, et le coefficient de diffusion de chaque particule dépend de sa masse. L'équation décrivant l'évolution du système de particules est un nouveau type d'équation sur un espace de dimension infinie et peut être interprétée comme un analogue infini-dimensionnel de l'équation satisfaite par le mouvement brownien à comportement réflectif et adhésif. Le modèle particulaire apparaît comme une solution particulière d'une version corrigée de l'équation de Dean–Kawasaki.

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*Keywords:* Wasserstein diffusion; Modified massive Arratia flow; Howitt–Warren flow; Sticky-reflected Brownian motion; Infinite-dimensional SDE with discontinuous coefficients

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# Lyapunov exponents for truncated unitary and Ginibre matrices

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**Abstract.** In this note, we show that the Lyapunov exponents of mixed products of random truncated Haar unitary and complex Ginibre matrices are asymptotically given by equally spaced 'picket-fence' statistics. We discuss how these statistics should originate from the connection between random matrix products and multiplicative Brownian motion on $\mathrm{GL}_n(\mathbb{C})$, analogous to the connection between discrete random walks and ordinary Brownian motion. Our methods are based on contour integral formulas for products of classical matrix ensembles from integrable probability.

**Résumé.** Dans cette note, nous montrons que les exposants de Lyapunov des produits mixtes de matrices aléatoires unitaires de Haar tronquées et de matrices de Ginibre complexes sont asymptotiquement donnés par des statistiques de type "palissade" équidistantes. Nous discutons comment ces statistiques devraient provenir de la connection entre les produits de matrices aléatoires et le mouvement brownien multiplicatif sur $\mathrm{GL}_n(\mathbb{C})$, analogue à celle entre les marches aléatoires discrètes et le mouvement brownien ordinaire. Nos méthodes sont basées sur des formules d'intégrale de contour pour les produits d'ensembles matriciels classiques à partir de probabilités intégrables.

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# Connecting eigenvalue rigidity with polymer geometry: Diffusive transversal fluctuations under large deviation

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**Abstract.** We consider exponential directed last passage percolation (LPP) on $\mathbb{Z}^2$, a paradigm model of the Kardar–Parisi–Zhang (KPZ) universality class, where $T_n$ denotes the last passage time from $(1,1)$ to $(n,n)$, and $\Gamma_n$ denotes the corresponding *polymer*, i.e., the optimal path attaining $T_n$. The typical fluctuation of the geodesic from the straight line joining its endpoints is known to be of order $n^{2/3}$, a feature of KPZ universality. Despite considerable interest, the behaviour of the polymer under large deviation events for $T_n$ had remained less understood. In this paper we consider the upper tail large deviation event $\mathcal{U}_\delta := \{T_n \geq (4+\delta)n\}$. We show that conditioning on $\mathcal{U}_\delta$ changes the transversal fluctuation exponent from $2/3$ to $1/2$, i.e., conditionally, the smallest strip around the diagonal that contains $\Gamma_n$ has width $n^{1/2+o(1)}$ with high probability. While earlier work by Deuschel and Zeitouni (*Combin. Probab. Comput.* **8** (1999) 247–263) had a $o(n)$ upper bound for the transversal fluctuation conditional on the upper tail large deviations in Poissonian last passage percolation, the exponent $1/2$ is new and is expected to be universal across various planar last passage percolation models in the KPZ universality class. Our proof combines several different ideas exploiting the correspondence between last passage times in the exponential LPP model and the largest eigenvalue of the Laguerre Unitary Ensemble (LUE), including a stochastic monotonicity result for determinantal point processes, as well as recent advances in understanding rigidity properties of eigenvalues to obtain a sharp finite size correction to the well known large deviation rate function for the largest eigenvalue.

**Résumé.** Nous considérons la percolation de dernier passage dirigée exponentielle (LPP) sur $\mathbb{Z}^2$, un modèle paradigmatique de la classe d'universalité de Kardar–Parisi–Zhang (KPZ), où $T_n$ désigne le temps de dernier passage de $(1,1)$ à $(n,n)$, et $\Gamma_n$ désigne le *polymère* correspondant, i.e. le chemin optimal atteignant $T_n$. La fluctuation typique de la géodésique à partir de la ligne droite joignant ses extrémités est connue pour et est d'ordre $n^{2/3}$, une caractéristique de l'universalité KPZ. Malgré un intérêt considérable, le comportement du polymère sous des événements de grande déviation pour $T_n$ était moins compris. Dans cet article, nous considérons l'événement de grande déviation vers des grandes valeurs $\mathcal{U}_\delta := \{T_n \geq (4+\delta)n\}$. Nous montrons que le conditionnement à $\mathcal{U}_\delta$ change l'exposant de la fluctuation transversale de $2/3$ à $1/2$, i.e. que, conditionnellement, la plus petite bande autour de la diagonale qui contient $\Gamma_n$ a une largeur $n^{1/2+o(1)}$ avec une grande probabilité. Alors que les travaux antérieurs de Deuschel et Zeitouni (*Combin. Probab. Comput.* **8** (1999) 247–263) avaient une borne supérieure $o(n)$ pour la fluctuation transversale conditionnelle aux grandes déviations vers des grandes valeurs dans la percolation de dernier passage poissonienne, l'exposant $1/2$ est nouveau et on s'attend à ce qu'il soit universel pour plusieurs modèles de percolation de dernier passage planaires dans la classe d'universalité KPZ. Notre preuve combine plusieurs idées différentes exploitant la correspondance entre les temps de dernier passage dans le modèle LPP exponentiel et la plus grande valeur propre de l'ensemble unitaire de Laguerre (LUE), y compris un résultat de monotonie stochastique pour les processus déterminantaux ponctuels, ainsi que des avancées récentes dans la compréhension des propriétés de rigidité des valeurs propres afin d'obtenir une correction précise de taille finie pour la fonction de taux de grande déviation bien connue pour la plus grande valeur propre.

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*Keywords:* Last passage percolation; Large deviations; Geodesics

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# Gaussian fluctuations and free energy expansion for Coulomb gases at any temperature

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**Abstract.** We obtain concentration estimates for the fluctuations of Coulomb gases in any dimension and in a broad temperature regime, including very small and very large temperature regimes which may depend on the number of points. We obtain a full Central Limit Theorem (CLT) for the fluctuations of linear statistics in dimension 2, valid for the first time down to microscales and for temperatures possibly tending to 0 or $\infty$ as the number of points diverges. We show that a similar CLT can also be obtained in any larger dimension conditional on a "no phase-transition" assumption, as soon as one can obtain a precise enough error rate for the expansion of the free energy – an expansion is obtained in any dimension, but the rate is so far not good enough to conclude. These CLTs can be interpreted as a convergence to the Gaussian Free Field. All the results are valid as soon as the test-function lives on a larger scale than the temperature-dependent minimal scale $\rho_\beta$ introduced in our previous work (*Ann. Probab.* **49** (2021) 46–121).

**Résumé.** On obtient des résultats de concentration pour les fluctuations du gaz de Coulomb en toute dimension et dans un large régime de température, incluant des températures très petites et très grandes qui peuvent dépendre du nombre de points. On obtient un Théorème Central Limite (TCL) complet pour les fluctuations des statistiques linéaires en dimension 2, valable pour la première fois jusqu'aux échelles microscopiques et pour des températures pouvant tentre vers 0 ou l'infini quand le nombre de points diverge. On montre qu'un TCL semblable peut aussi être obtenu en toute dimension sous une condition d'absence de transition de phase, dès lors qu'on peut obtenir une erreur suffisamment petite dans le développement de l'énergie libre – un tel développement est prouvé en toute dimension, mais l'erreur n'est pas suffisamment petite pour conclure. Ces TCL sont valables dès que la fonction-test vit à une échelle supérieure à l'échelle minimale $\rho_\beta$ dépendant de la température, introduite dans le précédent travail (*Ann. Probab.* **49** (2021) 46–121).

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*Keywords:* Coulomb gas; One-component plasma; Concentration; Central Limit Theorem; Gaussian Free Field

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