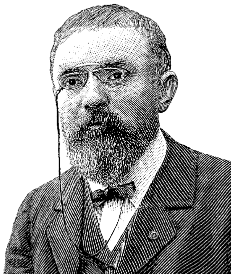

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Free energy upper bound for mean-field vector spin glasses

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Abstract. We consider vector spin glasses whose energy function is a Gaussian random field with covariance given in terms of the matrix of scalar products. For essentially any model in this class, we give an upper bound for the limit free energy, which is expected to be sharp. The bound is expressed in terms of an infinite-dimensional Hamilton–Jacobi equation.

Résumé. Nous considérons des verres de spins vectoriels dont la fonction d'énergie est un champ aléatoire gaussien avec une covariance s'exprimant en termes de la matrice des produits scalaires. Pour essentiellement tous les modèles de cette classe, nous donnons une limite supérieure pour l'énergie libre limite, qui devrait être exacte. La limite est exprimée en termes d'une équation de Hamilton–Jacobi de dimension infinie.

MSC2020 subject classifications: 82B44; 82D30

Keywords: Spin glass; Hamilton–Jacobi equation; Parisi formula

References

[1] A. Adhikari and C. Brennecke. Free energy of the quantum Sherrington–Kirkpatrick spin-glass model with transverse field. *J. Math. Phys.* **61** (8) (2020) 083302. MR4140500 https://doi.org/10.1063/5.0009291
[2] E. Agliari, A. Barra, R. Burioni and A. Di Biasio. Notes on the p-spin glass studied via Hamilton–Jacobi and smooth-cavity techniques. *J. Math. Phys.* **53** (6) (2012) 063304. MR2977685 https://doi.org/10.1063/1.4729233
[3] D. Alberici, P. Contucci and E. Mingione Deep Boltzmann machines: Rigorous results at arbitrary depth. Preprint. Available at arXiv:2004.04495. MR4289496 https://doi.org/10.1007/s00023-021-01027-2
[4] A. Barra, G. Del Ferraro and D. Tantari. Mean field spin glasses treated with PDE techniques. *Eur. Phys. J. B* **86** (7) (2013) 332. MR3089483 https://doi.org/10.1140/epjb/e2013-40334-6
[5] A. Barra, A. Di Biasio and F. Guerra. Replica symmetry breaking in mean-field spin glasses through the Hamilton–Jacobi technique. *J. Stat. Mech. Theory Exp.* **9** (2010) P09006. MR2727596 https://doi.org/10.1088/1742-5468/2010/09/p09006
[6] H.-B. Chen Hamilton–Jacobi equations for nonsymmetric matrix inference. Preprint. Available at arXiv:2006.05328.
[7] H.-B. Chen and J. Xia Hamilton–Jacobi equations for inference of matrix tensor products. Preprint. Available at arXiv:2009.01678. MR4421607 https://doi.org/10.1214/21-aihp1183
[8] W.-K. Chen. Phase transition in the spiked random tensor with Rademacher prior. *Ann. Statist.* **47** (5) (2019) 2734–2756. MR3988771 https://doi.org/10.1214/18-AOS1763
[9] P. Contucci and E. Mingione. A multi-scale spin-glass mean-field model. *Comm. Math. Phys.* **368** (3) (2019) 1323–1344. MR3951707 https://doi.org/10.1007/s00220-019-03308-8
[10] P. Erdős, A. Hajnal and J. Pach. A Ramsey-type theorem for bipartite graphs. *Geombinatorics* **10** (2) (2000) 64–68. MR1784373
[11] F. Guerra. Sum rules for the free energy in the mean field spin glass model. *Fields Inst. Commun.* **30** (2001) 161. MR1867553
[12] F. Guerra. Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.* **233** (1) (2003) 1–12. MR1957729 https://doi.org/10.1007/s00220-002-0773-5
[13] A. Jagannath, J. Ko and S. Sen. Max κ-cut and the inhomogeneous Potts spin glass. *Ann. Appl. Probab.* **28** (3) (2018) 1536–1572. MR3809471 https://doi.org/10.1214/17-AAP1337
[14] J. Ko. Free energy of multiple systems of spherical spin glasses with constrained overlaps. *Electron. J. Probab.* **25** (2020) 28. MR4073689 https://doi.org/10.1214/20-ejp431
[15] M. Mézard, G. Parisi and M. Virasoro. *Spin Glass Theory and Beyond: An Introduction to the Replica Method and Its Applications*, **9**. World Scientific, Singapore, 1987. MR1026102
[16] J.-C. Mourrat. Hamilton–Jacobi equations for mean-field disordered systems. *Ann. Henri Lebesgue*. To appear. MR4275243 https://doi.org/10.5802/ahl.77
[17] J.-C. Mourrat. Hamilton–Jacobi equations for finite-rank matrix inference. *Ann. Appl. Probab.* to appear. MR4149527 https://doi.org/10.1214/19-AAP1556

[18] J.-C. Mourrat Parisi's formula is a Hamilton–Jacobi equation in Wasserstein space. Preprint. Available at arXiv:1906.08471. MR4430924 https://doi.org/10.4153/s0008414x21000031
[19] J.-C. Mourrat Nonconvex interactions in mean-field spin glasses. Preprint. Available at arXiv:2004.01679. MR4408014 https://doi.org/10.2140/pmp.2021.2.281
[20] J.-C. Mourrat and D. Panchenko. Extending the Parisi formula along a Hamilton–Jacobi equation. *Electron. J. Probab.* **25** (2020) 23. MR4073684 https://doi.org/10.1214/20-ejp432
[21] D. Panchenko. Free energy in the generalized Sherrington–Kirkpatrick mean field model. *Rev. Math. Phys.* **17** (7) (2005) 793–857. MR2159369 https://doi.org/10.1142/S0129055X05002455
[22] D. Panchenko. The Parisi ultrametricity conjecture. *Ann. of Math. (2)* **177** (1) (2013) 383–393. MR2999044 https://doi.org/10.4007/annals.2013.177.1.8
[23] D. Panchenko. *The Sherrington–Kirkpatrick Model*. *Springer Monographs in Mathematics*. Springer, New York, 2013. MR3052333 https://doi.org/10.1007/978-1-4614-6289-7
[24] D. Panchenko. The free energy in a multi-species Sherrington–Kirkpatrick model. *Ann. Probab.* **43** (6) (2015) 3494–3513. MR3433586 https://doi.org/10.1214/14-AOP967
[25] D. Panchenko. Free energy in the Potts spin glass. *Ann. Probab.* **46** (2) (2018) 829–864. MR3773375 https://doi.org/10.1214/17-AOP1193
[26] D. Panchenko. Free energy in the mixed p-spin models with vector spins. *Ann. Probab.* **46** (2) (2018) 865–896. MR3773376 https://doi.org/10.1214/17-AOP1194
[27] G. Parisi. Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.* **43** (23) (1979) 1754. MR0702601 https://doi.org/10.1103/PhysRevLett.50.1946
[28] G. Parisi. A sequence of approximated solutions to the SK model for spin glasses. *J. Phys. A* **13** (4) (1980) L115–L121.
[29] S. T. Rachev and L. Rüschendorf. *Mass Transportation Problems. Vol. I.Theory*. *Probability and Its Applications (New York)*. Springer-Verlag, New York, 1998. MR1619171
[30] E. Subag TAP approach for multi-species spherical spin glasses I: General theory. Preprint. Available at arXiv:2111.07132.
[31] E. Subag On the second moment method and RS phase of multi-species spherical spin glasses. Preprint. Available at arXiv:2111.07133.
[32] E. Subag TAP approach for multi-species spherical spin glasses II: The free energy of the pure models. Preprint. Available at arXiv:2111.07134.
[33] M. Talagrand. The Parisi formula. *Ann. of Math. (2)* **163** (1) (2006) 221–263. MR2195134 https://doi.org/10.4007/annals.2006.163.221
[34] M. Talagrand. *Mean Field Models for Spin Glasses. Volume I*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete* **54**. Springer-Verlag, Berlin, 2011. MR2731561 https://doi.org/10.1007/978-3-642-15202-3
[35] M. Talagrand. *Mean Field Models for Spin Glasses. Volume II*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. **55**. Springer, Heidelberg, 2011. MR3024566
[36] F. Zhang. *Matrix Theory*, 2nd edition. *Universitext*. Springer, New York, 2011. MR2857760 https://doi.org/10.1007/978-1-4614-1099-7

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Winding number for stationary Gaussian processes using real variables

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Abstract. We consider the winding number of planar stationary Gaussian processes defined on the line. Under mild conditions, we obtain the asymptotic variance and the Central Limit Theorem for the winding number as the time horizon tends to infinity. In the asymptotic regime, our discrete approach is equivalent to the continuous one studied previously in the literature and our main result extends the existing ones. Our model allows for a general dependence of the coordinates of the process and non-differentiability of one of them. Furthermore, beyond our general framework, we consider as examples an approximation to the winding number of a process whose coordinates are both non-differentiable and the winding number of a process which slightly escapes from stationarity.

Résumé. Nous considérons le nombre de tours des processus gaussiens stationnaires planaires définis sur la droite. Dans des conditions faibles, on obtient la variance asymptotique et le théorème de la limite centrale pour le nombre de tours lorsque l'horizon temporel tend vers l'infini. Dans le régime asymptotique, notre approche discrète est équivalente à l'approche continue étudié précédemment dans la littérature et notre principal résultat étend les existants. Notre modèle permet une dépendance générale des coordonnées du processus et une non-différentiabilité de l'un d'entre eux. De plus, au-delà de notre cadre général, nous considérons comme exemples une approximation du nombre de tours d'un processus dont les coordonnées sont toutes deux non différentiables et le nombre du tours d'un processus qui s'écarte légèrement de la stationnarité.

MSC2020 subject classifications: 60G15; 60G10

Keywords: Gaussian process; Stationary process; Winding number; Wiener chaos expansions; Fourth moment theorem

References

[1] M. Arcones. Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.* **22** (4) (1994) 2242–2274. MR1331224

[2] J.-M. Azaïs, F. Dalmao and J. R. León. CLT for the zeros of classical random trigonometric polynomials. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2) (2016) 804–820. MR3498010 https://doi.org/10.1214/14-AIHP653

[3] J.-M. Azaïs and M. Wschebor. *Level Sets and Extrema of Random Processes and Fields*. John Wiley & Sons, Hoboken, NJ, USA, 2009. MR2478201 https://doi.org/10.1002/9780470434642

[4] J. Buckley and N. Feldheim. The winding of stationary Gaussian processes. *Probab. Theory Related Fields* **172** (1–2) (2018) 583–614. MR3851839 https://doi.org/10.1007/s00440-017-0816-7

[5] A. Comtet and Y. Tourigny. Explicit formulae in probability and in statistical physics. In *Memoriam Marc Yor—Séminaire de Probabilités XLVII* 505–519. *Lecture Notes in Math.* **2137**. Springer, Berlin, 2015. MR3444312 https://doi.org/10.1007/978-3-319-18585-9_22

[6] H. Cramér. On the theory of stationary random processes. *Ann. of Math. (2)* **41** (1940) 215–230. MR0000920 https://doi.org/10.2307/1968827

[7] H. Cramér and M. R. Leadbetter. *Stationary and Related Stochastic Processes*. *Sample Function Properties and Their Applications*. John Wiley & Sons, New York, 1967. MR0217860

[8] Y. Do, H. Nguyen, O. Nguyen and I. Pritsker. Central Limit Theorem for the number of real roots of random orthogonal polynomials. Available at https://arxiv.org/abs/2111.09015.

[9] D. Geman. On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Stat.* **43** (1972) 977–982. MR0301791 https://doi.org/10.1214/aoms/1177692560

[10] M. Kratz and J. R. León. Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: Crossings and extremes. *Stochastic Process. Appl.* **66** (2) (1997) 237–252. MR1440400 https://doi.org/10.1016/S0304-4149(96)00122-6

[11] M. Kratz and J. R. León. Central Limit Theorems for Level Functionals of Stationary Gaussian Processes and Fields. *Journal of Theoretical Probability* **14** (3) (2001). MR1860517 https://doi.org/10.1023/A:1017588905727

[12] P. Le Doussal, Y. Etzioni and B. Horovitz. Winding of planar Gaussian processes. *Journal of Statistical Mechanics: Theory and Experiment* **2009** (7) (2009).
[13] E. Lee and D. Messerschmitt *Digital Communication. Kluwer. Second Edition* (1994).
[14] P. Messulan and M. Yor. On D. Williams' "Pinching method" and some applications. *J. Lond. Math. Soc. (2)* **26** (1982) 348–364. MR0675178 https://doi.org/10.1112/jlms/s2-26.2.348
[15] I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge University Press, Cambridge, 2012. MR2962301 https://doi.org/10.1017/CBO9781139084659
[16] G. Peccati and C. A. Tudor. Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII* 247–262. *Lecture Notes in Math.* **1857**. Springer, Berlin, 2005. MR2126978 https://doi.org/10.1007/978-3-540-31449-3_17
[17] E. V. Slud. MWI representation of the number of curve-crossings by a differentiable Gaussian process, with applications. *Ann. Probab.* **22** (3) (1994) 1355–1380. MR1303648
[18] F. Spitzer. Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87** (1958) 187–197. MR0104296 https://doi.org/10.2307/1993096
[19] M. Taqqu. Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Z. Wahrsch. Verw. Gebiete* **40** (3) (1977) 203–238. MR0471045 https://doi.org/10.1007/BF00736047
[20] S. Vakeroudis. On hitting times of the winding processes of planar Brownian motion and of Ornstein–Uhlenbeck processes, via Bougerol's identity. *SIAM Theory Probab. Appl.* **56** (3) (2012) 485–507. MR3136465 https://doi.org/10.4213/tvp4407

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The conditional Gaussian multiplicative chaos structure underlying a critical continuum random polymer model on a diamond fractal

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Abstract. We discuss a Gaussian multiplicative chaos (GMC) structure underlying a family of random measures $\{\mathbf{M}_r\}_{r\in\mathbb{R}}$ on a space Γ of directed pathways crossing a diamond fractal with Hausdorff dimension two. The laws of these random continuum path measures arise in a critical weak-disorder limiting regime for discrete directed polymers on disordered hierarchical graphs. For the analogous subcritical continuum polymer model in which the diamond fractal has Hausdorff dimension less than two, the random path measures can be constructed as subcritical GMCs through couplings to a spatial Gaussian white noise. This construction fails in the critical dimension two where, formally, an infinite coupling strength to the environmental noise would be required to generate the disorder. We prove, however, that there is a conditional GMC interrelationship between the random measures in the family $\{\mathbf{M}_r\}_{r\in\mathbb{R}}$ such that the law of $\mathbf{M}_r$ can be constructed as a subcritical GMC with random reference measure $\mathbf{M}_R$ for any choice of $R\in(-\infty,r)$. A similar GMC structure plausibly would hold for a critical continuum $(2+1)$-dimensional directed polymer model.

Résumé. Nous discutons une structure de chaos multiplicatif gaussien (CMG) sous-jacente à une famille de mesures aléatoires $\{\mathbf{M}_r\}_{r\in\mathbb{R}}$ sur un espace Γ de chemins dirigés traversant un réseau fractal (de type diamant) avec une dimension de Hausdorff de deux. Les lois de ces mesures aléatoires de chemins continus apparaissent dans un régime limite critique de désordre faible pour les polymères discrets dirigés sur des graphes hiérarchiques désordonnés. Pour le modèle sous-critique analogue dans lequel le réseau fractal a une dimension de Hausdorff inférieure à deux, les mesures de chemins aléatoires peuvent être construites comme des CMG sous-critiques par couplages à un bruit blanc gaussien spatial. Cette construction échoue à la dimension critique deux où, formellement, un couplage infini avec l'environnement aléatoire serait nécessaire pour générer le désordre. Nous prouvons, cependant, qu'il existe une structure de correlation conditionnelle au niveau du CMG entre les mesures aléatoires dans la famille $\{\mathbf{M}_r\}_{r\in\mathbb{R}}$ telle que la loi de $\mathbf{M}_r$ peut être construite comme un CMG sous-critique avec une mesure de référence aléatoire $\mathbf{M}_R$ pour tout choix de $R\in(-\infty,r)$. Une structure similaire au niveau du CMG semblerait plausible pour un modèle critique $(2+1)$-dimensionnel de polymère dirigé.

MSC2020 subject classifications: Primary 60G57; 60G60; 82B44; 82D60; secondary 60G15; 82B21

Keywords: Gaussian multiplicative chaos; Disordered systems; Directed polymers; Diamond hierarchical lattice

References

[1] T. Alberts, K. Khanin and J. Quastel. The continuum directed random polymer. *J. Stat. Phys.* **154** (2014) 305–326. MR3162542 https://doi.org/10.1007/s10955-013-0872-z
[2] V. I. Bogachev. *Gaussian Measures*. *Mathematical Surveys and Monographs* **62**. American Mathematical Society, Providence, 1998. MR1642391 https://doi.org/10.1090/surv/062
[3] F. Caravenna, R. Sun and N. Zygouras. On the moments of the $(2+1)$-dimensional directed polymer and stochastic heat equation in the critical window. *Comm. Math. Phys.* **372** (2019) 385–440. MR4032870 https://doi.org/10.1007/s00220-019-03527-z
[4] F. Caravenna, R. Sun and N. Zygouras The critical 2D stochastic heat flow. Preprint, 2021. Available at arXiv:2109.03766. MR4017119 https://doi.org/10.1214/19-ejp353
[5] J. T. Clark. Continuum directed random polymers on disordered hierarchical diamond lattices. *Stochastic Process. Appl.* **130** (2020) 1643–1668. MR4058285 https://doi.org/10.1016/j.spa.2019.05.008
[6] J. T. Clark. Weak-disorder limit at criticality for random directed polymers on hierarchical graphs. *Comm. Math. Phys.* **386** (2021) 651–710. MR4294278 https://doi.org/10.1007/s00220-021-04149-0
[7] J. T. Clark. Continuum models of directed polymers on disordered diamond fractals in the critical case. *Ann. Appl. Probab.* To appear.
[8] G. Folland. *Real Analysis: Modern Techniques and Their Applications*, 2nd edition. Wiley, New York, 1999. MR1681462
[9] Y. Gu, J. Quastel and L. Tsai. Moments of the 2D SHE at criticality. *Probab. Math. Phys.* **2** (2021) 179–219. MR4404819 https://doi.org/10.2140/pmp.2021.2.179
[10] S. Janson. *Gaussian Hilbert Spaces*. University Press, Cambridge, 1997. MR1474726 https://doi.org/10.1017/CBO9780511526169

[11] J.-P. Kahane. Sur les chaos multiplicatif. *Ann. Sci. Math. Québec* **9** (1985) 105–150. MR0829798
[12] O. Kallenberg. *Random Measures, Theory and Applications*. Springer, New York, 2017. MR3642325 https://doi.org/10.1007/978-3-319-41598-7
[13] H. Lacoin and G. Moreno. Directed polymers on hierarchical lattices with site disorder. *Stochastic Process. Appl.* **120** (2010) 467–493. MR2594367 https://doi.org/10.1016/j.spa.2009.11.008
[14] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** (2014) 315–392. MR3274356 https://doi.org/10.1214/13-PS218
[15] A. Shamov. On Gaussian multiplicative chaos. *J. Funct. Anal.* **270** (2016) 3224–3261. MR3475456 https://doi.org/10.1016/j.jfa.2016.03.001

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A Maxwell principle for generalized Orlicz balls

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Dedicated to the memory of Nicole Tomczak-Jaegermann and Yehoram Gordon.

Abstract. In the 1980s, Diaconis and Freedman studied the low-dimensional projections of random vectors from the Euclidean unit sphere and the simplex in high dimensions, noting that the individual coordinates of these random vectors look like Gaussian and exponential random variables respectively. In subsequent works, Rachev and Rüschendorf and Naor and Romik unified these results by establishing a connection between ℓ_p^N balls and a p-generalized Gaussian distribution. In this paper, we study similar questions in a similar and significantly broader setting, looking at low-dimensional projections of random vectors uniformly distributed on sets of the form $B_{\phi,t}^N := \{(s_1, \dots, s_N) \in \mathbb{R}^N : \sum_{i=1}^N \phi(s_i) \le tN\}$, where $\phi : \mathbb{R} \to [0, \infty]$ is a function satisfying some fairly mild conditions. We find that there is a critical parameter t_{crit} at which there is a phase transition in the behaviour of the low-dimensional projections: for $t > t_{\mathrm{crit}}$ the coordinates of random vectors sampled from $B_{\phi,t}^N$ behave like independent uniform random variables, but for $t \le t_{\mathrm{crit}}$ however the Gibbs conditioning principle comes into play, and here there is a parameter $\beta_t > 0$ (the inverse temperature) such that the coordinates are approximately distributed according to a density proportional to $e^{-\beta_t \phi(s)}$.

Résumé. Dans les années 1980, Diaconis et Freedman ont étudié les projections en petites dimensions de vecteurs aléatoires de la sphère unitaire euclidienne et du simplexe en grandes dimensions, remarquant que les coordonnées individuelles de ces vecteurs aléatoires ressemblent respectivement à des variables aléatoires gaussiennes et exponentielles. Dans des travaux ultérieurs, Rachev et Rüschendorf et Naor et Romik ont unifié ces résultats en établissant une relation entre les boules ℓ_p^N et une distribution gaussienne p-généralisée. Dans cet article, nous étudions des questions analogues dans un cadre similaire et beaucoup plus large, en nous intéressant aux projections en petites dimensions de vecteurs aléatoires uniformément distribués sur des ensembles de la forme $B_{\phi,t}^N := \{(s_1, \dots, s_N) \in \mathbb{R}^N : \sum_{i=1}^N \phi(s_i) \le tN\}$, où $\phi : \mathbb{R} \to [0, \infty]$ est une fonction satisfaisant à des conditions assez générales. Nous montrons qu'il existe un paramètre critique t_{crit} pour lequel il y a une transition de phase dans le comportement des projections en petites dimensions : pour $t > t_{\mathrm{crit}}$ les coordonnées des vecteurs aléatoires échantillonnés dans $B_{\phi,t}^N$ se comportent comme des variables aléatoires uniformes indépendantes, mais pour $t \le t_{\mathrm{crit}}$ le principe de conditionnement de Gibbs entre en jeu, et il existe dans ce cas un paramètre $\beta_t > 0$ (l'inverse de la température) tel que les coordonnées sont approximativement distribuées selon une densité proportionnelle à $e^{-\beta_t \phi(s)}$.

MSC2020 subject classifications: Primary 60F05; secondary 52A20; 60F10

Keywords: Generalized Orlicz balls; Gibbs conditioning principle; Gibbs measures; Maxwell principle; Low-dimensional projections; Quantitative Cramér theorem; Large deviation principle

References

[1] D. Alonso-Gutiérrez and J. Prochno. Thin-shell concentration for random vectors in Orlicz balls via moderate deviations and Gibbs measures. *J. Funct. Anal.* **282** (2022), 109291. MR4330166 https://doi.org/10.1016/j.jfa.2021.109291
[2] R. R. Bahadur and R. R. Rao. On deviations of the sample mean. *Ann. Math. Stat.* **31** (1960) 1015–1027. MR0117775 https://doi.org/10.1214/aoms/1177705674
[3] J. R. Blum and K. P. Pathak. A note on the zero–one law. *Ann. Math. Stat.* **43** (1972) 1008–1009. MR0300314 https://doi.org/10.1214/aoms/1177692564
[4] E. Borel. *Introduction géométrique à quelques théories physiques*. *Cornell University Library Historical Math Monographs*. Gauthier-Villars, 1914.
[5] H. E. Daniels. Saddlepoint approximations in statistics. *Ann. Math. Stat.* **25** (1954) 631–650. MR0066602 https://doi.org/10.1214/aoms/1177728652
[6] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications* 2nd edition. Springer-Verlag, Berlin Heidelberg, 2010. MR2571413 https://doi.org/10.1007/978-3-642-03311-7

[7] P. Diaconis and D. Freedman. A dozen de Finetti-style results in search of a theory. *Ann. Inst. Henri Poincaré Probab. Stat.* **23** (1987) 397–423. MR0898502

[8] P. Diaconis and D. A. Freedman. Conditional limit theorems for exponential families and finite versions of de Finetti's theorem. *J. Theoret. Probab.* **1** (1988) 381–410. MR0958245 https://doi.org/10.1007/BF01048727

[9] W. Feller. *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd edition. John Wiley & Sons, Inc., New York–London–Sydney, 1971. MR0270403

[10] L. Fruehwirth and J. Prochno. Sanov-type large deviations and conditional limit theorems for high-dimensional Orlicz balls, 2021. arXiv preprint. Available at arXiv:2111.04691.

[11] J. Hörrmann, J. Prochno and C. Thäle. On the isotropic constant of random polytopes with vertices on an ℓ_p-sphere. *J. Geom. Anal.* **28** (2018) 405–426. MR3745865 https://doi.org/10.1007/s12220-017-9826-z

[12] Z. Kabluchko and J. Prochno. The maximum entropy principle and volumetric properties of Orlicz balls. *J. Math. Anal. Appl.* **495** (2021), 124687. MR4172842 https://doi.org/10.1016/j.jmaa.2020.124687

[13] Z. Kabluchko, J. Prochno and C. Thäle. High-dimensional limit theorems for random vectors in ℓ_p^n-balls. *Commun. Contemp. Math.* **21** (2019), 1750092, 30. MR3904638 https://doi.org/10.1142/S0219199717500924

[14] Z. Kabluchko, J. Prochno and V. Vysotsky. Yet another note on the arithmetic-geometric mean inequality. *Studia Math.* **253** (2020) 39–55. MR4074400 https://doi.org/10.4064/sm181014-16-3

[15] S. S. Kim, Y. T. Liao and K. Ramanan. An asymptotic thin shell condition and large deviations for random multidimensional projections. arXiv e-prints, 2019. Available at arXiv:1912.13447. MR4352552 https://doi.org/10.1016/j.aam.2021.102306

[16] S. S. Kim and K. Ramanan. A conditional limit theorem for high-dimensional ℓ^p-spheres. *J. Appl. Probab.* **55** (2018) 1060–1077. MR3899928 https://doi.org/10.1017/jpr.2018.71

[17] Y.-T. Liao and K. Ramanan. Geometric sharp large deviations for random projections of ℓ_p^n spheres and balls, 2020. arXiv preprint. Available at arXiv:2001.04053.

[18] A. Naor and D. Romik. Projecting the surface measure of the sphere of ℓ_p^n. *Ann. Inst. Henri Poincaré Probab. Stat.* **39** (2003) 241–261. MR1962135 https://doi.org/10.1016/S0246-0203(02)00008-0

[19] J. Prochno, C. Thäle and N. Turchi. The isotropic constant of random polytopes with vertices on convex surfaces. *J. Complexity* **54** (2019), 101394, 17. MR3983208 https://doi.org/10.1016/j.jco.2019.01.001

[20] S. T. Rachev and L. Rüschendorf. Approximate independence of distributions on spheres and their stability properties. *Ann. Probab.* **19** (1991) 1311–1337. MR1112418

[21] G. Schechtman and J. Zinn. On the volume of the intersection of two L_p^n balls. *Proc. Amer. Math. Soc.* **110** (1990) 217–224. MR1015684 https://doi.org/10.2307/2048262

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Gap at 1 for the percolation threshold of Cayley graphs

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Abstract. We prove that the set of possible values for the percolation threshold p_c of Cayley graphs has a gap at 1 in the sense that there exists $\varepsilon_0 > 0$ such that for every Cayley graph G one either has $p_c(G) = 1$ or $p_c(G) \leq 1 - \varepsilon_0$. The proof builds on the new approach of Duminil-Copin, Goswami, Raoufi, Severo & Yadin (*Duke Math. J.* **169** (2020) 3539–3563) to the existence of phase transition using the Gaussian free field, combined with the finitary version of Gromov's theorem on the structure of groups of polynomial growth of Breuillard, Green & Tao (*Publ. Math. Inst. Hautes Études Sci.* **116** (2012) 115–221).

Résumé. Nous prouvons que l'ensemble des valeurs possibles pour les seuils critiques de percolation p_c de graphes de Cayley a une « lacune » en 1, dans le sens qu'il existe $\varepsilon_0 > 0$ tel que pour tout graphe de Cayley G, on a soit $p_c(G) = 1$, soit $p_c(G) \leq 1 - \varepsilon_0$. La démonstration s'appuie sur la nouvelle approche de Duminil-Copin, Goswami, Raoufi, Severo & Yadin (*Duke Math. J.* **169** (2020) 3539–3563) pour prouver l'existence de la transition de phase en utilisant le champ libre gaussien, combinée avec la version finitaire du théorème de Gromov sur la structure des groupes à croissance polynomiale de Breuillard, Green & Tao (*Publ. Math. Inst. Hautes Études Sci.* **116** (2012) 115–221).

MSC2020 subject classifications: 60K35; 82B43; 82B26

Keywords: Percolation; Gaussian free field; Cayley graphs; Phase transition; Critical point

References

[1] M. Aizenman and G. Grimmett. Strict monotonicity for critical points in percolation and ferromagnetic models. *J. Stat. Phys.* **63** (1991) 817–835. MR1116036 https://doi.org/10.1007/BF01029985
[2] E. Babson and I. Benjamini. Cut sets and normed cohomology with applications to percolation. *Proc. Amer. Math. Soc.* **127** (1999) 589–597. MR1622785 https://doi.org/10.1090/S0002-9939-99-04995-3
[3] I. Benjamini, R. Pemantle and Y. Peres. Unpredictable paths and percolation. *Ann. Probab.* **26** (1998) 1198–1211. MR1634419 https://doi.org/10.1214/aop/1022855749
[4] I. Benjamini and O. Schramm. Percolation beyond $\mathbf{Z}^\mathbf{d}$, many questions and a few answers. *Electron. Commun. Probab.* **1** (1996) 71–82. MR1423907 https://doi.org/10.1214/ECP.v1-978
[5] E. Breuillard, B. Green and T. Tao. The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.* **116** (2012) 115–221. MR3090256 https://doi.org/10.1007/s10240-012-0043-9
[6] J. Bricmont, J. L. Lebowitz and C. Maes. Percolation in strongly correlated systems: The massless Gaussian field. *J. Stat. Phys.* **48** (1987) 1249–1268. MR0914444 https://doi.org/10.1007/BF01009544
[7] E. Candellero and A. Teixeira. Percolation and isoperimetry on roughly transitive graphs. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (2018) 1819–1847. MR3865659 https://doi.org/10.1214/17-AIHP857
[8] H. Duminil-Copin, S. Goswami, A. Raoufi, F. Severo and A. Yadin. Existence of phase transition for percolation using the Gaussian free field. *Duke Math. J.* **169** (2020) 3539–3563. MR4181032 https://doi.org/10.1215/00127094-2020-0036
[9] G. R. Grimmett. *Percolation*, 2nd edition. Springer-Verlag, Berlin, 1999. MR1707339 https://doi.org/10.1007/978-3-662-03981-6
[10] M. Gromov. Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Études Sci.* **53** (1981) 53–78. MR0623534
[11] T. Hutchcroft and M. Tointon Non-triviality of the phase transition for percolation on finite transitive graphs. Preprint, 2021. Available at arXiv:2104.05607.
[12] T. M. Liggett, R. H. Schonmann and A. M. Stacey. Domination by product measures. *Ann. Probab.* **25** (1997) 71–95. MR1428500 https://doi.org/10.1214/aop/1024404279
[13] R. Lyons. Random walks and the growth of groups. *C. R. Math. Acad. Sci. Paris* **320** (1995) 1361–1366. MR1338287
[14] R. Lyons, B. Morris and O. Schramm. Ends in uniform spanning forests. *Electron. J. Probab.* **13** (2008) 1702–1725. MR2448128 https://doi.org/10.1214/EJP.v13-566
[15] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, 2017. MR3616205 https://doi.org/10.1017/9781316672815

[16] S. Martineau and F. Severo. Strict monotonicity of percolation thresholds under covering maps. *Ann. Probab.* **47** (2019) 4116–4136. MR4038050 https://doi.org/10.1214/19-aop1355
[17] R. Muchnik and I. Pak. Percolation on Grigorchuk groups. *Comm. Algebra* **29** (2001) 661–671. MR1841989 https://doi.org/10.1081/AGB-100001531
[18] G. Pete. Probability and geometry on groups. Available at https://math.bme.hu/~gabor/PGG.pdf.
[19] A. Raoufi and A. Yadin. Indicable groups and $p_c < 1$. *Electron. Commun. Probab.* **22** (2017) 1–10. MR3607808 https://doi.org/10.1214/16-ECP40
[20] A. Teixeira. Percolation and local isoperimetric inequalities. *Probab. Theory Related Fields* **165** (2016) 963–984. MR3520023 https://doi.org/10.1007/s00440-015-0653-5
[21] R. Tessera and M. C. Tointon. A finitary structure theorem for vertex-transitive graphs of polynomial growth. *Combinatorica* **41** (2021) 263–298. MR4253426 https://doi.org/10.1007/s00493-020-4295-6

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An upper bound for p_c in range-R bond percolation in two and three dimensions

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Abstract. An upper bound for the critical probability of the range-R bond percolation in $d = 2$ and $d = 3$ is obtained by connecting the bond percolation with the SIR epidemic model, thus complementing the lower bound result in Frei and Perkins (*Electron. J. Probab.* **21** (2016) Paper No. 56, 1–22).

Résumé. Une borne supérieure pour la probabilité critique de la percolation par arêtes de portée R dans $d = 2$ et $d = 3$ est obtenue en connectant le modèle de percolation avec le modèle épidémique SIR, complétant ainsi le résultat dans Frei et Perkins (*Electron. J. Probab.* **21** (2016) Paper No. 56, 1–22) qui donne une borne inférieure.

MSC2020 subject classifications: Primary 60K35; secondary 60J68; 60J80; 92D30

Keywords: Long-range bond percolation; Critical probability; SIR epidemic; Branching random walk; Local time

References

[1] M. Bramson, R. Durrett and G. Swindle Statistical mechanics of crabgrass. *Ann. Probab.* **17** (2) (1989) 444–481. MR0985373
[2] R. Durrett. Ten lectures on particle systems. In *Lectures on Probability Theory (Saint Flour, 1993)* 97–201. *Lecture Notes in Math.* **1608**. Springer, Berlin, 1995. MR1383122 https://doi.org/10.1007/BFb0095747
[3] R. Durrett. *Probability: Theory and Examples*. *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2010. MR2722836 https://doi.org/10.1017/CBO9780511779398
[4] D. Freedman. On tail probabilities for martingales. *Ann. Probab.* **3** (1) (1975) 100–118. MR0380971 https://doi.org/10.1214/aop/1176996452
[5] S. Frei and E. Perkins. A lower bound for p_c in range-R bond percolation in two and three dimensions. *Electron. J. Probab.* **21** (2016) Paper No. 56, 1–22. MR3546393 https://doi.org/10.1214/16-EJP6
[6] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128** (1990) 333–391. MR1043524
[7] J. Hong. Renormalization of local times of super-Brownian motion. *Electron. J. Probab.* **23** (2018) Paper No. 109, 1–45. MR3878134 https://doi.org/10.1214/18-ejp231
[8] T. Hutchcroft. Power-law bounds for critical long-range percolation below the upper-critical dimension. *Probab. Theory Related Fields* **181** (2021) 533–570. MR4341081 https://doi.org/10.1007/s00440-021-01043-7
[9] H. Kesten. Asymptotics in high dimensions for percolation. In *Disorder in Physical Systems: A Volume in Honor of John Hammersley*. G. Grimmett and D. J. A. Welsh (Eds). Clarendon Press, Oxford, 1990. MR0869591
[10] S. Lalley, E. Perkins and X. Zheng. A phase transition for measure-valued SIR epidemic processes. *Ann. Probab.* **42** (1) (2014) 237–310. MR3161486 https://doi.org/10.1214/13-AOP846
[11] S. Lalley and X. Zheng. Spatial epidemics and local times for critical branching random walks in dimensions 2 and 3. *Probab. Theory Related Fields* **148** (3–4) (2010) 527–566. MR2678898 https://doi.org/10.1007/s00440-009-0239-1
[12] G. Lawler and V. Limic. *Random Walk: A Modern Introduction*. Cambridge University Press, Cambridge, 2010. MR2677157 https://doi.org/10.1017/CBO9780511750854
[13] M. D. Penrose. On the spread-out limit for bond and continuum percolation. *Ann. Probab.* **3** (1993) 253–276. MR1202526
[14] E. A. Perkins. A space-time property of a class of measure-valued branching diffusions. *Trans. Amer. Math. Soc.* **305** (2) (1988) 743–795. MR0924777 https://doi.org/10.2307/2000886
[15] E. A. Perkins. Dawson–Watanabe superprocesses and measure-valued diffusions. Lectures on probability theory and statistics. In *Lectures on Probability Theory and Statistics, No. 1781, Ecole d'Eté de Probabilités de Saint Flour 1999*. Springer, Berlin, 2002. MR1915445
[16] R. G. Pinsky. On the large time growth rate of the support of supercritical super-Brownian motion. *Ann. Probab.* **23** (1995) 1748–1754. MR1379166
[17] S. Ross. *Introduction to Probability Models*, 12th edition. Academic Press, London, 2019. MR3931305
[18] S. Sugitani. Some properties for the measure-valued branching diffusion processes. *J. Math. Soc. Japan* **41** (1989) 437–462. MR0999507 https://doi.org/10.2969/jmsj/04130437

[19] R. van der Hofstad and A. Sakai. Critical points for spread-out self-avoiding walk, percolation and the contact process above the upper critical dimensions. *Probab. Theory Related Fields* **132** (2005) 438–470. MR2197108 https://doi.org/10.1007/s00440-004-0405-4

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Local and global comparisons of the Airy difference profile to Brownian local time

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Abstract. There has recently been much activity within the Kardar–Parisi–Zhang universality class spurred by the construction of the canonical limiting object, the parabolic Airy sheet $\mathcal{S}:\mathbb{R}^2\to\mathbb{R}$ (Dauvergne, Ortmann and Virág (2018)). The parabolic Airy sheet provides a coupling of parabolic Airy$_2$ processes – a universal limiting geodesic weight profile in planar last passage percolation models – and a natural goal is to understand this coupling. Geodesic geometry suggests that the difference of two parabolic Airy$_2$ processes, i.e., a difference profile, encodes important structural information. This difference profile $\mathcal{D}$, given by $\mathbb{R}\to\mathbb{R}: x\mapsto \mathcal{S}(1,x)-\mathcal{S}(-1,x)$, was first studied by Basu, Ganguly and Hammond (2019), who showed that it is monotone and almost everywhere constant, with its points of non-constancy forming a set of Hausdorff dimension $1/2$. Noticing that this is also the Hausdorff dimension of the zero set of Brownian motion, we adopt a different approach. Establishing previously inaccessible fractal structure of $\mathcal{D}$, we prove, on a global scale, that $\mathcal{D}$ is absolutely continuous on compact sets to Brownian local time (of rate four) in the sense of increments, which also yields the main result of Basu, Ganguly and Hammond (2019) as a simple corollary. Further, on a local scale, we explicitly obtain Brownian local time of rate four as a local limit of $\mathcal{D}$ at a point of increase, picked by a number of methods, including at a typical point sampled according to the distribution function $\mathcal{D}$. Our arguments rely on the representation of $\mathcal{S}$ in terms of a last passage problem through the parabolic Airy line ensemble and an understanding of geodesic geometry at deterministic and random times.

Résumé. Il y a eu récemment beaucoup d'activité au sein de la classe d'universalité de Kardar–Parisi–Zhang stimulée par la construction de l'objet limite canonique, le drap d'Airy parabolique $\mathcal{S}:\mathbb{R}^2\to\mathbb{R}$ (Dauvergne, Ortmann et Virág (2018)). Le drap d'Airy parabolique fournit un couplage de processus Airy$_2$ paraboliques – un profil géodésique limite universel dans les modèles de percolation planaires au dernier passage – et un objectif naturel est de comprendre ce couplage. La géométrie géodésique suggère que la différence de deux processus Airy$_2$ paraboliques, c'est-à-dire un profil de la différence, encode des informations structurelles importantes. Ce profil de différence $\mathcal{D}$, donné par $\mathbb{R}\to\mathbb{R}: x\mapsto \mathcal{S}(1,x)-\mathcal{S}(-1,x)$, a d'abord été étudié par Basu, Ganguly et Hammond (2019), qui ont montré qu'il est monotone et presque partout constant, avec ses points de non-constance formant un ensemble de dimension Hausdorff $1/2$. Constatant qu'il s'agit également de la dimension de Hausdorff de l'ensemble des zéros du mouvement brownien, nous adoptons une approche différente. En établissant une structure fractale jusqu'alors inaccessible de $\mathcal{D}$, nous prouvons, à l'échelle globale, que $\mathcal{D}$ est absolument continu sur des ensembles compacts à temps local brownien (de taux quatre) au sens des accroissements, ce qui donne également le résultat principal de Basu, Ganguly et Hammond (2019) comme un simple corollaire. De plus, à l'échelle locale, nous obtenons explicitement le temps local brownien de taux quatre comme limite locale de $\mathcal{D}$ en un point d'accroissement, choisi par un certain nombre de méthodes, y compris en un point typique échantillonné selon la fonction de distribution $\mathcal{D}$. Nos arguments s'appuient sur la représentation de $\mathcal{S}$ en termes d'un problème de dernier passage au moyen de l'ensemble parabolique de lignes d'Airy et sur une compréhension de la géométrie géodésique en des temps déterministes et aléatoires.

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Keywords: Brownian local time; Weight difference profile; Airy sheet; Brownian Gibbs property

References

[1] T. Alberts, K. Khanin and J. Quastel. The continuum directed random polymer. *J. Stat. Phys.* **154** (1) (2014) 305–326. MR3162542 https://doi.org/10.1007/s10955-013-0872-z
[2] M. Balázs, O. Busani and T. Seppäläinen. Non-existence of bi-infinite geodesics in the exponential corner growth model. In *Forum of Mathematics, Sigma*, **8**. Cambridge University Press, 2020. MR4176750 https://doi.org/10.1017/fms.2020.31
[3] M. Balázs, O. Busani and T. Seppäläinen. Local stationarity in exponential last-passage percolation. *Probab. Theory Related Fields* **180** (1) (2021) 113–162. MR4265019 https://doi.org/10.1007/s00440-021-01035-7

[4] R. Basu, S. Ganguly and A. Hammond. Fractal geometry of Airy_2 processes coupled via the Airy sheet. *Ann. Probab.* **49** (1) (2021) 485–505. MR4203343 https://doi.org/10.1214/20-AOP1444

[5] R. Basu, S. Ganguly, M. Hegde and M. Krishnapur. Lower deviations in β-ensembles and law of iterated logarithm in last passage percolation. *Israel J. Math.* **242** (2021) 291–324. MR4282085 https://doi.org/10.1007/s11856-021-2135-z

[6] E. Bates, S. Ganguly and A. Hammond. Hausdorff dimensions for shared endpoints of disjoint geodesics in the directed landscape, 2019. arXiv preprint. Available at arXiv:1912.04164. MR4361743 https://doi.org/10.1214/21-ejp706

[7] V. Beffara. The dimension of the SLE curves. *Ann. Probab.* **36** (4) (2008) 1421–1452. MR2435854 https://doi.org/10.1214/07-AOP364

[8] J. Ben Hough, M. Krishnapur, Y. Peres and B. Virág. *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, **51**. American Mathematical Soc., 2009. MR2552864 https://doi.org/10.1090/ulect/051

[9] J. Calvert, A. Hammond and M. Hegde. Brownian structure in the KPZ fixed point, 2019. arXiv preprint. Available at arXiv:1912.00992.

[10] E. Cator and L. P. R. Pimentel. On the local fluctuations of last-passage percolation models. *Stochastic Process. Appl.* **125** (2) (2015) 538–551. MR3293293 https://doi.org/10.1016/j.spa.2014.08.009

[11] E. A. Cator, S. I. López and L. P. R. Pimentel. Attractiveness of Brownian queues in tandem. *Queueing Syst.* **92** (1–2) (2019) 25–45. MR3946497 https://doi.org/10.1007/s11134-019-09609-y

[12] I. Corwin and A. Hammond. Brownian Gibbs property for Airy line ensembles. *Invent. Math.* **195** (2) (2014) 441–508. MR3152753 https://doi.org/10.1007/s00222-013-0462-3

[13] I. Corwin, A. Hammond, M. Hegde and K. Matetski. Exceptional times when the KPZ fixed point violates Johansson's conjecture on maximizer uniqueness, 2021. arXiv preprint. Available at arXiv:2101.04205.

[14] I. Corwin, J. Quastel and D. Remenik. Renormalization fixed point of the KPZ universality class. *J. Stat. Phys.* **160** (4) (2015) 815–834. MR3373642 https://doi.org/10.1007/s10955-015-1243-8

[15] S. Das Temporal increments of the KPZ equation with general initial data. arXiv preprint, 2022. Available at arXiv:2203.00666.

[16] S. Das and P. Ghosal. Law of iterated logarithms and fractal properties of the KPZ equation, 2021. arXiv preprint. Available at arXiv:2101.00730.

[17] D. Dauvergne Last passage isometries for the directed landscape. arXiv preprint, 2021. Available at arXiv:2106.07566. MR4035169

[18] D. Dauvergne Non-uniqueness times for the maximizer of the kpz fixed point. arXiv preprint, 2022. Available at arXiv:2202.01700.

[19] D. Dauvergne, J. Ortmann and B. Virág The directed landscape. arXiv preprint, 2018. Available at arXiv:1812.00309. MR4035169

[20] D. Dauvergne, S. Sarkar and B. Virág. Three-halves variation of geodesics in the directed landscape, 2020. arXiv preprint. Available at arXiv:2010.12994. MR4035169

[21] D. Dauvergne and B. Virág. Bulk properties of the Airy line ensemble. *Ann. Probab.* **49** (4) (2021) 1738–1777. MR4260467 https://doi.org/10.1214/20-aop1492

[22] D. Dauvergne and L. Zhang Disjoint optimizers and the directed landscape. arXiv preprint, 2021. Available at arXiv:2102.00954. MR4035169

[23] S. Ganguly and L. Zhang. Fractal geometry of the space-time difference profile in the directed landscape via construction of geodesic local times, 2022. arXiv preprint. Available at arXiv:2204.01674.

[24] C. Garban, G. Pete and O. Schramm. The Fourier spectrum of critical percolation. *Acta Math.* **205** (1) (2010) 19–104. MR2736153 https://doi.org/10.1007/s11511-010-0051-x

[25] J. Hägg. Local Gaussian fluctuations in the Airy and discrete PNG processes. *Ann. Probab.* **36** (3) (2008) 1059–1092. MR2408583 https://doi.org/10.1214/07-AOP353

[26] A. Hammond. Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation, 2016. arXiv preprint. Available at arXiv:1609.02971. MR4403929 https://doi.org/10.1090/memo/1363

[27] A. Hammond. Modulus of continuity of polymer weight profiles in Brownian last passage percolation. *Ann. Probab.* **47** (6) (2019) 3911–3962. MR4038045 https://doi.org/10.1214/19-aop1350

[28] A. Hammond. A patchwork quilt sewn from Brownian fabric: Regularity of polymer weight profiles in Brownian last passage percolation. In *Forum of Mathematics, Pi*, **7**. Cambridge University Press, 2019. MR3987302 https://doi.org/10.1017/fmp.2019.2

[29] A. Hammond. Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *Proc. Lond. Math. Soc.* **120** (3) (2020) 370–433. MR4008374 https://doi.org/10.1112/plms.12292

[30] A. Hammond, G. Pete and O. Schramm. Local time on the exceptional set of dynamical percolation and the incipient infinite cluster. *Ann. Probab.* **43** (6) (2015) 2949–3005. MR3433575 https://doi.org/10.1214/14-AOP950

[31] L. P. R. Pimentel. Brownian aspects of the KPZ fixed point. In *In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius* 711–739. Springer, 2021. MR4237289 https://doi.org/10.1007/978-3-030-60754-8_29

[32] M. Ledoux. A law of the iterated logarithm for directed last passage percolation. *J. Theor. Probab.* **31** (4) (2018) 2366–2375. MR3866616 https://doi.org/10.1007/s10959-017-0775-z

[33] K. Matetski, J. Quastel and D. Remenik. The KPZ fixed point. *Acta Math.* (2021+). To appear. MR4346267 https://doi.org/10.4310/acta.2021.v227.n1.a3

[34] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, **44**. Cambridge University Press, 1999. MR1333890 https://doi.org/10.1017/CBO9780511623813

[35] P. Mörters and Y. Peres. *Brownian Motion*, **30**. Cambridge University Press, 2010. MR2604525 https://doi.org/10.1017/CBO9780511750489

[36] M. Prähofer and H. Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* **108** (5–6) (2002) 1071–1106. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR1933446 https://doi.org/10.1023/A:1019791415147

[37] J. Quastel and D. Remenik. Local behavior and hitting probabilities of the Airy_1 process. *Probab. Theory Related Fields* **157** (3–4) (2013) 605–634. MR3129799 https://doi.org/10.1007/s00440-012-0466-8

[38] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, **293**. Springer Science & Business Media, 2013. MR1083357 https://doi.org/10.1007/978-3-662-21726-9

[39] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math.* (2005) 883–924. MR2153402 https://doi.org/10.4007/annals.2005.161.883

[40] S. Sarkar and B. Virág. Brownian absolute continuity of the KPZ fixed point with arbitrary initial condition. *Ann. Probab.* **49** (4) (2021) 1718–1737. MR4260466 https://doi.org/10.1214/20-aop1491

[41] T. Seppäläinen and E. Sorensen. Global structure of semi-infinite geodesics and competition interfaces in Brownian last-passage percolation, 2021. arXiv preprint. Available at arXiv:2112.10729.

[42] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** (1) (1994) 151–174. MR1257246

[43] J. Warren. Dyson's Brownian motions, intertwining and interlacing. *Electron. J. Probab.* **12** (2007) 573–590. MR2299928 https://doi.org/10.1214/EJP.v12-406

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Asymptotics for the critical level and a strong invariance principle for high intensity shot noise fields

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Abstract. We study fine properties of the convergence of a high intensity shot noise field towards the Gaussian field with the same covariance structure. In particular we (i) establish a strong invariance principle, i.e. a quantitative coupling between a high intensity shot noise field and the Gaussian limit such that they are uniformly close on large domains with high probability, and (ii) use this to derive an asymptotic expansion for the critical level above which the excursion sets of the shot noise field percolate.

Résumé. On étudie les propriétés fines de convergence d'un champ shot noise haute fréquence vers le champ gaussien de même structure de covariance. En particulier, on (i) établit un *strong invariance principle*, i.e. un couplage quantitatif entre un champ shot noise haute fréquence et son champ limite gaussien tel qu'ils soient uniformément proches sur de larges domaines avec grande probabilité, et (ii) on en déduit un développement asymptotique du seuil critique de percolation du champ shot noise.

MSC2020 subject classifications: 60G60; 60K35

Keywords: Shot noise fields; Gaussian fields; Percolation; Strong invariance principle

References

[1] D. Ahlberg, V. Tassion and A. Teixeira. Sharpness of the phase transition for continuum percolation. *Probab. Theory Related Fields* **172** (2018) 525–581. MR3851838 https://doi.org/10.1007/s00440-017-0815-8
[2] M. Aljuaid and H. Yanikomeroglu. Investigating the Gaussian convergence of the distribution of the aggregate interference power in large wireless networks. *IEEE Trans. Veh. Technol.* **59** (2010) 4418–4424.
[3] J. Azaïs and M. Wschebor. *Level Sets and Extrema of Random Processes and Fields*. Wiley, Hoboken, NJ, 2009. MR2478201 https://doi.org/10.1002/9780470434642
[4] J. Beck. Lower bounds on the approximation of the multivariate empirical process. *Z. Wahrsch. Verw. Gebiete* **70** (1985) 289–306. MR0799151 https://doi.org/10.1007/BF02451433
[5] V. Beffara and D. Gayet. Percolation of random nodal lines. *Publ. Math.* **126** (2017) 131–176. MR3735866 https://doi.org/10.1007/s10240-017-0093-0
[6] P. Berthet and D. M. Mason. Revisiting two strong approximation results of Dudley and Philipp. In *High Dimensional Probability* 155–172. *IMS Lecture Notes–Monograph Series* **51**. Inst. Math. Statist., Beachwood, OH, 2006. MR2387767 https://doi.org/10.1214/074921706000000824
[7] H. Biermé and A. Desolneux. Crossings of smooth shot noise processes. *Ann. Appl. Probab.* **22** (2012) 2240–2281. MR3024968 https://doi.org/10.1214/11-AAP807
[8] H. Biermé and A. Desolneux. A Fourier approach for the level crossings of shot noise processes with jumps. *J. Appl. Probab.* **49** (2012) 100–113. MR2952884 https://doi.org/10.1239/jap/1331216836
[9] H. Biermé and A. Desolneux. Mean geometry for 2D random fields: Level perimeter and level total curvature integrals. *Ann. Appl. Probab.* **30** (2020) 561–607. MR4108116 https://doi.org/10.1214/19-AAP1508
[10] E. Broman and R. Meester. Phase transition and uniqueness of levelset percolation. *J. Stat. Phys.* **167** (2017) 1376–1400. MR3652518 https://doi.org/10.1007/s10955-017-1782-2
[11] M. Csörgő and P. Révész. A new method to prove Strassen-type laws of invariance principle. II. *Z. Wahrsch. Verw. Gebiete* **31** (1975) 261–269. MR1554018 https://doi.org/10.1007/BF00532866
[12] H. Duminil-Copin, A. Rivera, P. F. Rodriguez and H. Vanneuville, 2021. Existence of unbounded nodal hypersurface for smooth Gaussian fields in dimension $d \geq 3$. Preprint. Available at arXiv:2108.08008.
[13] B. Galerne, Y. Gousseau and J. Morel. Random phase textures: Theory and synthesis. *IEEE Trans. Image Process.* **20** (2011) 257–267. MR2789729 https://doi.org/10.1109/TIP.2010.2052822
[14] A. Gandolfi, M. Keane and L. Russo. On the uniqueness of the infinite occupied cluster in dependent two-dimensional site percolation. *Ann. Probab.* **16** (1988) 1147–1157. MR0942759

[15] G. R. Grimmett. *Percolation*. Springer, Berlin, Germany, 1999. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[16] L. Heinrich. Limit distributions of some stereological estimators in Wicksell's corpuscle problem. *Image Anal. Stereol.* **26** (2007) 63–71. MR2363472 https://doi.org/10.5566/ias.v26.p63-71

[17] L. Heinrich and V. Schmidt. Normal convergence of multidimension shot noise and rates of this convergence. *Adv. Appl. Probab.* **17** (1985) 709–730. MR0809427 https://doi.org/10.2307/1427084

[18] H. Kesten. *Percolation Theory for Mathematicians*. *Progress in Probability and Statistics* **2**. Springer, Berlin, 1982. MR0692943

[19] C. Kluppelberg and T. Mikosch. Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli* **1** (1995) 125–147. MR1354458 https://doi.org/10.2307/3318683

[20] L. Köhler-Schindler and V. Tassion, Crossing probabilities for planar percolation, 2020. Preprint. Available at arXiv:2011.04618.

[21] V. I. Koltchinskii. Komlos–Major–Tusnady approximation for the general empirical process and Haar expansions of classes of functions. *J. Theor. Probab.* **7** (1994) 73–118. MR1256392 https://doi.org/10.1007/BF02213361

[22] J. Komlós, P. Major and G. Tusnády. An approximation of partial sums of independent RV's and the sample DF. *Z. Wahrsch. Verw. Gebiete* **32** (1975) 111–131. MR0375412 https://doi.org/10.1007/BF00533093

[23] R. Lachièze-Rey and S. Muirhead. Percolation of the excursion sets of planar symmetric shot noise fields. *Stochastic Process. Appl.* **147** (2022) 175–209. MR4379380 https://doi.org/10.1016/j.spa.2022.01.013

[24] J. A. Lane. The Berry–Esseen bound for Poisson shot noise. *Adv. Appl. Probab.* **19** (1987) 512–514. MR0889950 https://doi.org/10.2307/1427432

[25] P. Massart. Strong approximation for multivariate empirical and related processes, via KMT constructions. *Ann. Probab.* **17** (1989) 266–291. MR0972785

[26] S. A. Molchanov and A. K. Stepanov. Percolation in random fields. II. *Theoret. Math. Phys.* **55** (1983) 592–599. MR0734878

[27] S. Muirhead, A. Rivera and H. Vanneuville The phase transition for planar Gaussian percolation models without FKG, 2020. With an appendix by L. Köhler-Schindler. Preprint. Available at arXiv:2010.11770.

[28] S. Muirhead and H. Vanneuville. The sharp phase transition for level set percolation of smooth planar Gaussian fields. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020) 1358–1390. MR4076787 https://doi.org/10.1214/19-AIHP1006

[29] F. Nazarov and M. Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *Math. Phys. Anal. Geom.* **12** (3) (2016) 205–278. MR3522141 https://doi.org/10.15407/mag12.03.205

[30] A. Papoulis. High density shot noise and Gaussianity. *J. Appl. Probab.* **8** (1971) 118–127. MR0278412 https://doi.org/10.2307/3211842

[31] L. D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.* **10** (1982) 496–499. MR0665603

[32] S. Popov and B. Ráth. On decoupling inequalities and percolation of the excursion sets of the Gaussian free field. *J. Stat. Phys.* **159** (2015) 312–320. MR3325312 https://doi.org/10.1007/s10955-015-1187-z

[33] P. Révész. On strong approximation of the multidimensional empirical process. *Ann. Probab.* **4** (1976) 729–743. MR0426116 https://doi.org/10.1214/aop/1176995981

[34] E. Rio. Local invariance principles and their application to density estimation. *Probab. Theory Related Fields* **98** (1994) 21–45. MR1254823 https://doi.org/10.1007/BF01311347

[35] E. Rio. Vitesse de convergence dans le principe d'invariance faible pour la fonction de répartition empirique multivariée. *C. R. Acad. Sci., Sér. 1 Math.* **322** (1996) 169–172. MR1373756

[36] A. Rivera and H. Vanneuville. Quasi-independence for nodal lines. *Ann. Henri Poincaré* **55** (2019) 1679–1711. MR4010948 https://doi.org/10.1214/18-aihp931

[37] A. Rivera and H. Vanneuville. The critical threshold for Bargmann–Fock percolation. *Ann. Henri Lebesgue* **3** (2020) 169–215. MR4060853 https://doi.org/10.5802/ahl.29

[38] F. Severo Sharp phase transition for Gaussian percolation in all dimensions, 2021. Preprint. Available at arXiv:2105.05219.

[39] G. Tusnády. A remark on approximation of the sample DF in the multidimensional case. *Period. Math. Hungar.* **8** (1977) 53–55. MR0443045 https://doi.org/10.1007/BF02018047

[40] A. Y. Zaitsev. Estimates of the Lévy–Prokhorov distance in the multivariate central limit theorem for random variables with finite exponential moments. *Theory Probab. Appl.* **31** (1987) 203–220. MR0850986

[41] A. Y. Zaitsev. On the Gaussian approximation of convolutions under multidimensional analogues of S.N. Bernstein's inequality conditions. *Z. Wahrsch. Verw. Gebiete* **74** (1987) 535–566. MR0876255 https://doi.org/10.1007/BF00363515

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Quasi-geometric rough paths and rough change of variable formula

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Abstract. Using some basic notions from the theory of Hopf algebras and quasi-shuffle algebras, we introduce rigorously a new family of rough paths: the quasi-geometric rough paths. We discuss their main properties. In particular, we will relate them with iterated Brownian integrals and the concept of "simple bracket extension", developed in the PhD thesis of David Kelly. As consequence of these results, we have a sufficient criterion to show for any $\gamma \in (0,1)$ and any sufficiently smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ a rough change of variable formula on any γ-Hölder continuous path $x\colon [0,T] \to \mathbb{R}^d$, i.e. an explicit expression of $\varphi(x_t)$ in terms of rough integrals.

Résumé. En utilisant certaines notions de base de la théorie des algèbres de Hopf et des algèbres de quasi-battage, nous introduisons formellement une nouvelle famille de chemins rugueux : les chemins rugueux quasi géométriques. Nous en examinons les propriétés principales. En particulier, nous les mettons en relation avec les intégrales itérées du mouvement brownien et avec le concept de « simple bracket extension », développé dans la thèse de David Kelly. Comme conséquence de ces résultats, nous disposons d'un critère suffisant pour montrer pour toute $\gamma \in (0,1)$ et toute fonction suffisamment lisse $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ une formule de changement de variable rugueuse pour tout chemin γ-Hölder $x\colon [0,T] \to \mathbb{R}^d$, c'est-à-dire une expression explicite de $\varphi(x_t)$ en termes d'intégrales rugueuses.

MSC2020 subject classifications: 60L20; 60L70

Keywords: Rough paths; Hopf algebras; Quasi-shuflle algebras; Change of variable formula

References

[1] G. B. Arous. Flots et series de Taylor stochastiques. *Probab. Theory Related Fields* **81** (1) (1989) 29–77. MR0981567 https://doi.org/10.1007/BF00343737
[2] C. Bellingeri, A. Djurdjevac, P. K. Friz and N. Tapia. Transport and continuity equations with (very) rough noise. *Part. Differ. Equ. Appl.* **2** (4) (2021) Paper No. 49, 26 pp. MR4338294 https://doi.org/10.1007/s42985-021-00101-y
[3] G. Bogfjellmo, R. Dahmen and A. Schmeding. Character groups of Hopf algebras as infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)* **66** (5) (2016) 2101–2155. MR3533279
[4] Y. Bruned, C. Curry and K. Ebrahimi-Fard. Quasi-shuffle algebras and renormalisation of rough differential equations. *Bull. Lond. Math. Soc.* **52** (1) (2020) 43–63. MR4072031 https://doi.org/10.1112/blms.12305
[5] J. C. Butcher. An algebraic theory of integration methods. *Math. Comp.* **26** (1972) 79–106. MR0305608 https://doi.org/10.2307/2004720
[6] P. Cartier. Jacobiennes généralisées, monodromie unipotente et intégrales itérées. In *Séminaire Bourbaki: volume 1987/88, exposés 686–699. Astérisque*. Société mathématique de France, 1988. MR0992201
[7] K.-T. Chen. Iterated integrals and exponential homomorphisms. *Proc. Lond. Math. Soc. (3)* **4** (1954) 502–512. MR0073174 https://doi.org/10.1112/plms/s3-4.1.502
[8] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.* **199** (1) (1998) 203–242. MR1660199 https://doi.org/10.1007/s002200050499
[9] R. Cont and N. Perkowski. Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity. *Trans. Amer. Math. Soc., Ser. B* **6** (2019) 161–186. MR3937343 https://doi.org/10.1090/btran/34
[10] C. Curry, K. Ebrahimi-Fard, S. J. A. Malham and A. Wiese. Lévy processes and quasi-shuffle algebras. *Stochastics* **86** (4) (2014) 632–642. MR3230071 https://doi.org/10.1080/17442508.2013.865131
[11] C. Curry, K. Ebrahimi-Fard, D. Manchon and H. Z. Munthe-Kaas. Planarly branched rough paths and rough differential equations on homogeneous spaces. *J. Differential Equations* **269** (11) (2020) 9740–9782. MR4120383 https://doi.org/10.1016/j.jde.2020.06.058
[12] C. Curry, K. Ebrahimi–Fard, S. J. A. Malham and A. Wiese. Algebraic structures and stochastic differential equations driven by Lévy processes. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **475** (2221) (2019) 20180567. MR3917537 https://doi.org/10.1098/rspa.2018.0567
[13] K. Ebrahimi-Fard, S. J. A. Malham, F. Patras and A. Wiese. The exponential Lie series for continuous semimartingales. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **471** (2184) (2015) 20150429, 19 pp. MR3452149 https://doi.org/10.1098/rspa.2015.0429
[14] K. Ebrahimi-Fard, S. J. A. Malham, F. Patras and A. Wiese. Flows and stochastic Taylor series in Itô calculus. *J. Phys. A: Math. Theor.* **48** (49) (2015) 495202. MR3434827 https://doi.org/10.1088/1751-8113/48/49/495202

[15] F. Fauvet and F. Menous. Ecalle's arborification coarborification transforms and Connes–Kreimer Hopf algebra. *Ann. Sci. Éc. Norm. Supér. (4)* **50** (1) (2017) 39–83. MR3621426 https://doi.org/10.24033/asens.2315

[16] H. Föllmer. Calcul d'Ito sans probabilités. *Sémin. Probab. Strasbourg* **15** (1981) 143–150. MR0622559

[17] P. K. Friz and M. Hairer. *A Course on Rough Paths, with Introduction to Regularity Structures*, 2nd extended edition. Springer International Publishing, Cham, 2020. MR4174393 https://doi.org/10.1007/978-3-030-41556-3

[18] P. K. Friz and A. Shekhar. General rough integration, Lévy rough paths and a Lévy–Kintchine-type formula. *Ann. Probab.* **45** (4) (2017) 2707–2765. MR3693973 https://doi.org/10.1214/16-AOP1123

[19] J. Gaines. The algebra of iterated stochastic integrals. *Stoch. Stoch. Rep.* **49** (3–4) (1994) 169–179. MR1785003 https://doi.org/10.1080/17442509408833918

[20] L. G. Gyurkó. Differential equations driven by π-rough paths. *Proc. Edinb. Math. Soc.* **59** (3) (2016) 741–758. MR3572768 https://doi.org/10.1017/S0013091515000474

[21] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.* **216** (1) (2004) 86–140. MR2091358 https://doi.org/10.1016/j.jfa.2004.01.002

[22] M. Gubinelli. Ramification of rough paths. *J. Differential Equations* **248** (4) (2010) 693–721. MR2578445 https://doi.org/10.1016/j.jde.2009.11.015

[23] M. Hairer. A theory of regularity structures. *Invent. Math.* **198** (2) (2014) 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4

[24] M. Hairer and D. Kelly. Geometric versus non-geometric rough paths. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** (1) (2015) 207–251. MR3300969 https://doi.org/10.1214/13-AIHP564

[25] M. E. Hoffman. Quasi-shuffle products. *J. Algebraic Combin.* **11** (1) (2000) 49–68. MR1747062 https://doi.org/10.1023/A:1008791603281

[26] Y. Hu and P. A. Meyer. Sur les intégrales multiples de Stratonovitch. In *Séminaire de Probabilités, XXII*. *Lecture Notes in Mathematics* **1321** 72–81. Springer, Berlin, 1988. MR0960509 https://doi.org/10.1007/BFb0084119

[27] D. Kelly. Itô corrections in stochastic equations. PhD thesis, University of Warwick, 2012. MR3271810

[28] T. J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** (2) (1998) 215–310. MR1654527 https://doi.org/10.4171/RMI/240

[29] T. J. Lyons and N. Victoir. An extension theorem to rough paths. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24** (5) (2007) 835–847. MR2348055 https://doi.org/10.1016/j.anihpc.2006.07.004

[30] D. Manchon. Hopf algebras in renormalisation. In *Handbook of Algebra* **5**. North-Holland, Amsterdam, 2008. MR2523455 https://doi.org/10.1016/S1570-7954(07)05007-3

[31] I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*. *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012. MR2962301 https://doi.org/10.1017/CBO9781139084659

[32] C. Reutenauer. *Free Lie Algebras*. *LMS Monographs*. Clarendon Press, New York, 1993. MR1231799

[33] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, 2004. MR1083357 https://doi.org/10.1007/978-3-662-21726-9

[34] G.-C. Rota and T. C. Wallstrom. Stochastic integrals: A combinatorial approach. *Ann. Probab.* **25** (3) (1997) 1257–1283. MR1457619 https://doi.org/10.1214/aop/1024404513

[35] N. Tapia and L. Zambotti. The geometry of the space of branched rough paths. *Proc. Lond. Math. Soc. (3)* **121** (2) (2020) 220–251. MR4093955 https://doi.org/10.1112/plms.12311

[36] W. H. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** (1) (1936) 251–282. MR1555421 https://doi.org/10.1007/BF02401743

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Convergence of trapezoid rule to rough integrals

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Abstract. Rough paths techniques give the ability to define solutions of stochastic differential equations driven by signals X which are not semimartingales and whose p-variation is finite only for large values of p. In this context, rough integrals are usually Riemann–Stieltjes integrals with correction terms that are sometimes seen as unnatural. As opposed to those somewhat artificial correction terms, our endeavor in this note is to produce a trapezoid rule for rough integrals driven by general d-dimensional Gaussian processes. Namely we shall approximate a generic rough integral $\int y\,dX$ by Riemann sums avoiding the usual higher order correction terms, making the expression easier to work with and more natural. Our approximations apply to all controlled processes y and to a wide range of Gaussian processes X including fractional Brownian motion with a Hurst parameter $H > 1/4$. As a corollary of the trapezoid rule, we also consider the convergence of a midpoint rule for integrals of the form $\int f(X)\,dX$.

Résumé. La théorie des trajectoires rugueuses ouvre la possibilité de résoudre des équations différentielles stochastiques dirigées par un signal général X. Cette théorie va au-delà du cas d'une semi-martingale, et concerne un signal X dont la p-variation est finie pour p arbitrairement grand. Dans ce contexte les intégrales rugueuses sont généralement définies comme des intégrales de Riemann–Stieljes, corrigées par des termes qui peuvent paraître non naturels. Dans la présente note nous proposons de remplacer ces termes quelque peu artificiels par une approximation trapézoïdale, dans le cas où X est un processus Gaussien d-dimensionnel général. Plus précisément, nous approchons une intégrale rugueuse de la forme $\int y\,dX$ par une somme de Riemann tout en évitant les termes correctifs d'ordre supérieur, ce qui rend l'expression plus facile à manipuler et plus naturelle. Nos approximations fonctionnent pour tous les processus contrôlés y, ainsi que pour une classe importante de processus Gaussiens X comprenant le mouvement brownien fractionnaire avec paramètre de Hurst $H > 1/4$. Nous énonçons aussi un résultat de convergence concernant la règle du point milieu pour des intégrales de la forme $\int f(X)\,dX$. Ce dernier résultat est un corollaire de notre règle trapézoïdale.

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Keywords: Rough paths; Weighted random sums; Limit theorems; Malliavin calculus

References

[1] G. Binotto, I. Nourdin and D. Nualart. Weak symmetric integrals with respect to the fractional Brownian motion. *Ann. Probab.* **46** (4) (2018) 2243–2267. MR3813991 https://doi.org/10.1214/17-AOP1227
[2] K. Chen. Iterated integrals and exponential homomorphisms. *Proc. Lond. Math. Soc.* **3** (4) (1954) 502–512. MR0073174 https://doi.org/10.1112/plms/s3-4.1.502
[3] J. Diehl, H. Oberhauser and S. Riedel. A Lévy area between Brownian motion and rough paths with applications to robust nonlinear filtering and rough partial differential equations. *Stochastic Process. Appl.* **125** (1) (2015) 161–181. MR3274695 https://doi.org/10.1016/j.spa.2014.08.005
[4] P. Friz, B. Gess, S. Archil Gulisashvili and S. Riedel. The Jain–Monrad criterion for rough paths and applications to random Fourier series and non-Markovian Hörmander theory. *Ann. Probab.* **44** (1) (2016) 684–738. MR3456349 https://doi.org/10.1214/14-AOP986
[5] P. Friz and M. Hairer. *A Course on Rough Paths. With an Introduction to Regularity Structures*. *Universitext*. Springer, Cham, 2014. MR3289027 https://doi.org/10.1007/978-3-319-08332-2
[6] P. Friz and N. Victoir. *Multidimensional Stochastic Processes as Rough Paths Theory and Applications*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge University Press, Cambridge, 2010. MR2604669 https://doi.org/10.1017/CBO9780511845079
[7] P. Friz and N. Victoir. A note on higher dimensional p-variation. *Electron. J. Probab.* **16** (68) (2011) 1880–1899. MR2842090 https://doi.org/10.1214/EJP.v16-951
[8] X. Geng. Private communication, 2021.
[9] B. Gess, C. Ouyang and S. Tindel. Density bounds for solutions to differential equations driven by Gaussian rough paths. *J. Theoret. Probab.* **33** (2020) 611–648. MR4091589 https://doi.org/10.1007/s10959-019-00967-0

[10] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois. m-Order integrals and generalized Itô's formula: The case of a fractional Brownian motion with any Hurst index. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** (4) (2005) 781–806. MR2144234 https://doi.org/10.1016/j.anihpb.2004.06.002
[11] B. Graham Sparse arrays of signatures for online character recognition. Preprint, 2013.
[12] D. Harnett and D. Nualart. Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stochastic Process. Appl.* **122** (10) (2012) 3460–3505. MR2956113 https://doi.org/10.1016/j.spa.2012.06.008
[13] C. Houdré and J. Villa. An example of infinite dimensional quasi-helix. In *Stochastic Models (Mexico City, 2002)* 195–201. *Contemporary Mathematics* **336**. Amer. Math. Soc., Providence, RI, 2003.
[14] Y. Hu, Y. Liu and D. Nualart. Rate of convergence and asymptotic error distribution of Euler approximation schemes for fractional diffusions. *Ann. Appl. Probab.* **26** (2) (2016) 1147–1207. MR3476635 https://doi.org/10.1214/15-AAP1114
[15] Y. Hu, Y. Liu and D. Nualart. Taylor schemes for rough differential equations and fractional diffusions. *Discrete Contin. Dyn. Syst. Ser. B* **21** (9) (2016) 3115–3162. MR3567804 https://doi.org/10.3934/dcdsb.2016090
[16] Y. Hu, Y. Liu and D. Nualart. Crank–Nicolson scheme for stochastic differential equations driven by fractional Brownian motions. *Ann. Appl. Probab.* **31** (2021) 39–83. MR4254473 https://doi.org/10.1214/20-aap1582
[17] M. Ledoux, Z. Qian and T. Zhang. Large deviations and support theorem for diffusion processes via rough paths. *Stochastic Process. Appl.* **102** (2) (2002) 265–283. MR1935127 https://doi.org/10.1016/S0304-4149(02)00176-X
[18] Y. Liu and S. Tindel. First-order Euler scheme for SDEs driven by fractional Brownian motions: The rough case. *Ann. Appl. Probab.* **29** (2) (2019) 758–826. MR3910017 https://doi.org/10.1214/17-AAP1374
[19] Y. Liu and S. Tindel. Discrete rough paths and limit theorems. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020) 1730–1774. MR4116706 https://doi.org/10.1214/19-AIHP1015
[20] T. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** (2) (1998) 215–310. MR1654527 https://doi.org/10.4171/RMI/240
[21] A. Neuenkirch and I. Nourdin. Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. *J. Theoret. Probab.* **20** (4) (2007) 871–899. MR2359060 https://doi.org/10.1007/s10959-007-0083-0
[22] I. Nourdin and A. Réveillac. Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: The critical case $H = 1/4$. *Ann. Probab.* **37** (6) (2008) 2200–2230. MR2573556 https://doi.org/10.1214/09-AOP473
[23] I. Nourdin, A. Réveillac and J. Swanson. The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter. *Electron. J. Probab.* **15** (70) (2010) 2117–2162. MR2745728 https://doi.org/10.1214/EJP.v15-843
[24] D. Nualart. *The Malliavin Calculus and Related Topics*, 2nd edition. Springer, Berlin, 2006. MR2200233
[25] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* **97** (3) (1993) 403–421. MR1245252 https://doi.org/10.1007/BF01195073
[26] N. Towghi. Multidimensional extension of L. C. Young's inequality. *J. Inequal. Pure Appl. Math.* **3** (2) (2002) Article 22, 13 pp. (electronic). MR1906391
[27] L. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** (1) (1936) 251–282. MR1555421 https://doi.org/10.1007/BF02401743

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Non compact estimation of the conditional density from direct or noisy data

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Abstract. In this paper, we propose a nonparametric estimation method for the conditional density function of Y given X, from independent and identically distributed observations $(X_i, Y_i)_{1\le i\le n}$. We consider a regression strategy related to projection subspaces of $\mathbb{L}^2$ generated by non compactly supported bases. This first study is then extended to the case where Y is not directly observed, but only $Z = Y + \varepsilon$, where ε is a noise with known density. In these two settings, we build and study collections of estimators, compute their rates of convergence on anisotropic space on non-compact supports, and prove related lower bounds. Then, we consider adaptive estimators for which we also prove risk bounds.

Résumé. Dans cet article, nous proposons une méthode d'estimation non-paramétrique de la densité conditionnelle de Y sachant X, à partir d'un échantillon d'observations $(X_i, Y_i)_{1\le i\le n}$, indépendantes et identiquement distribuées. Notre stratégie s'appuie sur un contraste de régression pour reconstruire une projection de la fonction cible sur des sous-espaces de $\mathbb{L}^2$ engendrés par des bases à support non compact. Cette étude est ensuite étendue au cas où la variable Y n'est pas directement observée, mais remplacée par $Z = Y + \varepsilon$ où ε est un bruit de densité connue. Dans ces deux contextes, nous construisons et étudions des collections d'estimateurs, calculons leurs vitesses de convergence sur des espaces anisotropiques de fonctions à support non compact. Des bornes inférieures associées sont également prouvées. Enfin, nous proposons des estimateurs adaptatifs, pour lesquels nous démontrons des bornes de risque de type oracle.

MSC2020 subject classifications: 62G05; 62G07; 62G08

Keywords: Anisotropic Sobolev spaces; Conditional density; Deconvolution; Hermite basis; Laguerre basis; Noisy data; Nonparametric estimation

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. *National Bureau of Standards Applied Mathematics Series* **55**. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. MR0167642

[2] N. Akakpo and C. Lacour. Inhomogeneous and anisotropic conditional density estimation from dependent data. *Electron. J. Stat.* **5** (2011) 1618–1653. MR2870146 https://doi.org/10.1214/11-EJS653

[3] R. Askey and S. Wainger. Mean convergence of expansions in Laguerre and Hermite series. *Amer. J. Math.* **87** (1965) 695–708. MR0182834 https://doi.org/10.2307/2373069

[4] D. Belomestny, F. Comte and V. Genon-Catalot. Sobolev–Hermite versus Sobolev nonparametric density estimation on $\mathbb{R}$. *Ann. Inst. Statist. Math.* **71** (1) (2019) 29–62. MR3898425 https://doi.org/10.1007/s10463-017-0624-y

[5] K. Bertin, C. Lacour and V. Rivoirard. Adaptive pointwise estimation of conditional density function. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2) (2016) 939–980. MR3498017 https://doi.org/10.1214/14-AIHP665

[6] L. Birgé and P. Massart. Minimum contrast estimators on sieves: Exponential bounds and rates of convergence. *Bernoulli* **4** (3) (1998) 329–375. MR1653272 https://doi.org/10.2307/3318720

[7] E. Brunel, F. Comte and C. Lacour. Adaptive estimation of the conditional density in presence of censoring. *Sankhyā* **69** (4) (2007) 734–763. MR2521231

[8] G. Chagny. Warped bases for conditional density estimation. *Math. Methods Statist.* **22** (4) (2013) 253–282. MR3146596 https://doi.org/10.3103/S1066530713040017

[9] A. Cohen, M. A. Davenport and D. Leviatan. On the stability and accuracy of least squares approximations. *Found. Comput. Math.* **13** (5) (2013) 819–834. MR3105946 https://doi.org/10.1007/s10208-013-9142-3

[10] S. X. Cohen and E. Le Pennec Partition-based conditional density estimation. *ESAIM Probab. Stat.* **17** (2013) 672–697. MR3126157 https://doi.org/10.1051/ps/2012017

[11] F. Comte, C.-A. Cuenod, M. Pensky and Y. Rozenholc. Laplace deconvolution on the basis of time domain data and its application to dynamic contrast-enhanced imaging. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** (1) (2017) 69–94. MR3597965 https://doi.org/10.1111/rssb.12159
[12] F. Comte and V. Genon-Catalot. Laguerre and Hermite bases for inverse problems. *J. Korean Statist. Soc.* **47** (3) (2018) 273–296. MR3840862 https://doi.org/10.1016/j.jkss.2018.03.001
[13] F. Comte and V. Genon-Catalot. Regression function estimation as a partly inverse problem. *Ann. Inst. Statist. Math.* **72** (4) (2020) 1023–1054. MR4113272 https://doi.org/10.1007/s10463-019-00718-2
[14] F. Comte and C. Lacour. Anisotropic adaptive kernel deconvolution. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2) (2013) 569–609. MR3088382 https://doi.org/10.1214/11-aihp470
[15] F. Comte and C. Lacour. Non compact estimation of the conditional density from direct or noisy data, 2021. Available at https://hal.archives-ouvertes.fr/hal-03276251.
[16] F. Comte and G. Mabon. Laguerre deconvolution with unknown matrix operator. *Math. Methods Statist.* **26** (4) (2017) 237–266. MR3740524 https://doi.org/10.3103/S1066530717040019
[17] J. G. De Gooijer and D. Zerom. On conditional density estimation. *Stat. Neerl.* **57** (2) (2003) 159–176. MR2028911 https://doi.org/10.1111/1467-9574.00226
[18] F. Dussap. Anisotropic multivariate deconvolution using projection on the Laguerre basis. *J. Statist. Plann. Inference* **215** (2021) 23–46. MR4230544 https://doi.org/10.1016/j.jspi.2021.02.005
[19] F. Dussap Nonparametric multiple regression by projection on non-compactly supported bases, 2022. Preprint hal-03506635.
[20] S. Efromovich. Conditional density estimation in a regression setting. *Ann. Statist.* **35** (6) (2007) 2504–2535. MR2382656 https://doi.org/10.1214/009053607000000253
[21] S. Efromovich. Oracle inequality for conditional density estimation and an actuarial example. *Ann. Inst. Statist. Math.* **62** (2) (2010) 249–275. MR2592098 https://doi.org/10.1007/s10463-008-0185-1
[22] J.-Q. Fan, L. Peng, Q.-W. Yao and W.-Y. Zhang. Approximating conditional density functions using dimension reduction. *Acta Math. Appl. Sin. Engl. Ser.* **25** (3) (2009) 445–456. MR2506985 https://doi.org/10.1007/s10255-008-8815-1
[23] J. Fan, Q. Yao and H. Tong. Estimation of conditional densities and sensitivity measures in nonlinear dynamical systems. *Biometrika* **83** (1) (1996) 189–206. MR1399164 https://doi.org/10.1093/biomet/83.1.189
[24] J. Fan and T. H. Yim. A crossvalidation method for estimating conditional densities. *Biometrika* **91** (4) (2004) 819–834. MR2126035 https://doi.org/10.1093/biomet/91.4.819
[25] F. Ferraty, A. Laksaci and P. Vieu. Estimating some characteristics of the conditional distribution in nonparametric functional models. *Stat. Inference Stoch. Process.* **9** (1) (2006) 47–76. MR2224840 https://doi.org/10.1007/s11203-004-3561-3
[26] A. Goldenshluger and O. Lepski. Bandwidth selection in kernel density estimation: Oracle inequalities and adaptive minimax optimality. *Ann. Statist.* **39** (3) (2011) 1608–1632. MR2850214 https://doi.org/10.1214/11-AOS883
[27] A. Goldenshluger and O. Lepski. On adaptive minimax density estimation on $\mathbb{R}^d$. *Probab. Theory Related Fields* **159** (3–4) (2014) 479–543. MR3230001 https://doi.org/10.1007/s00440-013-0512-1
[28] L. Györfi and M. Kohler. Nonparametric estimation of conditional distributions. *IEEE Trans. Inf. Theory* **53** (5) (2007) 1872–1879. MR2317148 https://doi.org/10.1109/TIT.2007.894631
[29] M. P. Holmes, A. G. Gray and C. L. Isbell Jr. Fast kernel conditional density estimation: A dual-tree Monte Carlo approach. *Comput. Statist. Data Anal.* **54** (7) (2010) 1707–1718. MR2608967 https://doi.org/10.1016/j.csda.2010.01.011
[30] R. J. Hyndman and Q. Yao. Nonparametric estimation and symmetry tests for conditional density functions. *J. Nonparametr. Stat.* **14** (3) (2002) 259–278. MR1905751 https://doi.org/10.1080/10485250212374
[31] D. A. Ioannides. Estimating the conditional mode of a stationary stochastic process from noisy observations. *Metrika* **50** (1) (1999) 19–35. MR1749580 https://doi.org/10.1007/s001840050033
[32] R. Izbicki and A. B. Lee. Nonparametric conditional density estimation in a high-dimensional regression setting. *J. Comput. Graph. Statist.* **25** (4) (2016) 1297–1316. MR3572041 https://doi.org/10.1080/10618600.2015.1094393
[33] R. Izbicki and A. B. Lee. Converting high-dimensional regression to high-dimensional conditional density estimation. *Electron. J. Stat.* **11** (2) (2017) 2800–2831. MR3679910 https://doi.org/10.1214/17-EJS1302
[34] A. Juditsky and S. Lambert-Lacroix. On minimax density estimation on $\mathbb{R}$. *Bernoulli* **10** (2) (2004) 187–220. MR2046772 https://doi.org/10.3150/bj/1082380217
[35] J. Kappus and G. Mabon. Adaptive density estimation in deconvolution problems with unknown error distribution. *Electron. J. Stat.* **8** (2) (2014) 2879–2904. MR3299125 https://doi.org/10.1214/14-EJS976
[36] T. Klein and E. Rio. Concentration around the mean for maxima of empirical processes. *Ann. Probab.* **33** (3) (2005) 1060–1077. MR2135312 https://doi.org/10.1214/009117905000000044
[37] A. Laksaci. Convergence en moyenne quadratique de l'estimateur à noyau de la densité conditionnelle avec variable explicative fonctionnelle. *Ann. I.S.U.P.* **51** (3) (2007) 69–80. MR2391425
[38] E. Lukacs. *Characteristic Functions*, 2nd edition, revised and enlarged. Hafner Publishing Co., New York, 1970. MR0346874
[39] G. Mabon. Adaptive deconvolution on the non-negative real line. *Scand. J. Stat.* **44** (3) (2017) 707–740. MR3687970 https://doi.org/10.1111/sjos.12272
[40] S. Mallat. *A Wavelet Tour of Signal Processing*, 3rd edition. Elsevier/Academic Press, Amsterdam, 2009. MR2479996
[41] C. Markett. Norm estimates for (C,δ) means of Hermite expansions and bounds for δ_{eff}. *Acta Math. Hungar.* **43** (3–4) (1984) 187–198. MR0733853 https://doi.org/10.1007/BF01958018
[42] A. Meister. *Deconvolution Problems in Nonparametric Statistics*, **193**. Springer Science & Business Media, 2009. MR2768576 https://doi.org/10.1007/978-3-540-87557-4
[43] M.-L. J. Nguyen. Nonparametric method for sparse conditional density estimation in moderately large dimensions, 2018. Available at https://hal.archives-ouvertes.fr/hal-01688664.
[44] M.-L. J. Nguyen, C. Lacour and V. Rivoirard. Adaptive greedy algorithm for moderately large dimensions in kernel conditional density estimation, 2021. Available at https://hal.archives-ouvertes.fr/hal-02085677.
[45] H. K. Otneim and D. Tjostheim. Conditional density estimation using the local Gaussian correlation. *Stat. Comput.* **28** (2) (2018) 303–321. MR3747565 https://doi.org/10.1007/s11222-017-9732-z

[46] P. Reynaud-Bouret, V. Rivoirard and C. Tuleau-Malot. Adaptive density estimation: A curse of support? *J. Statist. Plann. Inference* **141** (1) (2011) 115–139. MR2719482 https://doi.org/10.1016/j.jspi.2010.05.017

[47] M. Rosenblatt. Conditional probability density and regression estimators. In *Multivariate Analysis* 25–31. *II (Proc. Second Internat. Sympos., Dayton, Ohio, 1968)*. Academic Press, New York, 1969. MR0254987

[48] O. Sacko. Hermite density deconvolution. *ALEA Lat. Am. J. Probab. Math. Stat.* **17** (1) (2020) 419–443. MR4105925 https://doi.org/10.30757/alea.v17-17

[49] M. Sart Estimating the conditional density by histogram type estimators and model selection. *ESAIM Probab. Stat.* **21** (2017) 34–55. MR3630602 https://doi.org/10.1051/ps/2016026

[50] J.A. Tropp. User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.* **12** (4) (2012) 389–434. MR2946459 https://doi.org/10.1007/s10208-011-9099-z

[51] A. B. Tsybakov. *Introduction to Nonparametric Estimation*. *Springer Series in Statistics*. Springer, New York, 2009. MR2724359 https://doi.org/10.1007/b13794

[52] É. Youndjé. Propriétés de convergence de l'estimateur à noyau de la densité conditionnelle. *Rev. Roumaine Math. Pures Appl.* **41** (7–8) (1996) 535–566. MR1652527

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Inference via randomized test statistics

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Abstract. We show that external randomization may enforce the convergence of test statistics to their limiting distributions in particular cases. This results in a sharper inference. Our approach is based on a central limit theorem for weighted sums. We apply our method to a family of rank-based test statistics and a family of phi-divergence test statistics and prove that, with overwhelming probability with respect to the external randomization, the randomized statistics converge at the rate $O(1/n)$ (up to some logarithmic factors) to the limiting chi-square distribution in Kolmogorov metric.

Résumé. Nous montrons que l'ajout de randomisation externe peut guider la convergence des statistiques de test vers leurs lois limites dans certains cas particuliers. Il en résulte une inférence plus précise. Notre approche est basée sur un théorème central limite pour les sommes pondérées. Nous appliquons notre méthode à une famille de statistiques de test basées sur les rangs et à une famille de statistiques de test de phi-divergence et nous prouvons que, avec grande probabilité par rapport à la randomisation externe, les statistiques randomisées convergent à un taux $O(1/n)$ (en négligeant certains facteurs logarithmiques) vers la distribution limite du chi-deux dans la distance de Kolomogorov.

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Keywords: Phi-divergence test statistics; Power divergence test statistics; Rank-based test statistics; Central limit theorem; Weighted sums

References

[1] Z. Assylbekov. Convergence rate of multinomial goodness-of-fit statistics to chi-square distribution. *Hiroshima Math. J.* **40** (2010) 115–131. MR2642973
[2] Z. Assylbekov, V. Zubov and V. Ulyanov. On approximating some statistics of goodness-of-fit tests in the case of three-dimensional discrete data. *Sib. Math. J.* **52** (2011) 571–584. MR2883210 https://doi.org/10.1134/S0037446611040021
[3] S. Ayvazyan and V. Ulyanov. A multivariate CLT for "typical" weighted sums with rate of convergence of order O(1/n). In *Foundations of Modern Statistics – Festschrift in Honor of Vladimir Spokoiny*. *Springer Proceedings in Mathematics & Statistics*. Springer, Cham, 2022. In print. Available at arXiv:2112.05815.
[4] S. Bobkov. Edgeworth corrections in randomized central limit theorems. In *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 2017–2019 Volume I* 71–97, 2020. MR4175744 https://doi.org/10.1007/978-3-030-36020-7_5
[5] S. Bobkov, G. Chistyakov and F. Götze. Berry–Esseen bounds for typical weighted sums. *Electron. J. Probab.* **23** (2018) 22, Paper No. 92. MR3858920 https://doi.org/10.1214/18-EJP195
[6] S. Bobkov, G. Chistyakov and F. Götze. Normal approximation for weighted sums under a second-order correlation condition. *Ann. Probab.* **48** (2020) 1202–1219. MR4112712 https://doi.org/10.1214/19-AOP1388
[7] S. Bobkov, G. Chistyakov and F. Götze. Poincaré inequalities and normal approximation for weighted sums. *Electron. J. Probab.* **25** (2020) 1–31. MR4193896 https://doi.org/10.1214/20-ejp549
[8] G. Brown and A. Mood. On median tests for linear hypotheses. In *Berkeley Symposium on Mathematical Statistics and Probability* 159–166, 1951.
[9] N. Cressie and T. Read. Multinomial goodness-of-fit tests. *J. Roy. Statist. Soc. Ser. B* **46** (1984) 440–464. MR0790631
[10] C.-G. Esseen. Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law. *Acta Math.* **77** (1945) 1–125. MR0014626 https://doi.org/10.1007/BF02392223
[11] M. Freeman and J. Tukey. Transformations related to the angular and the square root. *Ann. Math. Stat.* **21** (1950) 607–611. MR0038028 https://doi.org/10.1214/aoms/1177729756
[12] M. Friedman. The use of ranks to avoid the assumption of normality implicit in the analysis of variance. *J. Amer. Statist. Assoc.* **32** (1937) 675–701.
[13] R. E. Gaunt. Stein's method for functions of multivariate normal random variables. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020) 1484–1513. MR4076792 https://doi.org/10.1214/19-AIHP1011

[14] R. E. Gaunt Bounds for the chi-square approximation of the power divergence family of statistics, 2021. Preprint. Available at arXiv:2107.00535.

[15] R. E. Gaunt, A. M. Pickett and G. Reinert. Chi-square approximation by Stein's method with application to Pearson's statistic. *Ann. Appl. Probab.* **27** (2017) 720–756. MR3655852 https://doi.org/10.1214/16-AAP1213

[16] R. E. Gaunt and G. Reinert Bounds for the chi-square approximation of Friedman's statistic by Stein's method, 2021. Preprint. Available at arXiv:2111.00949.

[17] F. Götze. Lattice point problems and values of quadratic forms. *Invent. Math.* **157** (2004) 195–226. MR2135188 https://doi.org/10.1007/s00222-004-0366-3

[18] F. Götze, A. Naumov, V. Spokoiny and V. Ulyanov. Large ball probabilities, Gaussian comparison and anti-concentration. *Bernoulli* **25** (2019) 2538–2563. MR4003557 https://doi.org/10.3150/18-BEJ1062

[19] F. Götze and V. Ulyanov. Asymptotic Distribution of χ^2-type Statistics. Preprint 03-033, Research group Spectral analysis, asymptotic distributions and stochastic dynamics. Bielefeld Univ., Bielefeld, 2003.

[20] F. Götze and A. Zaitsev. Explicit rates of approximation in the CLT for quadratic forms. *Ann. Probab.* **42** (2014) 354–397. MR3161488 https://doi.org/10.1214/13-AOP839

[21] D. Jensen. On approximating the distributions of Friedman's χ_r^2 and related statistics. *Metrika* **24** (1977) 75–86. MR0443173 https://doi.org/10.1007/BF01893394

[22] B. Klartag and S. Sodin. Variations on the Berry–Esseen theorem. *Theory Probab. Appl.* **56** (2012) 403–419. MR3136463 https://doi.org/10.4213/tvp4405

[23] L. Pardo. *Statistical Inference Based on Divergence Measures*. Chapman and Hall/CRC, New York, 2005. MR2183173

[24] K. Pearson. On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **50** (1900) 157–175.

[25] P. K. Sen. Asymptotically efficient tests by the method of n rankings. *J. Roy. Statist. Soc. Ser. B* **30** (1968) 312–317. MR0235680

[26] V. Ulyanov and V. Zubov. Refinement on the convergence of one family of goodness-of-fit statistics to chi-squared distribution. *Hiroshima Math. J.* **39** (2009) 133–161. MR2499200

[27] J. Yarnold. Asymptotic approximations for the probability that a sum of lattice random vectors lies in a convex set. *Ann. Math. Stat.* **43** (1972) 1566–1580. MR0372967 https://doi.org/10.1214/aoms/1177692389

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McKean SDEs with singular coefficients

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Abstract. The paper investigates existence and uniqueness for a stochastic differential equation (SDE) depending on the law density of the solution, involving a Schwartz distribution. Those equations, known as McKean SDEs, are interpreted in the sense of a suitable singular martingale problem. A key tool used in the analysis is the corresponding Fokker–Planck equation.

Résumé. Cet article explore existence et unicité pour une équation différentielle stochastique (EDS) dépendant de la loi de la solution, dont un coefficient contient une distribution de Schwartz. Ces équations sont connues sous le nom d'EDS de type McKean et sont interprétées à l'aide d'un problème de martingales approprié. Un outil fondamental de l'analyse est l'équation de Fokker–Planck correspondante.

MSC2020 subject classifications: 60H10; 60H30; 35C99; 35D99; 35K10

Keywords: Stochastic differential equations; Distributional drift; McKean; Martingale problem

References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, Heidelberg, 2011. MR2768550 https://doi.org/10.1007/978-3-642-16830-7
[2] V. Barbu and M. Röckner. Probabilistic representation for solutions to nonlinear Fokker–Planck equations. *SIAM J. Math. Anal.* **50** (2018) 4246–4260. MR3835244 https://doi.org/10.1137/17M1162780
[3] R. F. Bass and Z. Q. Chen. Brownian motion with singular drift. *Ann. Probab.* **31** (2003) 791–817. MR1964949 https://doi.org/10.1214/aop/1048516536
[4] J. M. Bony. Calcul symbolique et propagation des singularites pour les équations aux dérivées partielles non linéaires. *Ann. Sci. Éc. Norm. Supér.* **14** (1981) 209–246. MR0631751
[5] G. Cannizzaro and K. Chouk. Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.* **46** (2018) 1710–1763. MR3785598 https://doi.org/10.1214/17-AOP1213
[6] R. Carmona and F. Delarue. Probabilistic theory of mean field games with applications I. In *Probability Theory and Stochastic Modelling*, **84**. Springer International Publishing, Cham, 2018. MR3753660
[7] R. Carmona and F. Delarue. Probabilistic theory of mean field games with applications II. In *Probability Theory and Stochastic Modelling*, **84**. Springer International Publishing, Cham, 2018. MR3753660
[8] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *Ann. Probab.* **46** (2018) 2621–2679. MR3846835 https://doi.org/10.1214/17-AOP1235
[9] P. E. Chaudru de Raynal. Strong well-posedness of McKean–Vlasov stochastic differential equations with Hölder drift. *Stochastic Process. Appl.* **130** (2020) 79–107. MR4035024 https://doi.org/10.1016/j.spa.2019.01.006
[10] N. Dunford and J. T. Schwartz. *Linear Operators. Part I. General Theory*. *Wiley Classics Library*. John Wiley & Sons Inc., New York, 1988. With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication. MR1009162
[11] F. Flandoli, E. Issoglio and F. Russo. Multidimensional SDEs with distributional coefficients. *Trans. Amer. Math. Soc.* **369** (2017) 1665–1688. MR3581216 https://doi.org/10.1090/tran/6729
[12] F. Flandoli, F. Russo and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.* **40** (2003) 493–542. MR1988703
[13] F. Flandoli, F. Russo and J. Wolf. Some SDEs with distributional drift. II. Lyons–Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stoch. Equ.* **12** (2004) 145–184. MR2065168 https://doi.org/10.1163/156939704323074700
[14] M. Gubinelli, P. Imkeller and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** (2015) 75 pp. MR3406823 https://doi.org/10.1017/fmp.2015.2
[15] X. Huang and F. Y. Wang. Distribution dependent SDEs with singular coefficients. *Stochastic Process. Appl.* **129** (2019) 4747–4770. MR4013879 https://doi.org/10.1016/j.spa.2018.12.012
[16] X. Huang and F. Y. Wang. McKean–Vlasov SDEs with drifts discontinuous under Wasserstein distance. *Discrete Contin. Dyn. Syst.* **41** (2021) 1667–1679. MR4211198 https://doi.org/10.3934/dcds.2020336

[17] E. Issoglio. Transport equations with fractal noise – Existence, uniqueness and regularity of the solution. *J. Anal. Appl.* **32** (2013) 37–53. MR3061360 https://doi.org/10.4171/ZAA/1473

[18] E. Issoglio. A non-linear parabolic PDE with a distributional coefficient and its applications to stochastic analysis. *J. Differential Equations* **267** (2019) 5976–6003. MR3996791 https://doi.org/10.1016/j.jde.2019.06.014

[19] E. Issoglio and S. Jing. Forward–backward SDEs with distributional coefficients. *Stochastic Process. Appl.* **130** (2020) 47–78. MR4035023 https://doi.org/10.1016/j.spa.2019.01.001

[20] E. Issoglio and F. Russo. A Feynman–Kac result via Markov BSDEs with generalised drivers. *Bernoulli* **26** (2020) 728–766. MR4036050 https://doi.org/10.3150/19-BEJ1150

[21] E. Issoglio and F. Russo. SDEs with singular coefficients: The martingale problem view and the stochastic dynamics view. ArXiv preprint (2022). https://doi.org/10.48550/arXiv.2208.10799

[22] L. Izydorczyk, N. Oudjane and F. Russo. McKean Feynman–Kac probabilistic representations of non-linear partial differential equations. In *Geometry and Invariance in Stochastic Dynamics* 187–212. S. Ugolini, M. Fuhrman, E. Mastrogiacomo, P. Morando and B. Rüdiger (Eds). Springer International Publishing, Cham, 2021. MR4432540

[23] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. Henri Poincaré Probab. Stat.* **34** (1998) 727–766. MR1653393 https://doi.org/10.1016/S0246-0203(99)80002-8

[24] A. Le Cavil, N. Oudjane and F. Russo. Probabilistic representation of a class of non-conservative nonlinear partial differential equations. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** (2016) 1189–1233. MR3582914 https://doi.org/10.30757/alea.v13-43

[25] A. Le Cavil, N. Oudjane and F. Russo. Particle system algorithm and chaos propagation related to a non-conservative McKean type stochastic differential equations. *Stoch. Partial Differ. Equ. Anal. Comput.* **5** (2017) 1–37. MR3607725 https://doi.org/10.1007/s40072-016-0079-9

[26] J. Lieber, N. Oudjane and F. Russo. On the well-posedness of a class of McKean Feynman–Kac equations. *Markov Process. Related Fields* **25** (2019) 821–862. MR4246017

[27] M. Röckner and X. Zhang. Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli* **27** (2021) 1131–1158. MR4255229 https://doi.org/10.3150/20-bej1268

[28] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.* **35** (2007) 2213–2262. MR2353387 https://doi.org/10.1214/009117906000001178

[29] H. Triebel. *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*. *EMS Tracts in Mathematics* **11**. European Mathematical Society, Zürich, 2010. MR2667814 https://doi.org/10.4171/085

[30] H. Ye, J. Gao and Y. Ding. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **328** (2007) 1075–1081. MR2290034 https://doi.org/10.1016/j.jmaa.2006.05.061

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Reflection of stochastic evolution equations in infinite dimensional domains

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Abstract. In this paper, we establish the existence and the uniqueness of solutions of stochastic evolution equations (SEEs) with reflection in an infinite dimensional ball. Our framework is sufficiently general to include e.g. the stochastic Navier–Stokes equations.

Résumé. Nous établissons dans cet article l'existence et l'unicité des équations d'évolution stochastique avec réflexion dans une boule de dimension infinie. Notre méthodes sont assez générales et s'appliquent par exemple aux équations de Navier Stokes.

MSC2020 subject classifications: Primary 60H15; secondary 60J60; 35R60

Keywords: Stochastic evolution equations; Stochastic evolution equations with reflection; Random measures; Sobolev embedding

References

[1] V. Barbu, G. Da Prato and L. Tubaro. A reflection type problem for the stochastic 2D Navier–Stokes equations with periodic conditions. *Electron. Commun. Probab.* **16** (2011) 304–313. MR2819654 https://doi.org/10.1214/ECP.v16-1633
[2] V. Barbu, G. Da Prato and L. Tubaro. The stochastic reflection problem in Hilbert spaces. *Comm. Partial Differential Equations* **37** (2) (2012) 352–367. MR2876835 https://doi.org/10.1080/03605302.2011.596878
[3] V. Barbu and T. Precupanu. *Convexity and Optimization in Banach Spaces*, 4th edition. *Springer Monographs in Mathematics*. Springer, Dordrecht, 2012. MR3025420 https://doi.org/10.1007/978-94-007-2247-7
[4] Z. Brzeźniak and G. Dhariwal. Stochastic constrained Navier–Stokes equations on $\mathbb{T}^2$. *J. Differential Equations* **285** (2021) 128–174. MR228405 https://doi.org/10.1016/j.jde.2021.02.058
[5] Z. Brzeźniak and B. Ferrario. A note on stochastic Navier–Stokes equations with not regular multiplicative noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **5** (1) (2017) 53–80. MR3607727 https://doi.org/10.1007/s40072-016-0081-2
[6] Z. Brzeźniak and B. Ferrario. Stationary solutions for stochastic damped Navier–Stokes equations in $\mathbb{R}^d$. *Indiana Univ. Math. J.* **68** (1) (2019) 105–138. MR3922040 https://doi.org/10.1512/iumj.2019.68.7551
[7] Z. Brzeźniak and E. Motyl. Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains. *J. Differ. Equ.* **254** (4) (2013) 1627–1685. MR3003287 https://doi.org/10.1016/j.jde.2012.10.009
[8] Z. Brzeźniak and S. Peszat. Maximal inequalities and exponential estimates for stochastic convolutions in Banach spaces. In *Stochastic Processes, Physics and Geometry: New Interplays, I (Leipzig, 1999)* 55–64. *CMS Conf. Proc.* **28**. Amer. Math. Soc., Providence, RI, 2000. MR1803378
[9] I. Chueshov and A. Millet. Stochastic 2D hydrodynamical type systems: Well posedness and large deviations. *Appl. Math. Optim.* **61** (3) (2010) 379–420. MR2609596 https://doi.org/10.1007/s00245-009-9091-z
[10] P. Constantin and F. Ramos. Inviscid limit for damped and driven incompressible Navier–Stokes equations in $\mathbb{R}^2$. *Comm. Math. Phys.* **275** (2) (2007) 529–551. MR2335784 https://doi.org/10.1007/s00220-007-0310-7
[11] R. C. Dalang, C. Mueller and L. Zambotti. Hitting properties of parabolic SPDE's with reflection. *Ann. Probab.* **34** (4) (2006) 1423–1450. MR2257651 https://doi.org/10.1214/009117905000000792
[12] A. Debussche and L. Zambotti. Conservative stochastic Cahn–Hilliard equation with reflection. *Ann. Probab.* **35** (5) (2007) 1706–1739. MR2349572 https://doi.org/10.1214/009117906000000773
[13] N. Dinculeanu. *Vector Measures*. *International Series of Monographs in Pure and Applied Mathematics* **95**. Pergamon Press, Oxford–New York–Toronto, Ont.; VEB Deutscher Verlag der Wissenschaften, Berlin, 1967. MR0206190
[14] C. Donati-Martin and E. Pardoux. White noise driven SPDEs with reflection. *Probab. Theory Related Fields* **95** (1993) 1–24. MR1207304 https://doi.org/10.1007/BF01197335
[15] C. Donati-Martin and E. Pardoux. EDPS réfléchies et calcul de Malliavin. *Bull. Sci. Math.* **121** (5) (1997) 405–422. MR1465815
[16] A. Friedman. *Stochastic Differential Equations and Applications. Vol. 1. Probability and Mathematical Statistics* **28**. Academic Press, New York–London, 1975. MR0494490

[17] T. Funaki and S. Olla. Fluctuations for $\nabla\phi$ interface model on a wall. *Stochastic Process. Appl.* **94** (1) (2001) 1–27. MR1835843 https://doi.org/10.1016/S0304-4149(00)00104-6
[18] G. Gallavotti. *Foundations of Fluid Dynamics*. *Texts and Monographs in Physics*. Springer-Verlag, Berlin, 2002. MR1872661 https://doi.org/10.1007/978-3-662-04670-8
[19] S. Geis An introduction to probability theory, II. Available at http://users.jyu.fi/~geiss/lectures/probability-2.pdf.
[20] U. G. Haussmann and E. Pardoux. Stochastic variational inequalities of parabolic type. *Appl. Math. Optim.* **20** (1989) 163–192. MR0998402 https://doi.org/10.1007/BF01447653
[21] S. Łojasiewicz. *An Introduction to the Theory of Real Functions*, 3rd edition. John Wiley & Sons, Ltd., Chichester, 1988. With contributions by M. Kosiek, W. Mlak and Z. Opial. Translated from the Polish by G. H. Lawden. Translation edited by A. V. Ferreira. A Wiley-Interscience Publication. MR0952856
[22] D. Nualart and E. Pardoux. White noise driven by quaslinear SPDEs with reflection. *Probab. Theory Related Fields* **93** (1992) 77–89. MR1172940 https://doi.org/10.1007/BF01195389
[23] E. Pardoux. Stochastic partial differential equations and filtering of diffusion processes. *Stochastics* **3** (1979) 127–167. MR0553909 https://doi.org/10.1080/17442507908833142
[24] W. Rudin. *Real and Complex Analysis*, 3rd edition. McGraw-Hill, Inc., New York, 1987. MR0924157
[25] B. Schmalfuss. Qualitative properties for the stochastic Navier–Stokes equations. *Nonlinear Anal.* **28** (9) (1997) 1545–1563. MR1431206 https://doi.org/10.1016/S0362-546X(96)00015-6
[26] R. Temam *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd edition. *CBMS-NSF Regional Conference Series in Applied Mathematics* **66**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995. MR1318914 https://doi.org/10.1137/1.9781611970050
[27] R. Temam. *Navier–Stokes Equations. Theory and Numerical Analysis*, Reprint of the 1984th edition. AMS Chelsea Publishing, Providence, RI, 2001. MR1846644 https://doi.org/10.1090/chel/343
[28] T. Xu and T. Zhang. White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles. *Stochastic Process. Appl.* **119** (10) (2009) 3453–3470. MR2568282 https://doi.org/10.1016/j.spa.2009.06.005
[29] L. Zambotti. A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel bridge. *J. Funct. Anal.* **180** (2001) 195–209. MR1814427 https://doi.org/10.1006/jfan.2000.3685
[30] T. Zhang. White noise driven SPDEs with reflection: Strong Feller properties and Harnack inequalities. *Potential Anal.* **33** (2) (2010) 137–151. MR2658979 https://doi.org/10.1007/s11118-009-9162-4

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Pathwise regularization of the stochastic heat equation with multiplicative noise through irregular perturbation

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Abstract. Existence and uniqueness of solutions to the stochastic heat equation with multiplicative spatial noise is studied. In the spirit of pathwise regularization by noise, we show that a perturbation by a sufficiently irregular continuous path establish wellposedness of such equations, even when the drift and diffusion coefficients are given as generalized functions or distributions. In addition we prove regularity of the averaged field associated to a Lévy fractional stable motion, and use this as an example of a perturbation regularizing the multiplicative stochastic heat equation.

Résumé. L'existence et l'unicité des solutions de l'équation de la chaleur stochastique avec bruit spatial multiplicatif sont étudiées. Dans l'esprit de la régularisation par le bruit trajectorielle des équations différentielles ordinaires, nous montrons qu'une perturbation additive par des fonctions suffisamment irrégulières permet d'obtenir existence et unicité, et ce même lorsque les coefficients sont des fonctions généralisées. De plus, nous étudions la régularité du champ moyennisé associé aux processus fractionnaires de Lévy stables. Nous utilisons de tels processus comme exemples de perturbations régularisantes pour l'équation de la chaleur stochastique multiplicative.

MSC2020 subject classifications: Primary 60H50; 60H15; secondary 60L20

Keywords: Pathwise regularization by noise; Stochastic heat equation; Generalized parabolic Anderson model; Fractional Lévy processes

References

[1] S. Athreya, O. Butkovsky, K. Lê and L. Mytnik. Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation, 2020. Available at arXiv:2011.13498 [math].
[2] H. Bahouri, J.-Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **343**. Springer, Heidelberg, 2011. MR2768550 https://doi.org/10.1007/978-3-642-16830-7
[3] H. Bahouri, J.-Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, Berlin, 2011. MR2768550 https://doi.org/10.1007/978-3-642-16830-7
[4] I. Bailleul, A. Debussche and M. Hofmanová. Quasilinear generalized parabolic Anderson model equation. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** (1) (2019) 40–63. MR3916262 https://doi.org/10.1007/s40072-018-0121-1
[5] C. Bellingeri, P. K. Friz and M. Gerencsér. *Singular Paths Spaces and Applications*, 2020.
[6] J. Benedikt, V. E. Bobkov, P. Girg, L. Kotrla and P. Takac. Nonuniqueness of solutions of initial-value problems for parabolic p-Laplacian. *Electron. J. Differential Equations* **38** (2015), 2015. MR3335768
[7] O. Butkovsky and L. Mytnik. Regularization by noise and flows of solutions for a stochastic heat equation. *Ann. Probab.* **47** (1) (2019) 165–212. MR3909968 https://doi.org/10.1214/18-AOP1259
[8] R. Catellier and M. Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.* **126** (8) (2016) 2323–2366. MR3505229 https://doi.org/10.1016/j.spa.2016.02.002
[9] L. Coutin, R. Duboscq and A. Réveillac. The Itô–Tanaka Trick: A non-semimartingale approach, 2019.
[10] A. M. Davie. Differential equations driven by rough paths: An approach via discrete approximation. *Appl. Math. Res. Express. AMRX* **40** (2007) abm009. MR2387018
[11] A. Deya and S. T. Rough. Volterra equations. I. The algebraic integration setting. *Stoch. Dyn.* **9** (3) (2009) 437–477. MR2566910 https://doi.org/10.1142/S0219493709002737
[12] P. Friz. *Mini-Course on Rough Paths*, 2009.
[13] P. K. Friz and M. Hairer. *A Course on Rough Paths: With an Introduction to Regularity Structures*. *Universitext*. Springer, Cham, 2014. MR3289027 https://doi.org/10.1007/978-3-319-08332-2

[14] H. Fujita and S. Watanabe. On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations. *Comm. Pure Appl. Math.* **21** (6) (1968) 631–652. MR0234129 https://doi.org/10.1002/cpa.3160210609

[15] L. Galeati. Nonlinear young differential equations: A review. *J. Dynam. Differential Equations* (2021).

[16] L. Galeati and M. Gubinelli. Prevalence of rho-irregularity and related properties, 2020.

[17] L. Galeati and M. Gubinelli. Noiseless regularisation by noise. *Rev. Mat. Iberoam.* (2021). MR4404773 https://doi.org/10.4171/rmi/1280

[18] L. Galeati and F. A. Harang. Regularization of multiplicative sdes through additive noise, 2020.

[19] M. Gubinelli, P. Imkeller and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** (2015), e6. MR3406823 https://doi.org/10.1017/fmp.2015.2

[20] M. Gubinelli, P. Imkeller and N. Perkowski. A Fourier analytic approach to pathwise stochastic integration. *Electron. J. Probab.* **21** (2016). MR3485344 https://doi.org/10.1214/16-EJP3868

[21] M. Hairer and X.-M. Li. Averaging dynamics driven by fractional Brownian motion. *Ann. Probab.* **48** (4) (2020) 1826–1860. MR4124526 https://doi.org/10.1214/19-AOP1408

[22] M. Hairer and N. S. Pillai. Regularity of laws and ergodicity of hypoelliptic SDEs driven by rough paths. *Ann. Probab.* **41** (4) (2013) 2544–2598. MR3112925 https://doi.org/10.1214/12-AOP777

[23] F. Harang and A. Mayorcas. Pathwise regularisation of singular interacting particle systems and their mean field limits, 2020.

[24] F. A. Harang and C. Ling. Regularity of local times associated with Volterra–Lévy processes and path-wise regularization of stochastic differential equations. *J. Theor. Probab.* (2021).

[25] F. A. Harang and N. Perkowski. C infinity regularization of odes perturbed by noise. *Stoch. Dyn.* (2021), 2140010. MR4342752 https://doi.org/10.1142/S0219493721400104

[26] F. A. Harang and S. Tindel. Volterra equations driven by rough signals. *Stochastic Process. Appl.* **142(C)** (2021) 34–78. MR4309474 https://doi.org/10.1016/j.spa.2021.08.001

[27] Y. Hu, D. Nualart and J. Song. A nonlinear stochastic heat equation: Hölder continuity and smoothness of the density of the solution. In *Stochastic Processes and their Applications* 1083–1103, **123**, 2013. MR3005015 https://doi.org/10.1016/j.spa.2012.11.004

[28] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. *Analysis in Banach Spaces: Volume I: Martingales and Littlewood–Paley Theory*, 1st edition. Springer, Berlin, 2016. MR3617205

[29] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. In *Fractional Calculus and Applied Analysis* 7–51. *Fractional Calculus and Applied Analysis.* **20**. De Gruyter, Berlin, 2017. MR3613319 https://doi.org/10.1515/fca-2017-0002

[30] K. Lê. A stochastic sewing lemma and applications. *Electron. J. Probab.* **25** (2020) 1–55. MR4089788 https://doi.org/10.1214/20-ejp442

[31] M. Lifshits and T. Simon. Small deviations for fractional stable processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** (4) (2005) 725–752. MR2144231 https://doi.org/10.1016/j.anihpb.2004.05.004

[32] W. Liu and M. Röckner. *Stochastic Partial Differential Equations: An Introduction*. *Universitext*. Springer, Berlin, 2015. MR3410409 https://doi.org/10.1007/978-3-319-22354-4

[33] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic ϕ^4 model in the plane. *Ann. Probab.* **45** (4) (2017) 2398–2476. MR3693966 https://doi.org/10.1214/16-AOP1116

[34] C. Mueller, L. Mytnik and E. Perkins. Nonuniqueness for a parabolic SPDE with $\frac{3}{4}-\varepsilon $-Hölder diffusion coefficients. *Ann. Probab.* **42** (5) (2014) 2032–2112. MR3262498 https://doi.org/10.1214/13-AOP870

[35] L. Mytnik and E. A. Perkins. Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: The white noise case. *Probab. Theory Related Fields* **149** (2008) 1–96. MR2773025 https://doi.org/10.1007/s00440-009-0241-7

[36] L. Mytnik, E. A. Perkins and A. Sturm. On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Ann. Probab.* **34** (2005) 1910–1959. MR2271487 https://doi.org/10.1214/009117906000000331

[37] E. Neuman. Pathwise uniqueness of the stochastic heat equation with spatially inhomogeneous white noise. *Ann. Probab.* **46** (6) (2018) 3090–3187. MR3857853 https://doi.org/10.1214/17-AOP1239

[38] D. Nualart and Y. Ouknine. Regularization of quasilinear heat equations by a fractional noise. *Stoch. Dyn.* **4** (2) (2004) 201–221. MR2069689 https://doi.org/10.1142/S0219493704001012

[39] I. Pinelis. Optimum Bounds for the Distributions of Martingales in Banach Spaces. *Ann. Probab.* **22** (4) (1994) 1679–1706. MR1331198

[40] T. Rippl and A. Sturm. New results on pathwise uniqueness for the heat equation with colored noise. *Electron. J. Probab.* **18** (2012) 1–46. MR3101643 https://doi.org/10.1214/EJP.v18-2506

[41] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes. Stochastic Modeling: Stochastic Models with Infinite Variance*. Chapman & Hall, New York, 1994. MR1280932

[42] H. Triebel. *Theory of Function Spaces. III*. *Monographs in Mathematics* **100**. Birkhäuser Verlag, Basel, 2006. MR2250142

[43] X. Yang and X. Zhou. Pathwise uniqueness for an SPDE with Hölder continuous coefficient driven by α-stable noise. *Electron. J. Probab.* **22** (2017). MR3613697 https://doi.org/10.1214/16-EJP23

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Dissipation in parabolic SPDEs II: Oscillation and decay of the solution

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Abstract. We consider a stochastic heat equation of the type, $\partial_t u = \partial_x^2 u + \sigma(u)\dot{W}$ on $(0,\infty) \times [-1,1]$ with periodic boundary conditions and non-degenerate positive initial data, where $\sigma : \mathbb{R} \to \mathbb{R}$ is a non-random Lipschitz continuous function and $\dot{W}$ denotes space-time white noise. If additionally $\sigma(0) = 0$ then the solution is known to be strictly positive; see Mueller (*Stoch. Stoch. Rep.* **37** (1991) 225–245). In that case, we prove that the oscillation of the logarithm of the solution decays sublinearly as time tends to infinity. Among other things, it follows that, with probability one, all limit points of $t^{-1} \sup_{x\in[-1,1]} \log u(t,x)$ and $t^{-1} \inf_{x\in[-1,1]} \log u(t,x)$ must coincide. As a consequence of this fact, we prove that, when σ is linear, there is a.s. only one such limit point and hence the entire path decays almost surely at an exponential rate.

Résumé. On considère une équation de la chaleur stochastique du type, $\partial_t u = \partial_x^2 u + \sigma(u)\dot{W}$ sur $(0,\infty) \times [-1,1]$ avec des conditions aux limites périodiques et des données initiales positives non dégénérées, où $\sigma : \mathbb{R} \to \mathbb{R}$ est une fonction continue lipschitzienne non aléatoire et $\dot{W}$ désigne un bruit blanc spatio-temporel. Si en plus $\sigma(0) = 0$, alors la solution est strictement positive, voir Mueller (*Stoch. Stoch. Rep.* **37** (1991) 225–245). Dans ce cas, nous prouvons que l'oscillation du logarithme de la solution décroît de manière souslinéaire lorsque le temps tend vers l'infini. Entre autres, il s'ensuit que, avec probabilité un, tous les points limites de $t^{-1} \sup_{x\in[-1,1]} \log u(t,x)$ et $t^{-1} \inf_{x\in[-1,1]} \log u(t,x)$ doivent coïncider. En conséquence de ce fait, nous prouvons que, quand σ est linéaire, il n'y a presque sûrement qu'un seul point limite. Par conséquent, toute la trajectoire décroît exponentiellement presque sûrement.

MSC2020 subject classifications: Primary 60H15; secondary 35R60

Keywords: Stochastic heat equation; Almost sure Lyapunov exponents; Oscillation; Decay

References

[1] G. Amir, I. Corwin and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions. *Comm. Pure Appl. Math.* **64** (4) (2010) 466–537. MR2796514 https://doi.org/10.1002/cpa.20347
[2] R. F. Bass. The measurability of hitting times. *Electron. Commun. Probab.* **15** (2010) 99–105. MR2606507 https://doi.org/10.1214/ECP.v15-1535
[3] L. Bertini and G. Giacomin. On the long time behavior of the stochastic heat equation. *Probab. Theory Related Fields* **114** (1999) 279–289. MR1705123 https://doi.org/10.1007/s004400050226
[4] R. Carmona, L. Koralov and S. Molchanov. Asymptotics for the almost sure Lyapunov exponent for the solution of the parabolic Anderson problem. *Random Oper. Stoch. Equ.* **9** (1) (2001) 77–86. MR1910468 https://doi.org/10.1515/rose.2001.9.1.77
[5] R. A. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** (1994) 518. MR1185878 https://doi.org/10.1090/memo/0518
[6] N. N. Čentsov. Wiener random fields depending on several parameters. *Dokl. Akad. Nauk SSSR* **106** (1956) 607–609. MR0077824
[7] M. Cranston, T. S. Mountford and T. Shiga. Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comenian. (N.S.)* **71** (2) (2002) 321–355. MR1980378
[8] D. A. Freedman. On tail probabilities for martingales. *Ann. Probab.* **3** (1) (1975) 100–118. MR0380971 https://doi.org/10.1214/aop/1176996452
[9] P. Ghosal and Y. Lin Lyapunov exponents of the SHE for general initial data. Preprint, 2020.
[10] Y. Gu and T. Komorowski KPZ on the torus: Gaussian fluctuations. Preprint, 2021.
[11] Y. Gu and T. Komorowski High temperature behaviors of the directed polymer on a cylinder. Preprint, 2021. MR4379456 https://doi.org/10.1007/s10955-022-02899-2

[12] D. Khoshnevisan. *Analysis of Stochastic Partial Differential Equations*. *NSF – CBMS Regional Conf. Series in Math.* **119**. American Math. Soc., Providence, RI, 2014. MR3222416 https://doi.org/10.1090/cbms/119

[13] D. Khoshnevisan, K. Kim, C. Mueller and S.-Y. Shiu. Dissipation in parabolic SPDEs. *J. Stat. Phys.* **179** (2020) 502–534. MR4091567 https://doi.org/10.1007/s10955-020-02540-0

[14] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** (3) (1968) 499–510. MR0254907

[15] J. F. C. Kingman. Subadditive ergodic theory. *Ann. Probab.* **1** (6) (1973) 883–909. MR0356192 https://doi.org/10.1214/aop/1176996798

[16] C. Mueller. On the support of solutions to the heat equation with noise. *Stoch. Stoch. Rep.* **37** (4) (1991) 225–245. MR1149348 https://doi.org/10.1080/17442509108833738

[17] C. Mueller and D. Nualart. Regularity of the density for the stochastic heat equation. *Electron. J. Probab.* **13** (74) (2008) 2248–2258. MR2469610 https://doi.org/10.1214/EJP.v13-589

[18] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer, New York, 2006. MR2200233

[19] D. Nualart and E. Pardoux. Markov field properties of solutions of white noise driven quasi-linear parabolic PDEs. *Stoch. Stoch. Rep.* **48** (1994) 17–44. MR1786190 https://doi.org/10.1080/17442509408833896

[20] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.* **46** (2) (1994) 415–437. MR1271224 https://doi.org/10.4153/CJM-1994-022-8

[21] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV – 1984* 265–439. *Lecture Notes in Math.* **1180**. Springer, Berlin, 1986. MR0876085 https://doi.org/10.1007/BFb0074920

[22] T. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmaikin and D. D. Sokolov. Intermittency of passive fields in random media. *J. Exp. Theor. Phys.* (1985) 2061–2072 (in Russian).

[23] T. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmaikin and D. D. Sokolov. Intermittency, diffusion, and generation in a nonstationary random medium. *Sov. Sci. Rev. C Math. Phys.* **7** (1988) 1–110. MR1128327

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Dynamical random walk on the integers with a drift

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Abstract. We consider a particle performing a random walk on $\mathbb{Z}$ driven by expanding maps. We provide sufficient conditions for the position of the particle z_n to satisfy the Central Limit Theorem.

Résumé. On considère une particule qui suit une marche aléatoire sur $\mathbb{Z}$, dirigée par un systèmes dynamiques expansif. On fournit des conditions suffisantes pour que la position z_n de la particule satisfasse le Théorème Central Limite.

MSC2020 subject classifications: Primary 37A25; 37C30; secondary 60J15

Keywords: Central limit theorem; Decay of correlations; Random walk in random environment

References

[1] R. Aimino and C. Liverani. Deterministic walks in random environment. *Ann. Probab.* **48** (2020) 2212–2257. MR4152641 https://doi.org/10.1214/19-AOP1421
[2] R. Aimino and J. Rousseau. Concentration inequalities for sequential dynamical systems of the unit interval. *Ergodic Theory Dynam. Systems* **36** (2016) 2384–2407. MR3570016 https://doi.org/10.1017/etds.2015.19
[3] N. Berger, A. Drewitz and A. F. Ramirez. Effective polynomial ballisticity conditions for random walk in random environment. *Comm. Pure Appl. Math.* **67** (2014) 1947–1973. MR3272364 https://doi.org/10.1002/cpa.21500
[4] M. Biskup. Recent progress on the random conductance model. *Probab. Surv.* **8** (2011) 294–373. MR2861133 https://doi.org/10.1214/11-PS190
[5] C. Boldrighini, L. A. Bunimovich and G. Sinai Ya. On the Boltzmann equation for the Lorentz gas. *J. Stat. Phys.* **32** (1983) 477–501. MR0725107 https://doi.org/10.1007/BF01008951
[6] E. Bolthausen and I. Goldsheid. Recurrence and transience of random walks in random environments on a strip. *Comm. Math. Phys.* **214** (2000) 429–447. MR1796029 https://doi.org/10.1007/s002200000279
[7] E. Bolthausen and I. Goldsheid. Lingering random walks in random environment on a strip. *Comm. Math. Phys.* **278** (2008) 253–288. MR2367205 https://doi.org/10.1007/s00220-007-0390-4
[8] B. M. Brown. Martingale central limit theorems. *Ann. Math. Stat.* **42** (1971) 59–66. MR0290428 https://doi.org/10.1214/aoms/1177693494
[9] F. Comets, S. Popov, G. M. Schutz and M. Vachkovskaia. Billiards in a general domain with random reflections. *Arch. Ration. Mech. Anal.* **191** (2009) 497–537. MR2481068 https://doi.org/10.1007/s00205-008-0120-x
[10] F. Comets, S. Popov, G. M. Schutz and M. Vachkovskaia. Quenched invariance principle for Knudsen stochastic billiard in random tube. *Ann. Probab.* **38** (2010) 1019–1061. MR2674993 https://doi.org/10.1214/09-AOP504
[11] F. Comets, S. Popov, G. M. Schutz and M. Vachkovskaia. Knudsen gas in a finite random tube: Transport diffusion and first passage properties. *J. Stat. Phys.* **140** (2010) 948–984. MR2673342 https://doi.org/10.1007/s10955-010-0023-8
[12] J.-P. Conze and A. Raugi. Limit theorems for sequential expanding dynamical systems on [0, 1]. *Contemp. Math.* **430** (2007) 89–121. MR2331327 https://doi.org/10.1090/conm/430/08253
[13] G. Cristadoro, M. Lenci and M. Seri. Recurrence for quenched random Lorentz tubes. *Chaos* **20** (2010) 023115. MR2791168 https://doi.org/10.1063/1.3530427
[14] M. F. Demers and C. Liverani. Projective cones for generalized dispersing billiards. Available at arXiv:2104.06947.
[15] D. Dolgopyat and I. Goldsheid. Limit theorems for random walks on a strip in subdiffusive regimes. *Nonlinearity* **26** (2013) 1743–1782. MR3065931 https://doi.org/10.1088/0951-7715/26/6/1743
[16] D. Dolgopyat and I. Goldsheid. Local limit theorems for random walks in a random environment on a strip. *Pure Appl. Funct. Anal.* **5** (2020) 1297–1318. MR4196147
[17] D. Dolgopyat and L. Koralov. Motion in a random force field. *Nonlinearity* **22** (2009) 187–211. MR2470270 https://doi.org/10.1088/0951-7715/22/1/010
[18] D. Dragicevic, G. Froyland, C. Gonzalez-Tokman and S. Vaienti. A spectral approach for quenched limit theorems for random expanding dynamical systems. *Comm. Math. Phys.* **360** (2018) 1121–1187. MR3803820 https://doi.org/10.1007/s00220-017-3083-7
[19] R. Feres. Random walks derived from billiards. *MSRI Publ.* **54** (2007) 179–222. MR2369447 https://doi.org/10.1017/CBO9780511755187.008

[20] V. Gaposhkin. POn the dependence of the convergence rate in the strong law of large numbers for stationary processes on the rate of decay of the correlation function. *Theory Probab. Appl.* **26** (1982) 706–720. MR0636767

[21] I.Y. Goldsheid. Linear and sub-linear growth and the CLT for hitting times of a random walk in random environment on a strip. *Probab. Theory Related Fields* **141** (2008) 471–511. MR2391162 https://doi.org/10.1007/s00440-007-0091-0

[22] N. Haydn, M. Nicol, A. Torok and S. Vaienti. Almost sure invariance principle for sequential and non-stationary dynamical systems. *Trans. Amer. Math. Soc.* **369** (2017) 5293–5316. MR3646763 https://doi.org/10.1090/tran/6812

[23] I. A. Ibragimov and Y. V. Linnik. *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff Publishing, Groningen, 1971. MR0322926

[24] H. Kesten, M. V. Kozlov and F. Spitzer. A limit law for random walk in a random environment. *Compos. Math.* **30** (1975) 145–168. MR0380998

[25] H. Kesten and G. C. Papanicolaou. A limit theorem for turbulent diffusion. *Comm. Math. Phys.* **65** (1979) 97–128. MR0528186

[26] H. Kesten and G. C. Papanicolaou. A limit theorem for stochastic acceleration. *Comm. Math. Phys.* **78** (1981) 19–63. MR0597029

[27] R. Z. Khasminskii. Stochastic processes defined by differential equations with a small parameter. *Theory Probab. Appl.* **11** (1966) 211–228. MR0203788

[28] R. Z. Khasminskii. A limit theorem for solutions of differential equations with a random right hand part. *Theory Probab. Appl.* **11** (1966) 444–462. MR0203789

[29] Y. Kifer. Limit theorems for random transformations and processes in random environments. *Trans. Amer. Math. Soc.* **350** (1998) 1481–1518. MR1451607 https://doi.org/10.1090/S0002-9947-98-02068-6

[30] T. Komorowskiand L. Ryzhik. Diffusion in a weakly random Hamiltonian flow. *Comm. Math. Phys.* **263** (2006) 277–323. MR2207646 https://doi.org/10.1007/s00220-005-1500-9

[31] T. Komorowski and L. Ryzhik. The stochastic acceleration problem in two dimensions. *Israel J. Math.* **155** (2006) 157–204. MR2269429 https://doi.org/10.1007/BF02773954

[32] A. Lasota and J.-A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.* **186** (1973) 481–488. MR0335758 https://doi.org/10.2307/1996575

[33] M. Lenci and S. Troubetzkoy. Infinite-horizon Lorentz tubes and gases: Recurrence and ergodic properties. *Phys. D* **240** (2011) 1510–1515. MR2842891 https://doi.org/10.1016/j.physd.2011.06.020

[34] C. Liverani. Decay of correlations for piecewise expanding maps. *J. Stat. Phys.* **78** (1995) 1111–1129. MR1315241 https://doi.org/10.1007/BF02183704

[35] C. Lutsko and B. Tóth. Invariance principle for the random Lorentz gas beyond the Boltzmann–Grad limit. *Comm. Math. Phys.* **379** (2020) 589–632. MR4156218 https://doi.org/10.1007/s00220-020-03852-8

[36] P. Nandori, D. Szasz and T. Varju. A central limit theorem for time-dependent dynamical systems. *J. Stat. Phys.* **146** (2012) 1213–1220. MR2903045 https://doi.org/10.1007/s10955-012-0451-8

[37] M. Nicol, A. Török and S. Vaienti. Central limit theorems for sequential and random intermittent dynamical systems. *Ergodic Theory Dynam. Systems* **38** (2018) 1127–1153. MR3784257 https://doi.org/10.1017/etds.2016.69

[38] G. C. Papanicolaou and S. R. S. Varadhan. Diffusions with random coefficients. In *North-Holland Publishing Co., R. Rao* 547–552. G. Kallianpur, P. R. Krishnaiah and J. K. Ghosh (Eds). North-Holland Publishing Co., Amsterdam-New York, 1982. MR0659505

[39] G. Sinai Ya. The limiting behavior of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* **27** (1982) 256–268. MR0657919

[40] F. Solomon. Random walks in a random environment. *Ann. Probab.* **3** (1975) 1–31. MR0362503 https://doi.org/10.1214/aop/1176996444

[41] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.* **52** (1980) 569–615. MR0578142 https://doi.org/10.1103/RevModPhys.52.569

[42] H. Spohn. *Large Scale Dynamics of Interacting Particles*. Springer, Berlin-New York, 1991.

[43] A.-S. Sznitman. On a class of transient random walks in random environment. *Ann. Probab.* **29** (2001) 724–765. MR1849176 https://doi.org/10.1214/aop/1008956691

[44] A.-S. Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields* **122** (2002) 509–544. MR1902189 https://doi.org/10.1007/s004400100177

[45] A.-S. Sznitman and O. Zeitouni. An invariance principle for isotropic diffusions in random environment. *Invent. Math.* **164** (2006) 455–567. MR2221130 https://doi.org/10.1007/s00222-005-0477-5

[46] R. Treviño and A. Zelerowicz. Statistical properties of Lorentz gases on aperiodic tilings, part 1. Available at arXiv:2106.12550.

[47] M. Viana. *Stochastic Dynamics of Deterministic Systems*. *Brazillian Math. Colloquium*. IMPA, 1997.

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Kudō-continuity of conditional entropies

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Abstract. In this paper we introduce the notion of Kudō-continuity for real-valued functions on the space of all complete sub-σ-algebras of a standard probability space. This is an a priori strengthening of continuity with respect to strong convergence. We show that conditional entropies are Kudō-continuous, and discuss an application to the study of Furstenberg entropy spectra of SAT*-spaces.

Résumé. Dans cet article, nous introduisons la notion de Kudō-continuité pour les fonctions à valeurs réelles sur l'espace de toutes les sous-σ-algèbres complètes d'un espace de probabilité standard. A priori il s'agit d'un renforcement de la continuité par rapport à la convergence forte. Nous montrons que les entropies conditionnelles sont Kudō-continues, et discutons une application à l'étude des spectres d'entropie de Furstenberg des espaces SAT*.

MSC2020 subject classifications: Primary 60A10; secondary 28D20; 05C81

Keywords: Non-monotone sequences of σ-algebras; Conditional entropy

References

[1] Z. Artstein. Compact convergence of σ-fields and relaxed conditional expectation. *Probab. Theory Related Fields* **120** (3) (2001) 369–394. MR1843180 https://doi.org/10.1007/PL00008787
[2] D. Bakry, I. Gentil and M. Ledoux. *Analysis and Geometry of Markov Diffusion Operators*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **348**, xx+552 pp. Springer, Cham, 2014. MR3155209 https://doi.org/10.1007/978-3-319-00227-9
[3] L. Bowen, Y. Hartman and O. Tamuz. Generic stationary measures and actions. *Trans. Amer. Math. Soc.* **369** (7) (2017) 4889–4929. MR3632554 https://doi.org/10.1090/tran/6803
[4] R. Durrett. *Probability: Theory and Examples*, 4th edition. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**, x+428 pp. Cambridge University Press, Cambridge, 2010. MR2722836 https://doi.org/10.1017/CBO9780511779398
[5] A. Furman. Random walks on groups and random transformations. In *Handbook of Dynamical Systems* 931–1014, **1A**. North-Holland, Amsterdam, 2002. MR1928529 https://doi.org/10.1016/S1874-575X(02)80014-5
[6] Y. Hartman and A. Yadin. Furstenberg entropy of intersectional invariant random subgroups. *Compos. Math.* **154** (10) (2018) 2239–2265. MR3867301 https://doi.org/10.1112/s0010437x18007261
[7] W. Jaworski. Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups. *Pacific J. Math.* **170** (2) (1995) 517–533. MR1363877
[8] V. Kaimanovich. SAT actions and ergodic properties of the horosphere foliation. In *Rigidity in Dynamics and Geometry* 261–282. *Cambridge, 2000*. Springer, Berlin, 2002. MR1919405
[9] H. Kudō. A note on the strong convergence of σ-algebras. *Ann. Probab.* **2** (1) (1974) 76–83. MR0370674 https://doi.org/10.1214/aop/1176996752
[10] A. Nevo. The spectral theory of amenable actions and invariants of discrete groups. *Geom. Dedicata* **100** (2003) 187–218. MR2011122 https://doi.org/10.1023/A:1025839828396
[11] A. Nevo and R. Zimmer. Homogenous projective factors for actions of semi-simple Lie groups. *Invent. Math.* **138** (2) (1999) 229–252. MR1720183 https://doi.org/10.1007/s002220050377
[12] A. Nevo and R. Zimmer. Rigidity of Furstenberg entropy for semisimple Lie group actions. *Ann. Sci. Éc. Norm. Supér. (4)* **33** (3) (2000) 321–343. MR1775184 https://doi.org/10.1016/S0012-9593(00)00113-0
[13] O. Tamuz and T. Zheng On the spectrum of asymptotic entropies of random walks. Preprint. Available at https://arxiv.org/abs/1903.01312.

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Yaglom limit for unimodal Lévy processes

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Abstract. We prove universality of the Yaglom limit of Lipschitz cones among all unimodal Lévy processes sufficiently close to the isotropic α-stable Lévy process.

Résumé. On prouve l'universalité de la limite de Yaglom dans les cônes de Lipschitz parmi tous les processus de Lévy unimodaux suffisamment proches du processus de Lévy isotrope α-stable.

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Keywords: Yaglom limit; Lévy process; Lipschitz cone; Boundary limit

References

[1] A. Asselah, P. A. Ferrari, P. Groisman and M. Jonckheere. Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2016) 647–668. MR3498004 https://doi.org/10.1214/14-AIHP635
[2] R. Bañuelos and K. Bogdan. Symmetric stable processes in cones. *Potential Anal.* **21** (2004) 263–288. MR2075671 https://doi.org/10.1023/B:POTA.0000033333.72236.dc
[3] N. G. Bean, L. Bright, G. Latouche, C. E. M. Pearce, P. K. Pollett and P. G. Taylor. The quasi-stationary behavior of quasi-birth-and-death processes. *Ann. Appl. Probab.* **7** (1997) 134–155. MR1428753 https://doi.org/10.1214/aoap/1034625256
[4] N. G. Bean, M. M. O'Reilly and Z. Palmowski. Yaglom limit for stochastic fluid models. *Adv. Appl. Probab.* **53** (2021) 1–35. MR4322399 https://doi.org/10.1017/apr.2020.71
[5] N. G. Bean, P. K. Pollett and P. G. Taylor. The quasistationary distributions of level-independent quasi-birth-and-death processes. *Commun. Stat., Stoch. Models* **14** (1998) 389–406. Special issue in honor of Marcel F. Neuts. MR1617603 https://doi.org/10.1080/15326349808807478
[6] N. G. Bean, P. K. Pollett and P. G. Taylor. Quasistationary distributions for level-dependent quasi-birth-and-death processes. *Commun. Stat., Stoch. Models* **16** (2000) 511–541. MR1796345 https://doi.org/10.1080/15326340008807602
[7] J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.* **47** (1993) 17–35. MR1232850 https://doi.org/10.1016/0304-4149(93)90092-I
[8] J. Bertoin and R. A. Doney. On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22** (1994) 2152–2167. MR1331218
[9] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge University Press, Cambridge, 1989. MR0898871 https://doi.org/10.1017/CBO9780511721434
[10] K. Bogdan, T. Grzywny and M. Ryznar. Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.* **38** (2010) 1901–1923. MR2722789 https://doi.org/10.1214/10-AOP532
[11] K. Bogdan, T. Grzywny and M. Ryznar. Dirichlet heat kernel for unimodal Lévy processes. *Stochastic Process. Appl.* **124** (2014) 3612–3650. MR3249349 https://doi.org/10.1016/j.spa.2014.06.001
[12] K. Bogdan, T. Grzywny and M. Ryznar. Density and tails of unimodal convolution semigroups. *J. Funct. Anal.* **266** (2014) 3543–3571. MR3165234 https://doi.org/10.1016/j.jfa.2014.01.007
[13] K. Bogdan, T. Grzywny and M. Ryznar. Barriers, exit time and survival probability for unimodal Lévy processes. *Probab. Theory Related Fields* **162** (2015) 155–198. MR3350043 https://doi.org/10.1007/s00440-014-0568-6
[14] K. Bogdan, T. Kumagai and M. Kwaśnicki. Boundary Harnack inequality for Markov processes with jumps. *Trans. Amer. Math. Soc.* **367** (2015) 477–517. MR3271268 https://doi.org/10.1090/S0002-9947-2014-06127-8
[15] K. Bogdan, Z. Palmowski and L. Wang. Yaglom limit for stable processes in cones. *Electron. J. Probab.* **23** (2018) 1–19. MR3771748 https://doi.org/10.1214/18-EJP133
[16] L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.* **64** (1996) 39–54. MR1419491 https://doi.org/10.1016/S0304-4149(96)00081-6

[17] L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10** (28) (2005) 948–961. MR2164035 https://doi.org/10.1214/EJP.v10-261
[18] Z.-Q. Chen, P. Kim and R. Song. Dirichlet heat kernel estimates for rotationally symmetric Lévy processes. *Proc. Lond. Math. Soc. (3)* **109** (2014) 90–120. MR3237737 https://doi.org/10.1112/plms/pdt068
[19] W. Cygan, T. Grzywny and B. Trojan. Asymptotic behavior of densities of unimodal convolution semigroups. *Trans. Amer. Math. Soc.* **369** (2017) 5623–5644. MR3646773 https://doi.org/10.1090/tran/6830
[20] I. Czarna and Z. Palmowski. Parisian quasi-stationary distributions for asymmetric Lévy processes. *Statist. Probab. Lett.* **127** (2017) 75–84. MR3648297 https://doi.org/10.1016/j.spl.2017.03.011
[21] E. B. Dynkin. *Markov Processes. Vols. I, II. Die Grundlehren der Mathematischen Wissenschaften* **121, 122**. Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin–Göttingen–Heidelberg, 1965. Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. MR0193671
[22] P. A. Ferrari, H. Kesten, S. Martinez and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.* **23** (1995) 501–521. MR1334159
[23] P. A. Ferrari and N. Marić. Quasi stationary distributions and Fleming–Viot processes in countable spaces. *Electron. J. Probab.* **12** (24) (2007) 684–702. MR2318407 https://doi.org/10.1214/EJP.v12-415
[24] D. C. Flaspohler and P. T. Holmes. Additional quasi-stationary distributions for semi-Markov processes. *J. Appl. Probab.* **9** (1972) 671–676. MR0346932 https://doi.org/10.2307/3212336
[25] I. I. Gihman and A. V. Skorohod. *The Theory of Stochastic Processes. I*, english edition. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **210**. Springer-Verlag, Berlin–New York, 1980. Translated from the Russian by Samuel Kotz. MR0636254
[26] L. M. Graves. *The Theory of Functions of Real Variables*. McGraw-Hill Book Co., Inc., New York, 1946. MR0018708
[27] T. Grzywny. On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes. *Potential Anal.* **41** (2014) 1–29. MR3225805 https://doi.org/10.1007/s11118-013-9360-y
[28] T. Grzywny and M. Kwaśnicki. Potential kernels, probabilities of hitting a ball, harmonic functions and the boundary Harnack inequality for unimodal Lévy processes. *Stochastic Process. Appl.* **128** (2018) 1–38. MR3729529 https://doi.org/10.1016/j.spa.2017.04.004
[29] T. Grzywny, Ł. Leżaj and M. Miśta. Hitting probabilities for Lévy processes on the real line. *ALEA Lat. Am. J. Probab. Math. Stat.* **18** (2021) 727–760. MR4243514 https://doi.org/10.30757/alea.v18-27
[30] T. Grzywny and M. Ryznar. Potential theory of one-dimensional geometric stable processes. *Colloq. Math.* **129** (2012) 7–40. MR3007664 https://doi.org/10.4064/cm129-1-2
[31] T. Grzywny, M. Ryznar and B. Trojan. Asymptotic behaviour and estimates of slowly varying convolution semigroups. *Int. Math. Res. Not. IMRN* **23** (2019) 7193–7258. MR4039012 https://doi.org/10.1093/imrn/rnx324
[32] T. Grzywny and K. Szczypkowski. Lévy processes: Concentration function and heat kernel bounds. *Bernoulli* **26** (2020) 3191–3223. MR4140542 https://doi.org/10.3150/20-BEJ1220
[33] T. Grzywny and K. Szczypkowski. Estimates of heat kernels of non-symmetric Lévy processes. *Forum Math.* **33** (2021) 1207–1236. MR4308627 https://doi.org/10.1515/forum-2020-0364
[34] B. Haas and V. Rivero. Quasi-stationary distributions and Yaglom limits of self-similar Markov processes. *Stochastic Process. Appl.* **122** (2012) 4054–4095. MR2971725 https://doi.org/10.1016/j.spa.2012.08.006
[35] S. C. Harris, E. Horton, A. E. Kyprianou and M. Wang. Yaglom limit for critical nonlocal branching Markov processes. *Ann. Probab.* **50** (6) (2022) 2373–2408. https://doi.org/10.1214/22-AOP1585
[36] D. L. Iglehart. Random walks with negative drift conditioned to stay positive. *J. Appl. Probab.* **11** (1974) 742–751. MR0368168 https://doi.org/10.2307/3212557
[37] S. D. Jacka and G. O. Roberts. Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.* **32** (1995) 902–916. MR1363332 https://doi.org/10.2307/3215203
[38] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer-Verlag, Berlin, 1987. MR0959133 https://doi.org/10.1007/978-3-662-02514-7
[39] V. Knopova and R. L. Schilling. A note on the existence of transition probability densities of Lévy processes. *Forum Math.* **25** (2013) 125–149. MR3010850 https://doi.org/10.1515/form.2011.108
[40] T. Kulczycki and M. Ryznar. Gradient estimates of harmonic functions and transition densities for Lévy processes. *Trans. Amer. Math. Soc.* **368** (2016) 281–318. MR3413864 https://doi.org/10.1090/tran/6333
[41] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* **20** (2017) 7–51. MR3613319 https://doi.org/10.1515/fca-2017-0002
[42] M. Kwaśnicki and T. Juszczyszyn. Martin kernels for Markov processes with jumps. *Potential Anal.* **47** (2017) 313–335. MR3713580 https://doi.org/10.1007/s11118-017-9616-z
[43] A. E. Kyprianou and Z. Palmowski. Quasi-stationary distributions for Lévy processes. *Bernoulli* **12** (2006) 571–581. MR2248228 https://doi.org/10.3150/bj/1155735927
[44] E. K. Kyprianou. On the quasi-stationary distribution of the virtual waiting time in queues with Poisson arrivals. *J. Appl. Probab.* **8** (1971) 494–507. MR0292201 https://doi.org/10.2307/3212173
[45] A. Lambert. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.* **12** (14) (2007) 420–446. MR2299923 https://doi.org/10.1214/EJP.v12-402
[46] M. Mandjes, Z. Palmowski and T. Rolski. Quasi-stationary workload in a Lévy-driven storage system. *Stoch. Models* **28** (2012) 413–432. MR2959448 https://doi.org/10.1080/15326349.2012.699753
[47] S. Martínez and J. San Martín. Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.* **31** (1994) 911–920. MR1303922 https://doi.org/10.1017/s0021900200099447
[48] Z. Palmowski and M. Vlasiou. Speed of convergence to the quasi-stationary distribution for Lévy input fluid queues. *Queueing Syst.* **96** (2020) 153–167. MR4171931 https://doi.org/10.1007/s11134-020-09664-w
[49] W. E. Pruitt. The growth of random walks and Lévy processes. *Ann. Probab.* **9** (1981) 948–956. MR0632968
[50] Y.-X. Ren, R. Song and Z. Sun. A 2-spine decomposition of the critical Galton–Watson tree and a probabilistic proof of Yaglom's theorem. *Electron. Commun. Probab.* **23** (2018) Paper No. 42, 12. MR3841403 https://doi.org/10.1214/18-ECP143

[51] Y.-X. Ren, R. Song and Z. Sun. Limit theorems for a class of critical superprocesses with stable branching. *Stochastic Process. Appl.* **130** (2020) 4358–4391. MR4102269 https://doi.org/10.1016/j.spa.2020.01.001

[52] K.-i. Sato. *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author. MR1739520

[53] R. L. Schilling. *Measures, Integrals and Martingales*, 2nd edition. Cambridge University Press, Cambridge, 2017. MR3644418

[54] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3** (1966) 403–434. MR0207047 https://doi.org/10.2307/3212128

[55] H. Šikić, R. Song and Z. Vondraček. Potential theory of geometric stable processes. *Probab. Theory Related Fields* **135** (2006) 547–575. MR2240700 https://doi.org/10.1007/s00440-005-0470-3

[56] P. Sztonyk. On harmonic measure for Lévy processes. *Probab. Math. Statist.* **20** (2000) 383–390. MR1825650

[57] R. L. Tweedie. Quasi-stationary distributions for Markov chains on a general state space. *J. Appl. Probab.* **11** (1974) 726–741. MR0443090 https://doi.org/10.1017/s0021900200118169

[58] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.* **23** (1991) 683–700. MR1133722 https://doi.org/10.2307/1427670

[59] E. A. van Doorn and P. K. Pollett. Quasi-stationary distributions for discrete-state models. *European J. Oper. Res.* **230** (2013) 1–14. MR3063313 https://doi.org/10.1016/j.ejor.2013.01.032

[60] A. M. Yaglom. Certain limit theorems of the theory of branching random processes. *Dokl. Akad. Nauk SSSR* **56** (1947) 795–798. MR0022045

[61] J. Zhang, S. Li and R. Song. Quasi-stationarity and quasi-ergodicity of general Markov processes. *Sci. China Math.* **57** (2014) 2013–2024. MR3247530 https://doi.org/10.1007/s11425-014-4835-x

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Tail asymptotics for extinction times of self-similar fragmentations

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Abstract. We provide the exact large-time behavior of the tail distribution of the extinction time of a self-similar fragmentation process with a negative index of self-similarity, improving thus a previous result on the logarithmic asymptotic behavior of this tail. Two factors influence this behavior: the distribution of the largest fragment at the time of a dislocation and the index of self-similarity. As an application we obtain the asymptotic behavior of all positive moments of the largest fragment and compare it to the behavior of the positive moments of a tagged fragment, whose decrease is in general significantly slower. We illustrate our results on several examples, including fragmentations related to random real trees – for which we thus obtain the asymptotic behavior of the tail distribution of the height – such as the stable Lévy trees of Duquesne, Le Gall and Le Jan (including the Brownian tree of Aldous), the alpha-model of Ford and the beta-splitting model of Aldous.

Résumé. Nous décrivons le comportement en temps grand de la queue de distribution du temps d'extinction d'un processus de fragmentation auto-similaire avec un indice d'auto-similarité négatif, améliorant significativement ainsi un résultat précédent sur le comportement logarithmique de cette queue. Deux facteurs influencent ce comportement : la distribution du plus gros fragment lors d'une dislocation et l'indice d'auto-similarité. Comme conséquence, nous obtenons le comportement asymptotique des moments positifs du plus grand fragment et le comparons au comportement des moments positifs d'un fragment marqué, dont la décroissance est en général significativement plus lente. Nous illustrons nos résultats avec plusieurs exemples, dont des fragmentations liées à des arbres réels aléatoires – pour lesquels nous obtenons ainsi le comportement asymptotique de la queue de distribution de la hauteur – tels que les arbres de Lévy stables de Duquesne, Le Gall et Le Jan (y compris l'arbre brownien d'Aldous), le modèle alpha de Ford et le modèle dit "beta-splitting" d'Aldous.

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References

[1] D. Aldous. Probability distributions on cladograms. In *Random Discrete Structures* 1–18. *Minneapolis, MN, 1993. IMA Vol. Math. Appl.* **76**. Springer, New York, 1996. MR1395604 https://doi.org/10.1007/978-1-4612-0719-1_1
[2] J. Berestycki. Ranked fragmentations. *ESAIM Probab. Stat.* **6** (2002) 157–175. MR1943145 https://doi.org/10.1051/ps:2002009
[3] J. Bertoin. Homogeneous fragmentation processes. *Probab. Theory Related Fields* **121** (2001) 301–318. MR1867425 https://doi.org/10.1007/s004400100152
[4] J. Bertoin. Self-similar fragmentations. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** (2002) 319–340. MR1899456 https://doi.org/10.1016/S0246-0203(00)01073-6
[5] J. Bertoin. The asymptotic behavior of fragmentation processes. *J. Eur. Math. Soc. (JEMS)* **5** (2003) 395–416. MR2017852 https://doi.org/10.1007/s10097-003-0055-3
[6] J. Bertoin. *Random Fragmentation and Coagulation Processes*. *Cambridge Studies in Advanced Mathematics* **102**. Cambridge University Press, Cambridge, 2006. MR2253162 https://doi.org/10.1017/CBO9780511617768
[7] J. Bertoin, T. Budd, N. Curien and I. Kortchemski. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probab. Theory Related Fields* **172** (2018) 663–724. MR3877545 https://doi.org/10.1007/s00440-017-0818-5
[8] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.* **2** (2005) 191–212. MR2178044 https://doi.org/10.1214/154957805100000122
[9] M. D. Brennan and R. Durrett. Splitting intervals. *Ann. Probab.* **14** (1986) 1024–1036. MR0841602
[10] M. D. Brennan and R. Durrett. Splitting intervals II: Limit laws for lengths. *Probab. Theory Related Fields* **75** (1987) 109–127. MR0879556 https://doi.org/10.1007/BF00320085
[11] B. Chen, D. Ford and M. Winkel. A new family of Markov branching trees: The alpha-gamma model. *Electron. J. Probab.* **14** (15) (2009) 400–430. MR2480547 https://doi.org/10.1214/EJP.v14-616

[12] N. Curien, T. Duquesne, I. Kortchemski and I. Manolescu. Scaling limits and influence of the seed graph in preferential attachment trees. *J. Éc. Polytech. Math.* **2** (2015) 1–34. MR3326003 https://doi.org/10.5802/jep.15

[13] T. Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.* **31** (2003) 996–1027. MR1964956 https://doi.org/10.1214/aop/1048516543

[14] T. Duquesne and J.-F. Le Gall. *Random Trees, Lévy Processes and Spatial Branching Processes*. Société mathématique de France, Paris, France, 2002. MR1954248

[15] T. Duquesne and M. Wang. Decomposition of Lévy trees along their diameter. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 539–593. MR3634265 https://doi.org/10.1214/15-AIHP725

[16] A. Filippov. On the distribution of the sizes of particles which undergo splitting. *Theory Probab. Appl.* **6** (1961) 275–294.

[17] D. Ford Probabilities on cladograms: Introduction to the alpha model. Preprint. Available at arXiv:math/0511246. MR2708802

[18] B. Haas. Loss of mass in deterministic and random fragmentations. *Stochastic Process. Appl.* **106** (2003) 245–277. MR1989629 https://doi.org/10.1016/S0304-4149(03)00045-0

[19] B. Haas. Asymptotic behavior of solutions of the fragmentation equation with shattering: An approach via self-similar Markov processes. *Ann. Appl. Probab.* **20** (2010) 382–429. MR2650037 https://doi.org/10.1214/09-AAP622

[20] B. Haas. Precise asymptotics for the density and the upper tail of exponential functionals of subordinators. *Séminaire de Probabilités.* To appear.

[21] B. Haas and G. Miermont. The genealogy of self-similar fragmentations with negative index as a continuum random tree. *Electron. J. Probab.* **9** (4) (2004) 57–97. MR2041829 https://doi.org/10.1214/EJP.v9-187

[22] B. Haas and G. Miermont. Scaling limits of Markov branching trees with applications to Galton-Watson and random unordered trees. *Ann. Probab.* **40** (2012) 2589–2666. MR3050512 https://doi.org/10.1214/11-AOP686

[23] B. Haas, G. Miermont, J. Pitman and M. Winkel. Continuum tree asymptotics of discrete fragmentations and applications to phylogenetic models. *Ann. Probab.* **36** (2008) 1790–1837. MR2440924 https://doi.org/10.1214/07-AOP377

[24] B. Haas and V. Rivero. Quasi-stationary distributions and Yaglom limits of self-similar Markov processes. *Stochastic Process. Appl.* **122** (2012) 4054–4095. MR2971725 https://doi.org/10.1016/j.spa.2012.08.006

[25] B. Haas and R. Stephenson. Scaling limits of k-ary growing trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** (2015) 1314–1341. MR3414449 https://doi.org/10.1214/14-AIHP622

[26] D. P. Kennedy. The distribution of the maximum Brownian excursion. *J. Appl. Probab.* **13** (1976) 371–376. MR0402955 https://doi.org/10.1017/s0021900200094468

[27] J. F. C. Kingman. The coalescent. *Stochastic Process. Appl.* **13** (1982) 235–248. MR0671034 https://doi.org/10.1016/0304-4149(82)90011-4

[28] A. Kolmogorov. Uber das logarithmisch normale Verteilungsgesetz der Dimensionen der Teilchen bei Zerstuckelung. *C. R. Acad. Sci. URSS* **31** (1941) 99–101. MR0004415

[29] J.-F. Le Gall and Y. Le Jan. Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26** (1998) 213–252. MR1617047 https://doi.org/10.1214/aop/1022855417

[30] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. *Stochastic Process. Appl.* **116** (2006) 156–177. MR2197972 https://doi.org/10.1016/j.spa.2005.09.002

[31] E. McGrady and R. M. Ziff. Shattering transition in fragmentation. *Phys. Rev. Lett.* **58** (1987) 892. MR0927489 https://doi.org/10.1103/PhysRevLett.58.892

[32] G. Miermont. Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields* **127** (2003) 423–454. MR2018924 https://doi.org/10.1007/s00440-003-0295-x

[33] M. Minchev and M. Savov Asymptotic of densities of exponential functionals of subordinators. Preprint. Available at arXiv:2104.05381.

[34] J. Nussbaumer and A. Winter The algebraic α-ford tree under evolution. Preprint. Available at arXiv:2006.09316.

[35] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.* **27** (1999) 1870–1902. MR1742892 https://doi.org/10.1214/aop/1022677552

[36] V. Rivero. Tail asymptotics for exponential functionals of Lévy processes: The convolution equivalent case. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 1081–1102. MR3052404 https://doi.org/10.1214/12-AIHP477

[37] S. O. Stefansson. The infinite volume limit of Ford's alpha model. *Acta Phys. Pol. B* **2** (2009) 555–562.

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