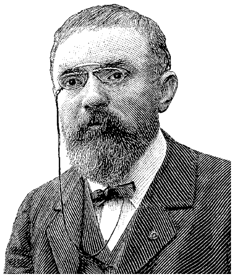

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Dmitry Ioffe (April 5, 1963–October 1, 2020)

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The Schonmann projection: How Gibbsian is it?

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Dedicated to the memory of our dear friend Dima Ioffe

Abstract. We study the one-dimensional projection of the extremal Gibbs measures of the two-dimensional Ising model – the "Schonmann projection". These measures are known to be non-Gibbsian at low temperatures, since their conditional probabilities as a function of the (two-sided) boundary conditions are not continuous. We prove the conjecture that they are g-measures, which means that their conditional probabilities have a continuous dependence on one-sided boundary conditions.

Résumé. Nous étudions la projection unidimensionnelle des mesures extrémales de Gibbs du modèle d'Ising bidimensionnel – la "projection Schonmann". Ces mesures sont connues pour être non-Gibbsiennes à basses températures, puisque leurs probabilités conditionnelles en fonction des conditions au bord (bilatérales) ne sont pas continues. Nous prouvons la conjecture que néanmoins ce sont des g-mesures, ce qui signifie que leurs probabilités conditionnelles dépendent de façon continue des conditions au bord unilatérales.

MSC2020 subject classifications: Primary 82B20; 60K35 secondary82B41

Keywords: Non-Gibbsian measure; g-measure; Schonmann Projection; Entropic Repulsion

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What does a typical metric space look like?

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To the memory of Dima Ioffe, our friend and colleague. Mathematical physicist, probabilist and a dear person who freely shared his good advice and insight. His passing is a great loss to our community.

Abstract. The collection $\mathcal{M}_n$ of all metric spaces on n points whose diameter is at most 2 can naturally be viewed as a compact convex subset of $\mathbb{R}^{\binom{n}{2}}$, known as the metric polytope. In this paper, we study the metric polytope for large n and show that it is close to the cube $[1,2]^{\binom{n}{2}} \subseteq \mathcal{M}_n$ in the following two senses. First, the volume of the polytope is not much larger than that of the cube, with the following quantitative estimates:

$$\left(\frac{1}{6}+o(1)\right)n^{3/2} \le \log \mathrm{Vol}(\mathcal{M}_n) \le O\left(n^{3/2}\right).$$

Second, when sampling a metric space from $\mathcal{M}_n$ uniformly at random, the minimum distance is at least $1-n^{-c}$ with high probability, for some $c>0$. Our proof is based on entropy techniques. We discuss alternative approaches to estimating the volume of $\mathcal{M}_n$ using exchangeability, Szemerédi's regularity lemma, the hypergraph container method, and the Kővári–Sós–Turán theorem.

Résumé. La collection $\mathcal{M}_n$ de tous les espaces métriques à n points de diamètre au plus 2 peut être vue naturellement comme un convexe compact de $\mathbb{R}^{\binom{n}{2}}$, appelé le polytope métrique. Dans cet article, nous étudions le polytope métrique lorsque n est grand, et montrons qu'il est proche du cube $[1,2]^{\binom{n}{2}} \subseteq \mathcal{M}_n$ aux deux sens suivants. Tout d'abord, le volume du polytope n'est pas beaucoup plus grand que celui du cube, avec les estimées quantitatives suivantes :

$$\left(\frac{1}{6}+o(1)\right)n^{3/2} \le \log \mathrm{Vol}(\mathcal{M}_n) \le O\left(n^{3/2}\right).$$

Ensuite, lorsqu'on échantillonne uniformément au hasard un espace métrique de $\mathcal{M}_n$, la distance minimale est au moins $1-n^{-c}$ avec grande probabilité, pour un $c>0$. Notre preuve utilise des techniques d'entropie. Nous discutons également d'autres approches permettant d'estimer le volume de $\mathcal{M}_n$, utlisant l'échangeabilité, le lemme de régularité de Szemerédi, la méthode des conteneurs d'hypergraphes, et le théorème de Kővári–Sós–Turán.

MSC2020 subject classifications: Primary 60D05; secondary 51F99; 05A16

Keywords: Finite metric space; Metric polytope; Random metric space; Entropy method

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Entropy inequalities for random walks and permutations

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Dedicated to the memory of Dima Ioffe

Abstract. We consider a new functional inequality controlling the rate of relative entropy decay for random walks, the interchange process and more general block-type dynamics for permutations. The inequality lies between the classical logarithmic Sobolev inequality and the modified logarithmic Sobolev inequality, roughly interpolating between the two as the size of the blocks grows. Our results suggest that the new inequality may have some advantages with respect to the latter well known inequalities when multi-particle processes are considered. We prove a strong form of tensorization for independent particles interacting through synchronous updates. Moreover, for block dynamics on permutations we compute the optimal constants in all mean field settings, namely whenever the rate of update of a block depends only on the size of the block. Along the way we establish the independence of the spectral gap on the number of particles for these mean field processes. As an application of our entropy inequalities we prove a new subadditivity estimate for permutations, which implies a sharp upper bound on the permanent of arbitrary matrices with nonnegative entries, thus resolving a well known conjecture.

Résumé. On considère une nouvelle inégalité fonctionnelle contrôlant le taux de décroissance de l'entropie relative de marche aléatoire, processus d'échange ou plus généralement, de dynamique en bloc pour les permutations. Cette inégalité se situe entre l'inégalité logarithmique de Sobolev classique et l'inégalité logarithmique de Sobolev modifiée, interpolant les deux lorsque la taille des blocs augmente. Notre résultat suggère que cette nouvelle inégalité pourrait avoir des avantages par rapport à ces dernières dans le cadre de processus à multiples particules. On prouve une forme de tensorisation forte pour des particules indṕendantes à mise à jour synchronisées. De plus, pour toutes les dynamiques en bloc à champ moyen, quand les taux d'actualisation dépendent uniquement de la taille des blocs, on établit la constante optimale. Au passage, on prouve aussi l'indépendance du trou spectral par rapport au nombre de particules. En application de notre inégalité entropique, on prouve une estimée de sous-additivité pour les permutations, impliquant une borne supérieure optimale pour le permanent de matrices non negatives arbitraire, ce qui résout une conjecture bien connue.

MSC2020 subject classifications: 82B20; 82C20; 39B62

Keywords: Entropy; Logarithmic Sobolev inequalities; Spectral gap; Permutations

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Space–time fluctuation of the Kardar–Parisi–Zhang equation in $d \geq 3$ and the Gaussian free field

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Dedicated to the memory of Dima Ioffe

Abstract. We study the solution h_ε of the Kardar–Parisi–Zhang (KPZ) equation in $\mathbb{R}^d \times [0,\infty)$ for $d \geq 3$:

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon, \quad h_\varepsilon(0,x) = 0.$$

Here $\beta > 0$ is a parameter called the disorder strength, $\xi_\varepsilon = \xi \star \phi_\varepsilon$ is a spatially smoothened (at scale ε) Gaussian space–time white noise and C_ε is a divergent constant as $\varepsilon \to 0$. When β is sufficiently small and $\varepsilon \to 0$, $h_\varepsilon(t,x) - \mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x) \to 0$ in probability where $\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)$ is the *stationary solution* of the KPZ equation – more precisely, $\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)$ solves the above equation with a random initial condition (that is independent of the driving noise ξ) and its marginal law is constant in (ε,t,x). In the present article we quantify the rate of the above convergence in this regime and show that the fluctuations $(\varepsilon^{1-\frac{d}{2}}[h_\varepsilon(t,x) - \mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)])_{x\in\mathbb{R}^d, t>0}$ *about* the stationary solution converge pointwise (with finite dimensional distributions in space and time) to a Gaussian free field convoluted with the deterministic heat equation. We also identify the fluctuations *of* the stationary solution itself and show that the rescaled averages $\int_{\mathbb{R}^d} \mathrm{d}x \varphi(x) \varepsilon^{1-\frac{d}{2}}[\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x) - \mathbb{E}\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)]$ converge to that of the *stationary solution* of the stochastic heat equation with additive noise, but with (random) Gaussian free field marginals (instead of flat initial condition).

Résumé. Nous étudions la solution h_ε de l'équation de Kardar–Parisi–Zhang (KPZ) sur $\mathbb{R}^d \times [0,\infty)$ avec $d \geq 3$:

$$\frac{\partial}{\partial t} h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta \varepsilon^{\frac{d-2}{2}} \xi_\varepsilon, \quad h_\varepsilon(0,x) = 0.$$

Ici $\beta > 0$ est un paramètre appelé la force du désordre, $\xi_\varepsilon = \xi \star \phi_\varepsilon$ est un bruit blanc gaussien espace-temps régularisé en espace (à l'échelle ε) et C_ε est une constante qui diverge lorsque $\varepsilon \to 0$. Lorsque β est suffisamment petit et $\varepsilon \to 0$, $h_\varepsilon(t,x) - \mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x) \to 0$ en probabilité où $\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)$ est la *solution stationnaire* de l'équation KPZ – plus précisément, $\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)$ est solution de l'équation ci-dessus avc une condition initiale aléatoire (laquelle est indépendante du bruit ξ) et dont la loi marginale est constante en (ε,t,x). Dans cet article nous quantifions la vitesse de la convergence ci-dessus et nous montrons que les fluctuations $(\varepsilon^{1-\frac{d}{2}}[h_\varepsilon(t,x) - \mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)])_{x\in\mathbb{R}^d, t>0}$ *autour* de la solution stationnaire converge ponctuellement (de manière jointe pour un nombre fini de points de l'espace et du temps) vers un champ libre gaussien convolé à l'équation de la chaleur déterministe. Nous identifions également les fluctuations de l'équation stationnaire autour de sa moyenne et montrons que $\int_{\mathbb{R}^d} \mathrm{d}x \varphi(x) \varepsilon^{1-\frac{d}{2}}[\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x) - \mathbb{E}\mathfrak{h}_\varepsilon^{\mathrm{st}}(t,x)]$ converge vers la *solution stationnaire* de l'équation de la chaleur avec bruit additif, dont la loi marginale est donnée par le champ libre gaussien (au lieu de la condition initiale plate).

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Keywords: SPDE; Kardar–Parisi–Zhang equation; Stochastic heat equation; Rate of convergence; Edwards–Wilkinson limit; Gaussian free field; Directed polymers; Random environment

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On the limiting law of line ensembles of Brownian polymers with geometric area tilts

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Dedicated to the memory of Dima Ioffe

Abstract. We study the line ensembles of non-crossing Brownian bridges above a hard wall, each tilted by the area of the region below it with geometrically growing pre-factors. This model, which mimics the level lines of the $(2+1)$D SOS model above a hard wall, was studied in two works from 2019 by Caputo, Ioffe and Wachtel. In those works, the tightness of the law of the top k paths, for any fixed k, was established under either zero or free boundary conditions, which in the former setting implied the existence of a limit via a monotonicity argument. Here we address the open problem of existence of a limit under free boundary conditions: we prove that as the interval length, followed by the number of paths, go to ∞, the top k paths converge to the same limit as in the zero boundary case, as conjectured by Caputo, Ioffe and Wachtel.

Résumé. Nous étudions l'ensemble de lignes déterminé par des mouvements Browniens non-intersectant au-dessus d'un mur solide. Ce modèle, qui imite les lignes de niveaux du modèle $(2+1)$D SOS au-dessus d'un mur, a été étudié en 2019 par Caputo, Ioffe et Wachtel. Dans ces travaux, la tension de la loi des k lignes hautes, pour chaque k fixe, a été obtenue sous des conditions nulles au bord ou des conditions libres au bord. Dans le premier cas, ca implique l'existence d'une limite par un argument de monotonicité. Nous abordons ici le problème ouvert d'existence d'une limite sous des conditions libres au bord : nous démontrons que quand la longueur de l'intervalle, suivi par le nombre de lignes, tend vers l'infinie, les k lignes hautes convergent vers la même limite que dans le cas de conditions nulles au bord, comme conjecturé par Caputo, Ioffe et Wachtel.

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Random walks in cones revisited

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Dedicated to the memory of Dima Ioffe

Abstract. In this paper we continue our study of a multidimensional random walk with zero mean and finite variance killed on leaving a cone. We suggest a new approach that allows one to construct a positive harmonic function in Lipschitz cones under minimal moment conditions. This approach allows also to obtain more accurate information about the behaviour of the harmonic function not far from the boundary of the cone. We also prove limit theorems under new moment conditions.

Résumé. Dans cet article, nous poursuivons notre étude des marches aléatoires multidimensionnelles ayant dérive nulle, une variance finie, et tuées à la sortie d'un cône. Nous proposons une nouvelle approche, permettant de construire une fonction harmonique positive lorsque le cône possède une régularité Lipschitz et sous des conditions minimales de moments des accroissements de la marche aléatoire. Cette approche permet également de décrire précisément le comportement de la fonction harmonique au voisinage du bord du cône. Nous prouvons finalement des théorèmes limites, sous ces nouvelles hypothèses de moments.

MSC2020 subject classifications: Primary 60G50; secondary 60G40; 60F17

Keywords: Random walk; Exit time; Harmonic function; Conditioned process

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Ornstein–Zernike behavior for Ising models with infinite-range interactions

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In memory of Dima Ioffe a brilliant mathematician and, above all, a great friend

Abstract. We prove Ornstein–Zernike behavior for the large-distance asymptotics of the two-point function of the Ising model above the critical temperature under essentially optimal assumptions on the interaction. The main contribution of this work is that the interactions are not assumed to be of finite range. To the best of our knowledge, this is the first proof of OZ asymptotics for a nontrivial model with infinite-range interactions.

Our results actually apply to the Green function of a large class of "self-repulsive in average" models, including a natural family of self-repulsive polymer models that contains, in particular, the self-avoiding walk, the Domb–Joyce model and the killed random walk.

We aimed at a pedagogical and self-contained presentation.

Résumé. Nous prouvons, sous des hypothèses essentiellement optimales sur l'interaction, que le comportement asymptotique de la fonction à 2-point du modèle d'Ising au-dessus de sa température critique prend la forme prédite par Ornstein et Zernike. La contribution principale de ce travail est que nous ne supposons pas l'interaction de portée finie. À notre connaissance, il s'agit de la première preuve du comportement Ornstein–Zernike pour un modèle non trivial avec des interactions de portée infinie.

Nos résultats s'appliquent plus généralement à la fonction de Green d'une grande classe de modèle « auto-répulsifs en moyenne », incluant une famille naturelle de modèles de polymère auto-répulsif à laquelle appartiennent, en particulier, la marche aléatoire auto-évitante, le modèle de Domb–Joyce et la marche aléatoire tuée.

Nous nous sommes efforcés de rendre notre présentation aussi pédagogique et complète que possible.

MSC2020 subject classifications: Primary 60K35; 82B20; secondary 82D60

Keywords: Ising model; Polymers; Long-range interactions; Green function; Ornstein–Zernike asymptotics

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On the rate of convergence to coalescing Brownian motions

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In memory of a dear friend Dima Ioffe

Abstract. In this paper we study the rate of convergence of the iterates of i.i.d. random piecewise constant monotone maps to the time-1 transport map for the process of coalescing Brownian motions. We prove that the rate of convergence is given by a power law. The time-1 map for the coalescing Brownian motions can be viewed as a fixed point for a natural renormalization transformation acting in the space of probability laws for random piecewise constant monotone maps. Our result implies that this fixed point is exponentially stable.

Résumé. Dans cet article nous étudions le taux de convergence des itérées de fonctions aléatoires monotones et constantes par morceaux i.i.d. vers l'application de transport du mouvement Brownien coalescents au temps 1. Nous prouvons que le taux de convergence est donné par une loi de puissance. Cette application de transport peut être vue comme le point fixe d'une transformation de renormalisation naturelle agissant sur l'espace des lois de probabilité pour les fonctions aléatoires monotones et constantes par morceaux. Notre résultat implique que ce point fixe est exponentiellement stable.

MSC2020 subject classifications: 60J65; 60J05; 37H30; 37A25

Keywords: Coalescing Brownian motions; Rate of convergence; Coupling; Random monotone maps

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Long-range models in 1D revisited

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Abstract. In this short note, we revisit a number of classical results on long-range 1D percolation, Ising model and Potts models (*Comm. Math. Phys.* **84** (1982) 87–101; *Comm. Math. Phys.* **104** (1986) 547–571; *J. Stat. Phys.* **50** (1988) 1–40; *Comm. Math. Phys.* **118** (1988) 303–336). More precisely, we show that for Bernoulli percolation, FK percolation and Potts models, there is symmetry breaking for the $1/r^2$-interaction at large β, and that the phase transition is necessarily discontinuous. We also show, following the notation of (*J. Stat. Phys.* **50** (1988) 1–40) that $\beta^*(q) = 1$ for all $q \geq 1$.

Résumé. Dans cette courte note, nous revisitons un certain nombre de résultats classiques sur la percolation unidimensionnelle à longue portée, le modèle d'Ising et les modèles de Potts (*Comm. Math. Phys.* **84** (1982) 87–101 ; *Comm. Math. Phys.* **104** (1986) 547–571 ; *J. Stat. Phys.* **50** (1988) 1–40 ; *Comm. Math. Phys.* **118** (1988) 303–336). Plus précisément, nous montrons que pour la percolation de Bernoulli, la percolation FK et les modèles de Potts, il y a une rupture de symétrie pour l'interaction en $1/r^2$ lorsque β est suffisamment grand, et la transition de phase est nécessairement discontinue. Nous montrons également, en suivant la notation de (*J. Stat. Phys.* **50** (1988) 1–40), que $\beta^*(q) = 1$ pour tous les $q \geq 1$.

MSC2020 subject classifications: 60K35; 82B20; 82B43

Keywords: Percolation; Long-range; Renormalization; Critical; One-dimension

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Random field induced order in two dimensions

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In memory of Dima Ioffe (1963–2020), friend and mentor.

Abstract. In this article we prove that a classical XY model subjected to weak i.i.d. random field pointing in a fixed direction exhibits residual magnetic order in $\mathbb{Z}^2$ and aligns perpendicular to the random field direction. The paper is a sequel to (*J. Stat. Phys.* **142** (2011) 11–42) where the three-dimensional case was treated. Our approach is based on a multi-scale Peierls contour argument developed in (*J. Stat. Phys.* **142** (2011) 11–42). On the microscopic scale we extract energetic costs from the occurrence of contours, which themselves are defined on a macroscopic scale. The technical challenges in $\mathbb{Z}^2$ stem from difficulties controlling the size and roughness of the fluctuation fields which model the short length-scale oscillations of near-optimizers of the random field Hamiltonian.

Résumé. Dans cet article nous prouvons qu'un modèle XY classique soumis à un faible champ aléatoire i.i.d. pointant dans une direction fixe présente un ordre magnétique résiduel en $\mathbb{Z}^2$ et s'aligne perpendiculairement à la direction du champ aléatoire. Cet article est une suite de (*J. Stat. Phys.* **142** (2011) 11–42) où le cas tridimensionnel a été traité. Notre approche est basée sur un argument de contour de Peierls multi-échelle développé dans (*J. Stat. Phys.* **142** (2011) 11–42). À l'échelle microscopique, nous extrayons les coûts énergétiques de l'apparition de contours, eux-mêmes définis à l'échelle macroscopique. Les défis techniques de $\mathbb{Z}^2$ proviennent des difficultés à contrôler la taille et la rugosité des champs de fluctuation qui modélisent les oscillations à courte échelle des quasi-optimiseurs de l'hamiltonien avec champ aléatoire.

MSC2020 subject classifications: 82D40

Keywords: XY model; Random field; Mermin-Wagner theorem; Residual magnetic ordering; Spin-flop transition

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Near-maxima of the two-dimensional discrete Gaussian free field

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Abstract. We consider the Discrete Gaussian Free Field (DGFF) in domains $D_N \subseteq \mathbb{Z}^2$ arising, via scaling by N, from nice domains $D \subseteq \mathbb{R}^2$. We study the statistics of the values order $\sqrt{\log N}$ below the absolute maximum. Encoded as a point process on $D \times \mathbb{R}$, the scaled spatial distribution of these near-extremal level sets in D_N and the field values (in units of $\sqrt{\log N}$ below the absolute maximum) tends, as $N \to \infty$, in law to the product of the critical Liouville Quantum Gravity (cLQG) Z^D and the Rayleigh law. The convergence holds jointly with the extremal process, for which Z^D enters as the intensity measure of the limiting Poisson point process, and that of the DGFF itself; the cLQG defined by the limit field then coincides with Z^D. While the limit near-extremal process is measurable with respect to the limit continuum GFF, the limit extremal process is not. Our results explain why the various ways to "norm" the lattice cLQG measure lead to the same limit object, modulo overall normalization.

Résumé. Nous considérons le champ libre gaussien discret (DGFF) dans des domaines $D_N \subseteq \mathbb{Z}^2$ qu'on obtient, via une mise à l'échelle par N, à partir de domaines raisonnables $D \subseteq \mathbb{R}^2$. Nous étudions les statistiques des valeurs d'ordre $\sqrt{\log N}$ en dessous du maximum absolu. Encodés dans un processus ponctuel sur $D \times \mathbb{R}$, la distribution spatiale mise à l'échelle de ces ensembles de niveaux proches du maximum sur D_N et les valeurs du champ (en unités de $\sqrt{\log N}$ en dessous du maximum absolu) convergent, pour $N \to \infty$, en loi vers le produit de la gravité quantique critique de Liouville (cLQG) Z^D et de la loi de Rayleigh. La convergence est valable conjointement avec le processus extrémal, pour lequel Z^D représente la mesure d'intensité du processus ponctuel de Poisson qui apparait à la limite, et avec le DGFF lui-même ; le cLQG défini par le champ limite coïncide alors avec Z^D. Bien que le processus limite des valeurs proches du maximum soit mesurable par rapport au GFF (continu) limite, le processus limite des valeurs proches du maximum ne l'est pas. Nos résultats expliquent pourquoi les différentes manières de *régulariser* la mesure cLQG du réseau conduisent au même objet limite, à une normalisation globale près.

MSC2020 subject classifications: 60G70; 60G60; 60G15; 60G57

Keywords: Gaussian Free Field; Log correlated fields; Liouville quantum gravity; Extreme value theory

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Multivariate normal approximation for traces of orthogonal and symplectic matrices

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Abstract. We show that the distance in total variation between $\left(\operatorname{Tr} U, \frac{1}{\sqrt{2}} \operatorname{Tr} U^2, \ldots, \frac{1}{\sqrt{m}} \operatorname{Tr} U^m\right)$ and a real Gaussian vector, where U is a Haar distributed orthogonal or symplectic matrix of size $2n$ or $2n+1$, is bounded by $\Gamma\left(2\frac{n}{m}+1\right)^{-\frac{1}{2}}$ times a correction. The correction term is explicit and holds for all $n \geq m^4$, for m sufficiently large. For $n \geq m^3$ we obtain the bound $\left(\frac{n}{m}\right)^{-c\sqrt{\frac{n}{m}}}$ with an explicit constant c. Our method of proof is based on an identity of Toeplitz + Hankel determinants due to Basor and Ehrhardt, see (*Oper. Matrices* **3** (2009) 167–86), which is also used to compute the joint moments of the traces.

Résumé. Nous montrons que la distance en variation totale entre $\left(\operatorname{Tr} U, \frac{1}{\sqrt{2}} \operatorname{Tr} U^2, \ldots, \frac{1}{\sqrt{m}} \operatorname{Tr} U^m\right)$ et un vecteur gaussien réel, où U est une matrice orthogonale ou symplectique distribuée selon la mesure de Haar et de taille $2n$ ou $2n+1$, est bornée par $\Gamma\left(2\frac{n}{m}+1\right)^{-\frac{1}{2}}$ fois une correction. Cette correction est explicite et valable pour tout $n \geq m^4$, pour m suffisamment grand. Lorsque $n \geq m^3$ nous obtenons la borne $\left(\frac{n}{m}\right)^{-c\sqrt{\frac{n}{m}}}$ où c est une constante explicite. Notre méthode de démonstration repose sur une identité de déterminants du type Toeplitz + Hankel due à Basor et Ehrhardt, voir (*Oper. Matrices* **3** (2009) 167–86), qui est aussi utilisée pour calculer les moments joints des traces.

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Keywords: Multivariate Gaussian approximation; Toeplitz determinants; Hankel determinants

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Short- and long-time path tightness of the continuum directed random polymer

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Abstract. We consider the point-to-point continuum directed random polymer (CDRP) model that arises as a scaling limit from $(1+1)$-dimensional directed polymers in the intermediate disorder regime. We show that the annealed law of a point-to-point CDRP of length t converges to the Brownian bridge under diffusive scaling when $t \downarrow 0$. In case that t is large, we show that the transversal fluctuations of point-to-point CDRP are governed by the $2/3$ exponent. More precisely, as t tends to infinity, we prove tightness of the annealed path measures of point-to-point CDRP of length t upon scaling the length by t and fluctuations of paths by $t^{2/3}$. The $2/3$ exponent is tight such that the one-point distribution of the rescaled paths converges to the geodesic of the directed landscape. This point-wise convergence can be enhanced to process-level modulo a conjecture. Our short- and long-time tightness results also extend to point-to-line CDRP. In the course of proving our main results, we establish quantitative versions of quenched modulus of continuity estimates for long-time CDRP which are of independent interest.

Résumé. Nous considérons le modèle de polymère dirigé continu point à point (CDRP) qui se présente comme une limite d'échelle des polymères dirigés de dimension $(1+1)$ dans le régime du désordre intermédiaire. Nous montrons que la loi recuite d'un CDRP point à point de longueur t sous une renormalisation diffusive converge vers le pont brownien lorsque $t \downarrow 0$. Dans le cas où t est grand, nous montrons que les fluctuations transversales d'un CDRP point à point sont régies par l'exposant $2/3$. Plus précisément, lorsque t tend vers l'infini, nous prouvons la tension de la mesure recuite des trajectoires de CDRP point-à-point de longueur t en renormalisant la longueur par t et les fluctuations des trajectoires par $t^{2/3}$. L'exposant $2/3$ est tendu de telle sorte que la loi en un point des trajectoires renormalisées converge vers la géodésique du paysage dirigé. Cette convergence ponctuelle peut être améliorée au niveau du processus modulo une conjecture. Nos résultats de tension à temps court et à temps long s'étendent également au CDRP point-à-ligne. Au cours de la démonstration de nos principaux résultats, nous établissons des versions quantitatives des estimations trempées du module de continuité pour le CDRP à temps long, qui présentent un intérêt indépendant.

MSC2020 subject classifications: Primary 60K37; 82B21; secondary 82D60

Keywords: Directed Polymer; Kardar–Parisi–Zhang equation; Stochastic heat equation; Brownian bridge

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Martingale solutions to the stochastic thin-film equation in two dimensions

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Abstract. We construct solutions to the stochastic thin-film equation with quadratic mobility and Stratonovich gradient noise in the physically relevant dimension $d = 2$ and allow in particular for solutions with non-full support. The construction relies on a Trotter–Kato time-splitting scheme, which was recently employed in $d = 1$. The additional analytical challenges due to the higher spatial dimension are overcome using α-entropy estimates and corresponding tightness arguments.

Résumé. Nous construisons des solutions de l'équation aux dérivées partielles stochastique des couches minces avec une mobilité quadratique et un forçage stochastique gradient de type Stratonovich en dimension $d = 2$, physiquement pertinente. Les solutions à support non plein sont autorisées. La construction repose sur une méthode de Trotter–Kato en fractionnant l'intervalle de temps, récemment utilisée dans le cas $d = 1$. Les difficultés supplémentaires, dues à la dimension spatiale supérieure, sont surmontées à l'aide d'estimations de l'α-entropie et d'arguments de tension correspondants.

MSC2020 subject classifications: 35R60; 76A20

Keywords: Thin-film equation; Noise; α-Entropy estimates; Stochastic compactness method

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Stochastic maximal $L^p(L^q)$-regularity for second order systems with periodic boundary conditions

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Abstract. In this paper we consider an SPDE where the leading term is a second order operator with periodic boundary conditions, coefficients which are measurable in (t, ω), and Hölder continuous in space. Assuming stochastic parabolicity conditions, we prove $L^p((0, T) \times \Omega, t^\kappa \, dt; H^{\sigma,q}(\mathbb{T}^d))$-estimates. The main novelty is that we do not require $p = q$. Moreover, we allow arbitrary $\sigma \in \mathbb{R}$ and weights in time. Such mixed regularity estimates play a crucial role in applications to nonlinear SPDEs which is clear from our previous work. To prove our main results we develop a general perturbation theory for SPDEs. Moreover, we prove a new result on pointwise multiplication in spaces with fractional smoothness.

Résumé. Dans cet article, nous considérons une EDPS où le terme dominant est un opérateur du second ordre avec des conditions aux limites périodiques, des coefficients mesurables en (t, ω) et de regularité Hölderienne en espace. En supposant des conditions de parabolicité stochastique, nous prouvons des estimations de type $L^p((0, T) \times \Omega, t^\kappa \, dt; H^{\sigma,q}(\mathbb{T}^d))$. La principale nouveauté est que nous n'avons pas besoin de $p = q$. De plus, nous autorisons $\sigma \in \mathbb{R}$ à être arbitraire et des poids en temps. De telles estimations de régularité mixtes jouent un rôle crucial dans les applications aux EDPS non linéaires, ce qui ressort clairement de nos travaux précédents. Pour prouver nos principaux résultats, nous développons une méthode générale perturbative pour les EDPS. De plus, nous prouvons un nouveau résultat sur la multiplication ponctuelle dans des espaces à régularité fractionnaire.

MSC2020 subject classifications: Primary 60H15; secondary 60H15; 35B65; 42B37; 46F10; 47D06

Keywords: Stochastic maximal regularity; Stochastic evolution equations; Second order operators; Periodic boundary conditions; Perturbation theory; Pointwise multipliers

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The large-time and vanishing-noise limits for entropy production in nondegenerate diffusions

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Abstract. We investigate the behaviour of a family of entropy production functionals associated to stochastic differential equations of the form

$$\mathrm{d}X_s = -\nabla V(X_s)\,\mathrm{d}s + b(X_s)\,\mathrm{d}s + \sqrt{2\epsilon}\,\mathrm{d}W_s,$$

where b is a globally Lipschitz nonconservative vector field keeping the system out of equilibrium, with emphasis on the large-time limit and then the vanishing-noise limit. Different members of the family correspond to different choices of boundary terms. Our analysis yields a law of large numbers and a local large deviation principle which does not depend on the choice of boundary terms and which exhibits a Gallavotti–Cohen symmetry. We use techniques from the theory of semigroups and from semiclassical analysis to reduce the description of the asymptotic behaviour of the functional to the study of the leading eigenvalue of a quadratic approximation of a deformation of the infinitesimal generator near critical points of V.

Résumé. Nous étudions le comportement d'une famille de fonctionnelles de production d'entropie associée aux équations différentielles stochastiques de la forme

$$\mathrm{d}X_s = -\nabla V(X_s)\,\mathrm{d}s + b(X_s)\,\mathrm{d}s + \sqrt{2\epsilon}\,\mathrm{d}W_s,$$

où b est un champ vectoriel Lipschitzien non conservatif qui maintient le système hors équilibre, en mettant l'accent sur la limite en temps long et la limite du bruit disparaissant, dans cet ordre. Les différents membres de la famille correspondent à des choix différents de termes de bords. Notre analyse donne une loi des grands nombres et un principe local de grandes déviations qui ne dépend pas du choix des termes de bords et qui présente une symétrie de Gallavotti–Cohen. Nous utilisons des techniques issues de la théorie des semigroupes et de l'analyse semi-classique pour réduire la description du comportement asymptotique de la fonctionnelle à l'étude à celle de la valeur propre principale d'une approximation quadratique d'une déformation du générateur infinitésimal près des points critiques de V.

MSC2020 subject classifications: Primary 82C31; 82C35; secondary 60H10; 47D08

Keywords: Time reversal; Large deviations; Leading eigenvalue; Feynman–Kac semigroup; Semiclassical limit

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Λ-Linked coupling for Langevin diffusions

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Abstract. We ask when a sample path construction is Markovian and when it forms a Λ-link. This question leads us to develop (1) a general construction of intertwined duality via Liggett duality, and (2) a realization of diffusion processes in the form of stochastic flow. We can present examples of Λ-link for the Langevin diffusion when a higher dimension is considered. In particular, a constructive framework requires the Skorohod equations in all examples. An example includes Pitman's theorem for one-dimensional Brownian motion.

Résumé. Nous demandons quand un processus construit à partir d'une trajectoire échantillonnée est markovien et quand il forme un Λ-lien. Cette question nous amène à développer (1) une construction générale de la dualité entrelacée via la dualité de Liggett, et (2) une réalisation de processus de diffusion sous forme de flux stochastique. Nous pouvons présenter des exemples de Λ-lien pour la diffusion de Langevin lorsqu'une dimension supérieure est considérée. En particulier, un cadre constructif exige les équations de Skorohod dans tous les exemples. Un exemple inclut le théorème de Pitman pour le mouvement brownien unidimensionnel.

MSC2020 subject classifications: Primary 60J25; secondary 60J60

Keywords: Intertwining duality; Langevin diffusion; Stochastic flow; Sample path construction; Time-reversed Markov process

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Large deviations for singularly interacting diffusions

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Abstract. In this paper we prove a large deviation principle (LDP) for the empirical measure of a general system of mean-field interacting diffusions with singular drift (as the number of particles tends to infinity) and show convergence to the associated McKean–Vlasov equation. Along the way, we prove an extended version of the Varadhan Integral Lemma for a discontinuous change of measure and subsequently a LDP for Gibbs and Gibbs-like measures with singular potentials.

Résumé. Dans cet article, nous prouvons un principe des grandes déviations (PGD) pour la mesure empirique d'un système général de diffusions interagissant en champ moyen avec une dérive singulière (lorsque le nombre de particules tend vers l'infini) et montrons la convergence vers l'équation de McKean–Vlasov associée. En cours de route, nous prouvons une version étendue du lemme intégral de Varadhan pour un changement discontinu de mesure et par la suite un PGD pour les mesures de type Gibbs avec des potentiels singuliers.

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Reflected random walks and unstable Martin boundary

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Abstract. We introduce a family of two-dimensional reflected random walks in the positive quadrant and study their Martin boundary. While the minimal boundary is systematically equal to a union of two points, the full Martin boundary exhibits an instability phenomenon, in the following sense: if some parameter associated to the model is rational (resp. non-rational), then the Martin boundary is countable, homeomorphic to $\mathbb{Z} \cup \{\pm\infty\}$ (resp. uncountable, homeomorphic to $\mathbb{R} \cup \{\pm\infty\}$). Such instability phenomena are very rare in the literature. Along the way of proving this result, we obtain several precise estimates for the Green functions of reflected random walks with escape probabilities along the boundary axes and an arbitrarily large number of inhomogeneity domains. Our methods mix probabilistic techniques and an analytic approach for random walks with large jumps in dimension two.

Résumé. Nous introduisons une famille de marches aléatoires en dimension deux, réfléchies au bord du quart de plan positif, et étudions leur frontière de Martin. Tandis que leur frontière minimale est systématiquement une union de deux points, nous montrons que la frontière de Martin complète est intrinsèquement instable, au sens suivant : lorsqu'un certain paramètre associé au modèle s'avère rationnel (respectivement non rationnel), la frontière de Martin est alors dénombrable et homéomorphe à $\mathbb{Z} \cup \{\pm\infty\}$ (respectivement non dénombrable et homéomorphe à $\mathbb{R} \cup \{\pm\infty\}$). De tels phénomènes d'instabilité sont rares dans la littérature. Les démonstrations contiennent plusieurs estimées précises pour des fonctions de Green de marches aléatoires réfléchies avec probabilité de fuite le long des axes, possédant en outre un nombre infini de domaines d'inhomogénéité. Nos méthodes mélangent des techniques probabilistes avec une approche analytique pour des marches aléatoires avec grands pas en dimension deux.

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Maxima of a random model of the Riemann zeta function over intervals of varying length

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Abstract. We consider a model of the Riemann zeta function on the critical axis and study its maximum over intervals of length $(\log T)^{\theta}$, where θ is either fixed or tends to zero at a suitable rate. It is shown that the deterministic level of the maximum interpolates smoothly between the ones of log-correlated variables and of i.i.d. random variables, exhibiting a smooth transition 'from $\frac{3}{4}$ to $\frac{1}{4}$' in the second order. This provides a natural context where extreme value statistics of log-correlated variables with *time-dependent variance and rate* occur. A key ingredient of the proof is a precise upper tail tightness estimate for the maximum of the model on intervals of size one, that includes a Gaussian correction. This correction is expected to be present for the Riemann zeta function and pertains to the question of the correct order of the maximum of the zeta function in large intervals.

Résumé. Nous considérons un modèle aléatoire des valeurs de la fonction zêta de Riemann sur sa droite critique pour en étudier le maximum sur des intervalles de longueur $(\log T)^{\theta}$, où θ est soit fixé, soit tend vers zéro à une vitesse spécifiquement calibrée. Nous établissons que la valeur déterministe attendue de ce maximum passe continûment de celle obtenue pour des variables log-corrélées à celle de variables i.i.d., effectuant une transition 'de 3/4 à 1/4' au second ordre. Il s'agit d'un contexte naturel pour l'étude des valeurs extrêmes de variables log-corrélées avec un taux et une variance variables en temps. Un ingrédient-clef de notre preuve est une estimation fine de la queue de distribution du maximum sur des intervalles de longueur unité, qui inclut une correction de type gaussien. Un telle correction est également escomptée dans les statistiques du maximum de la véritable fonction zêta sur des intervalles aléatoires de taille unité, ce qui est lié à la question analogue du maximum de la fonction zêta sur des intervalles plus longs de la droite critique.

MSC2020 subject classifications: Primary 11M06; 60G70; secondary 60J80

Keywords: Extreme value theory; Riemann zeta function; Branching random walk

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Asymptotic enumeration and limit laws for multisets: The subexponential case

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Abstract. For a given combinatorial class $\mathcal{C}$ we study the class $\mathcal{G} = \textsc{Mset}(\mathcal{C})$ satisfying the multiset construction, that is, any object in $\mathcal{G}$ is uniquely determined by a set of $\mathcal{C}$-objects paired with their multiplicities. For example, $\textsc{Mset}(\mathbb{N})$ is (isomorphic to) the class of number partitions of positive integers, a prominent and well-studied case. The multiset construction appears naturally in the study of unlabelled objects, for example graphs or various structures related to number partitions. Our main result establishes the asymptotic size of the set $\mathcal{G}_{n,N}$ that contains all multisets in $\mathcal{G}$ having size n and being comprised of N objects from $\mathcal{C}$, as n *and* N tend to infinity and when the counting sequence of $\mathcal{C}$ is governed by subexponential growth. Moreover, we study the component distribution of random objects from $\mathcal{G}_{n,N}$ and we discover a phenomenon that we baptise *extreme condensation*: taking away the largest component as well as all the components of the smallest possible size, we are left with an object which converges in distribution as $n, N \to \infty$. The distribution of the limiting object is also retrieved. Moreover and rather surprisingly, in stark contrast to analogous results for labelled objects, the results here hold uniformly in N.

Résumé. Étant donnée une classe combinatoire $\mathcal{C}$, nous étudions la classe $\mathcal{G} = \textsc{Mset}(\mathcal{C})$ satisfaisant la construction multiset, c'est-à-dire que tout objet dans $\mathcal{G}$ est déterminé de manière unique par un ensemble d'objets $\mathcal{C}$ appariés avec leurs multiplicités. Par exemple, $\textsc{Mset}(\mathbb{N})$ est (isomorphe à) la classe des partitions de nombres entiers positifs, un cas important et bien étudié. La construction multiset apparaît naturellement dans l'étude des objets non étiquetés, par exemple les graphes ou diverses structures liées aux partitions de nombres. Notre résultat principal établit la taille asymptotique de l'ensemble $\mathcal{G}_{n,N}$ qui contient tous les multisets dans $\mathcal{G}$ ayant une taille n et étant composés de N objets de $\mathcal{C}$, quand n *et* N tendent vers l'infini et lorsque la suite de comptage de $\mathcal{C}$ est gouvernée par une croissance sous-exponentielle. De plus, nous étudions la loi des composantes des objets aléatoires de $\mathcal{G}_{n,N}$ et nous découvrons un phénomène que nous baptisons *condensation extrême* : en enlevant la plus grande composante ainsi que toutes les composantes de la plus petite taille possible, on se retrouve avec un objet dont la loi converge quand $n, N \to \infty$. On récupère également la loi de l'objet limite. De plus, et de manière assez surprenante, en contraste saisissant avec les résultats analogues pour les objets étiquetés, les résultats obtenus ici sont vrais uniformément en N.

MSC2020 subject classifications: 05A16; 05A18; 05C30; 60C05; 60F05

Keywords: Asymptotic enumeration; Multisets; Weighted integer partitions; Limit theorem; Condensation; Benjamini–Schramm convergence

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Complexity of bipartite spherical spin glasses

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Abstract. This paper characterizes the annealed complexity of bipartite spherical spin glasses, both pure and mixed. This means we give exact variational formulas for the asymptotics of the expected numbers of critical points and of local minima. This problem was initially considered by Auffinger and Chen (*J. Stat. Phys.* **157** (2014) 40–59), who gave upper and lower bounds on this complexity. We find two surprising connections between pure bipartite and pure single-species spin glasses, which were studied by Auffinger, Ben Arous, and Černý (*Comm. Pure Appl. Math.* **66** (2013) 165–201). First, the local minima of any pure bipartite model lie primarily in a low-energy band, similar to the single-species case. Second, for a more restricted set of pure (p, q) bipartite models, the complexity matches exactly that of a pure $p+q$ single-species model.

Résumé. Cet article caractérise la complexité annealed des verres de spin sphériques bipartis, à la fois purs et mixtes. Nous donnons des formules exactes et variationnelles pour la limite des nombres de points critiques et de minima locaux. Ce problème a d'abord été considéré par Auffinger et Chen (*J. Stat. Phys.* **157** (2014) 40–59), qui ont donné des bornes supérieures et inférieures sur cette complexité. Nous trouvons deux connexions surprenantes entre les verres de spin purs et bipartis, et les purs et monospécifiques, qui ont été étudiés par Auffinger, Ben Arous, et Černý (*Comm. Pure Appl. Math.* **66** (2013) 165–201). D'abord, les minima locaux de n'importe quel modèle pur et biparti se situent principalement dans une bande de basse énergie, similaire au cas monospécifique. De plus, pour un ensemble restreint des modèles purs et bipartis de type (p, q), la complexité correspond exactement à celle d'un modèle pur et monospécifique de type $p+q$.

MSC2020 subject classifications: Primary 82B44; secondary 60G15; 60B20

Keywords: Bipartite spin glasses; Spherical spin glasses; Landscape complexity; Kac–Rice formula; Dyson equation

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Scaling limits of external multi-particle DLA on the plane and the supercooled Stefan problem

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Abstract. We consider (a variant of) the external multi-particle diffusion-limited aggregation (MDLA) process of ROSENSTOCK and MARQUARDT on the plane. Based on the findings of (*Ann. Probab.* **24** (1996) 559–598, *Arch. Ration. Mech. Anal.* **233** (2019) 643–699, Delarue, Nadtochiy and Shkolnikov (2019)) in one space dimension it is natural to conjecture that the scaling limit of the growing aggregate in such a model is given by the growing solid phase in a suitable "probabilistic" formulation of the single-phase supercooled Stefan problem for the heat equation. To address this conjecture, we first prove that the limit points of diffusively scaled MDLA systems are well-defined and described by absorbed Brownian motions. Then, we show that these limit points satisfy the equation that characterizes the growth rate of the solid phase in the supercooled Stefan problem with an inequality, which can be strict in general. This result provides the first rigorous answer to a question that has received much attention in the physics literature. In the course of the proof, we establish two additional results interesting in their own right: (i) the stability of a "crossing property" of planar Brownian motion and (ii) a rigorous connection between the probabilistic solutions to the supercooled Stefan problem and its classical and weak solutions.

Résumé. Nous considérons (une variante) du processus d'agrégation limitée par diffusion externe multi-particule (MDLA) de ROSENSTOCK et MARQUARDT dans le plan. Sur la base des résultats de (*Ann. Probab.* **24** (1996) 559–598, *Arch. Ration. Mech. Anal.* **233** (2019) 643–699, Delarue, Nadtochiy and Shkolnikov (2019)) en dimension un, il est naturel de conjecturer que la limite d'échelle de l'agrégat en croissance dans un tel modèle est donnée par la phase solide en croissance dans une formulation "probabiliste" appropriée du problème de Stefan de surfusion monophasé pour l'équation de la chaleur. Pour répondre à cette conjecture, nous prouvons d'abord que les points limites des systèmes MDLA à échelle diffusive sont bien définis et décrits par des mouvements browniens absorbés. Puis, nous montrons que ces points limites satisfont l'équation qui caractérise le taux de croissance de la phase solide dans le problème de Stefan de surfusion avec une inégalité, qui peut être stricte en général. Ce résultat fournit la première réponse rigoureuse à une question qui a reçu beaucoup d'attention dans la littérature physique. Au cours de la preuve, nous établissons deux résultats supplémentaires intéressants en soi : (i) la stabilité d'une "propriété de croisement" du mouvement brownien plan et (ii) une connexion rigoureuse entre les solutions probabilistes du problème de Stefan surfusion et ses solutions classiques et faibles.

MSC2020 subject classifications: 82C24; 35R35; 80A22; 55M25

Keywords: Crossing property of planar Brownian motion; Multi-particle diffusion-limited aggregation; Painleve-Kuratowski convergence; Probabilistic formulation; Scaling limit; Supercooled Stefan problem

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TASEP with a moving wall

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Abstract. We consider a totally asymmetric simple exclusion on $\mathbb{Z}$ with the step initial condition, under the additional restriction that the first particle cannot cross a deterministically moving wall. We prove that such a wall may induce asymptotic fluctuation distributions of particle positions of the form

$$\mathbb{P}\Big(\sup_{\tau\in\mathbb{R}}\big\{\mathrm{Airy}_2(\tau)-g(\tau)\big\}\le S\Big)$$

with arbitrary barrier functions g. This is the same class of distributions that arises as one-point asymptotic fluctuations of TASEPs with arbitrary initial conditions. Examples include Tracy–Widom GOE and GUE distributions, as well as a crossover between them, all arising from various particles behind a linearly moving wall.

We also prove that if the right-most particle is second class, and a linearly moving wall is shock-inducing, then the asymptotic distribution of the position of the second class particle is a mixture of the uniform distribution on a segment and the atomic measure at its right end.

Résumé. On considère un processus d'exclusion simple totalement asymétrique sur $\mathbb{Z}$ avec les sites $\{1,2,\dots\}$ vides et les autres occupés comme condition initiale, sous la restriction supplémentaire que la première particule ne peut pas traverser un mur qui évolue de façon déterministe. Nous prouvons que une telle paroi peut induire des distributions de fluctuations asymptotiques des positions de particules de la forme

$$\mathbb{P}\Big(\sup_{\tau\in\mathbb{R}}\big\{\mathrm{Airy}_2(\tau)-g(\tau)\big\}\le S\Big)$$

avec des fonctions barrières g arbitraires. Il s'agit de la même classe de distributions trouvée dans l'étude des fluctuations asymptotiques en un point des TASEP avec des conditions initiales arbitraires. Les exemples incluent les distributions Tracy–Widom GOE et GUE, ainsi que des interpolations de ces distributions, tous issus de diverses particules derrière un mur en mouvement linéaire.

Nous prouvons également que si la particule la plus à droite est de seconde classe, et une paroi mobile avec dynamique linéaire génère un choc, alors la distribution asymptotique de la position de la particule de deuxième classe est un mélange de la distribution uniforme sur un segment et de la mesure atomique à son extrémité droite.

MSC2020 subject classifications: 60K35

Keywords: Exclusion process; KPZ universality; Shock formation

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Isoperimetric lower bounds for critical exponents for long-range percolation

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Abstract. We study independent long-range percolation on $\mathbb{Z}^d$ where the vertices x and y are connected with probability $1 - e^{-\beta\|x-y\|^{-d-\alpha}}$ for $\alpha > 0$. Provided the critical exponents δ and $2-\eta$ defined by $\delta = \lim_{n\to\infty} \frac{-\log(n)}{\log(\mathbb{P}_{\beta_c}(|K_0|\geq n))}$ and $2-\eta = \lim_{x\to\infty} \frac{\log(\mathbb{P}_{\beta_c}(0\leftrightarrow x))}{\log(\|x\|)} + d$ exist, where K_0 is the cluster containing the origin, we show that

$$\delta \geq \frac{d+(\alpha\wedge 1)}{d-(\alpha\wedge 1)} \quad \text{and} \quad 2-\eta \geq \alpha \wedge 1.$$

The lower bound on δ is believed to be sharp for $d=1$, $\alpha \in [\frac{1}{3}, 1)$ and for $d=2$, $\alpha \in [\frac{2}{3}, 1]$, whereas the lower bound on $2-\eta$ is sharp for $d=1$, $\alpha \in (0,1)$, and for $\alpha \in (0,1]$ for $d>1$, and is not believed to be sharp otherwise. Our main tool is a connection between the critical exponents and the isoperimetry of cubes inside $\mathbb{Z}^d$.

Résumé. Nous étudions la percolation indépendante de longue portée sur $\mathbb{Z}^d$: les sommets x et y sont connectés avec probabilité $1 - e^{-\beta\|x-y\|^{-d-\alpha}}$ pour $\alpha > 0$. En supposant que les exposants critiques δ et $2-\eta$ définis par $\delta = \lim_{n\to\infty} \frac{-\log(n)}{\log(\mathbb{P}_{\beta_c}(|K_0|\geq n))}$ et $2-\eta = \lim_{x\to\infty} \frac{\log(\mathbb{P}_{\beta_c}(0\leftrightarrow x))}{\log(\|x\|)} + d$ existent, où K_0 est l'amas contenant l'origine, nous montrons que

$$\delta \geq \frac{d+(\alpha\wedge 1)}{d-(\alpha\wedge 1)} \quad \text{et} \quad 2-\eta \geq \alpha \wedge 1.$$

La borne inférieure sur δ est censée être précise pour $d=1$, $\alpha \in [\frac{1}{3}, 1)$ et pour $d=2$, $\alpha \in [\frac{2}{3}, 1]$, alors que la borne inférieure sur $2-\eta$ est précise pour $d=1$, $\alpha \in (0,1)$, et pour $\alpha \in (0,1]$ pour $d>1$: elle n'est probablement pas précise dans les autres cas. Notre outil principal est une relation entre les exposants critiques et l'isopérimétrie des cubes dans $\mathbb{Z}^d$.

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Keywords: Long-range percolation; Phase transition; Critical exponents

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Stochastic transport equation with singular drift

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Abstract. We prove existence, uniqueness and Sobolev regularity of weak solution of the Cauchy problem of the stochastic transport equation with drift in a large class of singular vector fields containing, in particular, the L^d class, the weak L^d class, as well as some vector fields that are not even in $L^{2+\varepsilon}_{\rm loc}$ for any $\varepsilon > 0$.

Résumé. On obtient des résultats sur l'existence, l'unicité et la régularité en sens de Sobolev de la solution faible au problème de Cauchy pour l'équation de transport stochastique avec dérive dans une large classe de champs vectoriels singuliers contenant, en particulier, la classe L^d, la classe L^d faible, ainsi que certains champs vectoriels qui n'appartiennent même pas à $L^{2+\varepsilon}_{\rm loc}$ pour $\varepsilon > 0$ arbitraire.

MSC2020 subject classifications: Primary 35R60; secondary 35A21

Keywords: Stochastic transport equation; Singular drift; Stochastic differential equations; Gradient estimates

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