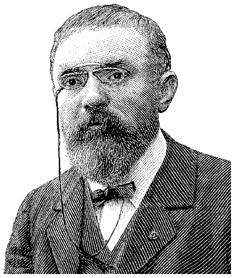

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Existence and uniqueness of the conformally covariant volume measure on conformal loop ensembles

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Abstract. We prove the existence and uniqueness of the canonical conformally covariant volume measure on the carpet/gasket of a conformal loop ensemble (CLE_κ, $\kappa \in (8/3, 8)$) which respects the Markov property for CLE. The starting point for the construction is the existence of the canonical measure on CLE in the context of Liouville quantum gravity (LQG) previously constructed by the first author with Sheffield and Werner. As a warm-up, we construct the natural parameterization of SLE_κ for $\kappa \in (4, 8)$ using LQG which serves to complement earlier work of Benoist on the case $\kappa \in (0, 4)$.

Résumé. Nous prouvons l'existence et l'unicité de la mesure de volume canonique et conformément covariante sur le tapis/tamis d'un ensemble de boucles conformes (CLE_κ, $\kappa \in (8/3, 8)$), et qui vérifie la propriété de Markov pour le CLE. Le point de départ de la construction est l'existence de la mesure canonique du CLE dans le contexte de la gravité quantique de Liouville (LQG) précédemment construite par le premier auteur ainsi que Sheffield et Werner. En guise de préambule, nous construisons la paramétrisation naturelle de SLE_κ pour $\kappa \in (4, 8)$ en utilisant LQG, ce qui complète le travail antérieur de Benoist dans le cas $\kappa \in (0, 4)$.

MSC2020 subject classifications: 60J67; 60G57

Keywords: Conformal loop ensembles; Schramm–Loewner evolution; Liouville quantum gravity

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On natural measures of SLE- and CLE-related random fractals

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Abstract. In this paper, we construct the natural measure on several types of random fractals, namely, the cut points and boundary-touching points of the Schramm–Loewner Evolution (SLE), as well as the pivotal points and the carpet/gasket of the Conformal Loop Ensemble (CLE), and then prove the uniqueness up to scalar multiples in each case. As an application, we show the equivalence between natural measures defined in this paper for CLE_6 pivotal points and gasket and scaling limits of their respective discrete analogs (i.e., the pivotal and area measure of the critical planar Bernoulli percolation constructed in [Garban-Pete-Schramm, J. Amer. Math. Soc., 2013]). Although the existence and uniqueness for the natural measures of the CLE carpet and gasket have already been proved in [Miller-Schoug, arXiv:2201.01748], we provide a different argument via the coupling of CLE and Liouville quantum gravity (LQG).

Résumé. Dans cet article, nous construisons la mesure naturelle sur plusieurs fractales aléatoires, à savoir les points de coupure et les points bord-touchants de l'evolution de Schramm–Loewner (SLE), les points pivots et le tapis/tamis de l'ensemble des boucles conformes (CLE). Puis nous prouvons l'unicité, à une constante multiplicative près, dans chaque cas. Comme application, nous montrons également l'équivalence entre les mesures naturelles définies dans cet article pour les points pivots et le tamis du CLE_6 et les limites d'échelle de leurs analogues discrets respectifs (c'est-à-dire la mesure pivot et la mesure de l'aire de la percolation de Bernoulli planaire critique construite dans [Garban-Pete-Schramm, J. Amer. Math. Soc., 2013]). Bien que l'existence et l'unicité des mesures naturelles du tapis et du tamis du CLE aient déjà été prouvées dans [Miller-Schoug, arXiv:2201.01748], nous proposons ici un argument différent via le couplage du CLE et la gravité quantique de Liouville (LQG).

MSC2020 subject classifications: Primary 60J67; 60G57; secondary 28A80

Keywords: Schramm–Loewner evolution; Liouville quantum gravity

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Critical Gaussian multiplicative chaos revisited

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Abstract. We present new, short and self-contained proofs of the convergence (with an adequate renormalization) of four different sequences to the critical Gaussian Multiplicative Chaos: the derivative martingale, the critical martingale, the exponential of the mollified field and the subcritical Gaussian Multiplicative Chaos.

Résumé. Nous présentons des preuves courtes, originales et auto-contenues de la convergence (sous renormalization idoine) de quatres processus aléatoires vers le chaos multiplicatif gaussien critique : la martingale dérivée, la martingale critique, l'exponentielle du champs régularisé et le chaos multiplicatif sous-critique.

MSC2020 subject classifications: 60F99; 60G15; 82B99

Keywords: Random distributions; log-correlated fields; Gaussian Multiplicative Chaos

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When is the convex hull of a Lévy path smooth?

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Abstract. We characterise, in terms of their transition laws, the class of one-dimensional Lévy processes whose graph has a continuously differentiable (planar) convex hull. We show that this phenomenon is exhibited by a broad class of infinite variation Lévy processes and depends subtly on the behaviour of the Lévy measure at zero. We introduce a class of strongly eroded Lévy processes, whose Dini derivatives vanish at every local minimum of the trajectory for all perturbations with a linear drift, and prove that these are precisely the processes with smooth convex hulls. We study how the smoothness of the convex hull can break and construct examples exhibiting a variety of smooth/non-smooth behaviours. Finally, we conjecture that an infinite variation Lévy process is either strongly eroded or abrupt, a claim implied by Vigon's point-hitting conjecture. In the finite variation case, we characterise the points of smoothness of the hull in terms of the Lévy measure.

Résumé. Nous caractérisons, en fonction de leurs lois de transition, la classe des processus de Lévy unidimensionnels dont le graphe a une enveloppe convexe (planaire) continûment dérivable. Nous montrons que ce phénomène est présent pour une large classe de processus de Lévy à variation infinie et dépend subtilement du comportement de la mesure de Lévy au voisinage de zéro. Nous introduisons une classe de processus de Lévy fortement érodés, dont les dérivées de Dini s'annulent à chaque minimum local de la trajectoire pour toutes les perturbations à dérive linéaire, et prouvons que ce sont précisément les processus à enveloppes convexes lisses. Nous étudions comment la régularité de l'enveloppe convexe peut cesser et construisons des exemples présentant une variété de comportements lisses/non lisses. Enfin, nous conjecturons qu'un processus de Lévy à variation infinie est soit fortement érodé, soit abrupt, une assertion impliquée par la conjecture *point-hitting* de Vigon. Dans le cas de la variation finie, nous caractérisons les points lisses de l'enveloppe en fonction de la mesure de Lévy.

MSC2020 subject classifications: 60G51

Keywords: Convex hull; Lévy process; Smoothness of convex minorant

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Effect of small noise on the speed of reaction-diffusion equations with non-Lipschitz drift

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Abstract. We consider the [0, 1]-valued solution $(u_{t,x} : t \geq 0, x \in \mathbb{R})$ to the one dimensional stochastic reaction diffusion equation with Wright-Fisher noise $\partial_t u = \partial_x^2 u + f(u) + \epsilon\sqrt{u(1-u)}\dot{W}$. Here, W is a space-time white noise, $\epsilon > 0$ is the noise strength, and f is a continuous function on [0, 1] satisfying $\sup_{z\in[0,1]} |f(z)|/\sqrt{z(1-z)} < \infty$. We assume the initial data satisfies $1 - u_{0,-x} = u_{0,x} = 0$ for x large enough. Recently, it was proved in (*Comm. Math. Phys.* **384** (2021) 699–732) that the front of u_t propagates with a finite deterministic speed $V_{f,\epsilon}$, and under slightly stronger conditions on f, the asymptotic behavior of $V_{f,\epsilon}$ was derived as the noise strength ϵ approaches ∞. In this paper we complement the above result by obtaining the asymptotic behavior of $V_{f,\epsilon}$ as the noise strength ϵ approaches 0: for a given $p \in [1/2, 1)$, if $f(z)$ is non-negative and is comparable to z^p for sufficiently small z, then $V_{f,\epsilon}$ is comparable to $\epsilon^{-2\frac{1-p}{1+p}}$ for sufficiently small ϵ.

Résumé. Nous considérons la solution $(u_{t,x} : t \geq 0, x \in \mathbb{R})$ à valeur dans l'intervalle [0, 1] de l'équation stochastique de réaction-diffusion unidimensionnelle avec un bruit de Wright-Fisher $\partial_t u = \partial_x^2 u + f(u) + \epsilon\sqrt{u(1-u)}\dot{W}$. Où W est un bruit blanc en espace et en temps, $\epsilon > 0$ est l'intensité du bruit et f est une fonction continue sur [0, 1] telle que $\sup_{z\in[0,1]} |f(z)|/\sqrt{z(1-z)} < \infty$. Nous supposons que la condition initiale satisfait $1 - u_{0,-x} = u_{0,x} = 0$ pour x suffisamment grand. Il a été récemment prouvé dans (*Comm. Math. Phys.* **384** (2021) 699–732) que le front de u_t se propage à une vitesse $V_{f,\epsilon}$ déterministe finie et, sous des conditions légèrement plus fortes pour f, le comportement asymptotique de $V_{f,\epsilon}$ est entièrement déterminé par l'intensité du bruit ϵ lorsqu'il tend vers l'infini. Dans cet article, nous complétons ces résultats en décrivant le comportement asymptotique de $V_{f,\epsilon}$ lorsque ϵ tend vers 0 : pour $p \in [1/2, 1)$, si $f(z)$ est positif et se comporte comme z^p pour z suffisamment petit, alors $V_{f,\epsilon}$ est équivalent à $\epsilon^{-2\frac{1-p}{1+p}}$ quand ϵ est suffisamment petit.

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Keywords: Reaction-diffusion equations; Stochastic partial differential equations; White noise; Traveling waves

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Prevalence of ρ-irregularity and related properties

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Abstract. We show that generic Hölder continuous functions are ρ-irregular. The property of ρ-irregularity has been first introduced by Catellier and Gubinelli (*Stochastic Process. Appl.* **126** (2016) 2323–2366) and plays a key role in the study of well-posedness for some classes of perturbed ODEs and PDEs. Genericity here is understood in the sense of *prevalence*. As a consequence we obtain several results on regularisation by noise "without probability", i.e. without committing to specific assumptions on the statistical properties of the perturbations. We also establish useful criteria for stochastic processes to be ρ-irregular and study in detail the geometric and analytic properties of ρ-irregular functions.

Résumé. On montre que génériquement les fonctions Höldériennes sont ρ-irrégulières. La propriété de ρ-irrégularité a été introduite par Catellier et Gubinelli (*Stochastic Process. Appl.* **126** (2016) 2323–2366) et elle joue un rôle important dans l'étude du caractère bien posé de certaines équations différentielles ordinaires et aux dérivées partielles perturbées. La généricité ici est à entendre dans le sens de la *prévalence*. Par conséquent, on obtient des résultats de régularisation par bruit « sans probabilité », c'est-à-dire, sans utiliser d'hypothèses spécifiques sur la nature aléatoire des perturbations. Nous établissons également des critères utiles pour que les processus stochastiques soient ρ-irréguliers et nous étudions en détail les propriétés géométriques et analytiques des fonctions ρ-irrégulières.

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Keywords: Prevalence; ρ-irregularity; Regularisation by noise; Pathwise approach; Roughness

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A stochastic reconstruction theorem

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Abstract. In a recent landmark paper, Khoa Lê (*Electron. J. Probab.* **25** (2020) 38) established a stochastic sewing lemma which since has found many applications in stochastic analysis. He further conjectured that a similar result may hold in the context of the reconstruction theorem within Hairer's regularity structures. The purpose of this article is to provide such a stochastic reconstruction theorem. We also discuss two variations of this theorem, motivated by different constructions of stochastic integration against white noise. Our formulation makes use of the distributional viewpoint of Caravenna–Zambotti (*EMS Surv. Math. Sci.* **7** (2020) 207–251).

Résumé. Dans un récent article important, Khoa Lê (*Electron. J. Probab.* **25** (2020) 38) a établi un lemme de couture stochastique qui a depuis trouvé de nombreuses applications en analyse stochastique. Khoa Lê a également émis l'hypothèse qu'un résultat similaire pourrait s'appliquer dans le contexte du théorème de reconstruction au sein des structures de régularité d'Hairer. Le but de cet article est de fournir un tel théorème de reconstruction stochastique. Nous discutons également deux variations de ce théorème, motivées par différentes constructions d'intégration stochastique pour un bruit blanc en espace-temps. Notre formulation utilise le point de vue distributionnel de Caravenna–Zambotti (*EMS Surv. Math. Sci.* **7** (2020) 207–251).

MSC2020 subject classifications: 60L99

Keywords: Reconstruction theorem; Regularity structure; Stochastic sewing

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A bilinear flory equation

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Abstract. We consider coagulation equations of Flory type where particles are represented by finite dimensional vectors and the coagulation rate between two particles of types x and y is given by a bilinear form $y \cdot Ax$, generalising the multiplicative kernel. For these coagulation rates, a gelation transition occurs at a finite time $t_{\mathrm{g}} \in (0, \infty)$, which can be given exactly in terms of an eigenvalue problem in finite dimensions. We prove a hydrodynamic limit for the corresponding stochastic coagulant, including the phase transition for the largest particle, and exploit a coupling to random graphs to extend analysis of the limiting process beyond the gelation time.

Résumé. On considère des équations de coagulation de type Flory où les particules sont représentées par des vecteurs de dimension finie et le taux de coagulation entre deux particules de types x et y est donné par une forme bilinéaire $y \cdot Ax$, généralisant le noyau multiplicatif. Pour ces taux de coagulation, une transition de gélification se produit à un temps fini $t_{\mathrm{g}} \in (0, \infty)$, qui peut être donné exactement en termes d'un problème aux valeurs propres en dimensions finies. Nous prouvons une limite hydrodynamique pour le coagulant stochastique correspondant, y compris la transition de phase pour la plus grande particule, et exploitons un couplage avec des graphes aléatoires pour étendre l'analyse du processus limite au-delà du temps de gélification.

MSC2020 subject classifications: Primary 60K35; secondary 82B40

Keywords: Coagulation Equation; Flory Equation; Stochastic Particle System; Random Graphs

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Percolation phase transition on planar spin systems

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Abstract. In this article we study the continuity and sharpness of the phase transition for percolation models defined on top of planar spin systems. The two examples that we treat in detail concern the Glauber dynamics for the Ising model and a Dynamic Bootstrap process. For both of these models we prove that their phase transition is continuous and sharp, providing also quantitative estimates on the two point connectivity. The techniques that we develop in this work can be applied to a variety of different percolation models based on spin-flip dynamics. We also discuss some of the problems that can be tackled in a similar fashion.

Résumé. Dans cet article, nous étudions la continuité et la caractère abrupt de la transition de phase pour des modèles de percolation définis au-dessus de systèmes de spin planaires. Les deux exemples que nous traitons en détail concernent la dynamique de Glauber pour le modèle d'Ising et un processus dynamique de type «bootstrap». Pour ces deux modèles, nous prouvons que leur transition de phase est continue et abrupte, fournissant également des estimations quantitatives sur la connectivité à deux points. Les techniques que nous développons dans ce travail peuvent être appliquées à une variété de modèles de percolation différents basés sur la dynamique spin-flip. Nous discutons également certains des problèmes qui peuvent être résolus de la même manière.

MSC2020 subject classifications: 60K35; 82C22; 82C27

Keywords: Percolation; Spin systems; Dependence

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Spectral analysis of a class of Lévy-type processes and connection with some spin systems

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Abstract. We consider a class of Lévy-type processes on which spectral analysis technics can be made to produce optimal results, in particular for the decay rate of their survival probability and for the spectral gap of their ground state transform. This class is defined by killed symmetric Lévy processes under general random time-changes satisfying some integrability assumptions. Our results reveal a connection between those processes and a family of spin systems. This connection, where the free energy and correlations play an essential role, is, up to our knowledge, new, and relates some key properties of objects from the two families. When the underlying Lévy process is a Brownian motion, the associated spin system turns out to have interactions of a rather nice form that are a natural alternative to the quadratic interactions from the lattice Gaussian Free Field. More general Lévy processes give rise to more general interactions in the associated spin systems.

Résumé. Nous considérons une classe de processus de type Lévy pour laquelle il est possible de mettre en oeuvre des méthodes d'analyse spectrale produisant des résultats optimaux, en particulier pour le taux de décroissance de leur probabilité de survie et pour le trou spectral de leur transformée par l'état fondamental. Cette classe est définie par des processus de Lévy symétriques tués sous des changements de temps aléatoires généraux satisfaisant des hypothèses d'intégralité. Nos résultats révèlent une connection entre ces processus et une famille de systèmes de spins. Cette connection, où l'énergie libre et les corrélations jouent un role essentiel, est à notre connaissance nouvelle et relie des propriétés clés des objets des deux familles. Lorsque le processus de Lévy sous-jacent est un mouvement brownien, le système de spins associé se trouve avoir des interactions d'une forme particulière qui offre une alternative naturelle aux interactions quadratiques du Champ Libre Gaussien discret. Des processus de Lévy plus généraux donnent lieu à des interactions plus générales dans le système de spins associé.

MSC2020 subject classifications: Primary 60G53; 60G51; 60J25; 60J35; 82B20; secondary 60J55; 82B31

Keywords: Lévy-type processes; Lévy processes with random time changes; Spectral analysis; Ground state; Ground state transform; Spectral gap; Survival probability; Spin systems; Interface models; Partition function; Free energy; Infinite-volume Gibbs states; Correlations

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On log-concave approximations of high-dimensional posterior measures and stability properties in non-linear inverse problems

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Abstract. The problem of efficiently generating random samples from high-dimensional and non-log-concave posterior measures arising in nonlinear regression problems is considered. Extending investigations from (*J. Eur. Math. Soc. (JEMS)* **26** (2024) 1031–1112), local and global stability properties of the model are identified under which such posterior distributions can be approximated in Wasserstein distance by suitable log-concave measures. This allows the use of fast gradient based sampling algorithms, for which convergence guarantees are established that scale polynomially in all relevant quantities (assuming 'warm' initialisation). The scope of the general theory is illustrated in a non-linear inverse problem from integral geometry for which new stability results are derived.

Résumé. On s'intéresse au problème de génération efficace d'échantillons à partir de mesures *a posteriori* de grande dimension et non log-concave, qui se pose dans des problèmes de régressions non linéaires. En développant les recherches de ((*J. Eur. Math. Soc. (JEMS)* **26** (2024) 1031–1112), on identifie des propriétés de stabilité locale et globale du modèle, sous lesquelles de telles distributions *a posteriori* peuvent être approximées dans la distance de Wasserstein par des mesures log-concaves appropriées. Ceci permet l'utilisation d'algorithmes d'échantillonnage rapides basés sur des gradients, pour lesquels on établit des garanties de convergence qui évoluent à une échelle polynomiale en toutes les quantités pertinentes (en supposant un *démarrage à chaud*). On illustre la portée de cette théorie générale dans un problème inverse non linéaire de géométrie intégrale, pour lequel on obtient de nouveaux résultats de stabilité.

MSC2020 subject classifications: Primary 62G05; 35R30; secondary 65Y20

Keywords: Non-linear inverse problems; Bayesian non-parametric statistics; Non-Abelian X-ray transform; Polynomial time sampling

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Inference for ergodic McKean–Vlasov stochastic differential equations with polynomial interactions

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Abstract. We consider a family of one-dimensional McKean–Vlasov stochastic differential equations with no potential term and with interaction term modeled by an odd increasing polynomial. We assume that the observed process is in stationary regime and that the sample path is continuously observed on a time interval $[0, 2T]$. Due to the McKean–Vlasov structure, the drift function depends on the unknown marginal law of the process in addition to the unknown parameters present in the interaction function. This is why the exact likelihood function does not lead to computable estimators. We overcome this difficulty by a two-step approach: We use the observations on $[0, T]$ to build empirical estimates of moments of the stationary distribution and on $[T, 2T]$ to build an approximate likelihood. We derive explicit estimators of the interaction term parameters, which are proved to be consistent and asymptotically normal with rate $\sqrt{T}$ as T grows to infinity. Examples illustrating the theory are proposed.

Résumé. Considérons une équation différentielle stochastique de McKean–Vlasov unidimensionnelle de potentiel nul et dont l'interaction est un polynôme impair croissant. Nous supposons que le processus est en régime stationnaire et qu'il est observé de façon continue sur $[0, 2T]$. La dérive dépend de la loi marginale inconnue du processus en plus des paramètres inconnus présents dans la fonction d'interaction. De ce fait, la vraisemblance ne permet pas d'obtenir des estimateurs calculables. Pour résoudre ce problème, nous procédons en deux temps : avec la trajectoire sur $[0, T]$, nous construisons des estimateurs empiriques des moments de la loi invariante, puis avec la trajectoire sur $[T, 2T]$ nous définissons une vraisemblance approchée. Nous obtenons ainsi des estimateurs explicites dont nous prouvons la convergence et la normalité asymptotique à la vitesse $\sqrt{T}$. La théorie est illustrée par plusieurs exemples.

MSC2020 subject classifications: 60J60; 60J99; 62F12; 62M05

Keywords: McKean–Vlasov stochastic differential equation; Continuous observations; Parametric inference; Invariant distribution; Asymptotic properties of estimators; Approximate likelihood; Long time asymptotics

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Higher order fluctuations of extremal eigenvalues of sparse random matrices

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Abstract. We consider extremal eigenvalues of sparse random matrices, a class of random matrices including the adjacency matrices of Erdős–Rényi graphs $\mathcal{G}(N, p)$. Recently, it was shown that the leading order fluctuations of extremal eigenvalues are given by a single random variable associated with the total degree of the graph (*Ann. Probab.* **48** (2020) 916–962; *Probab. Theory Related Fields* **180** (2021) 985–1056). We construct a sequence of random correction terms to capture higher (sub-leading) order fluctuations of extremal eigenvalues in the regime $N^\epsilon < pN < N^{1/3-\epsilon}$. Using these random correction terms, we prove a local law up to a shifted edge and recover the rigidity of extremal eigenvalues under some corrections for $pN > N^\epsilon$.

Résumé. Nous considérons les valeurs propres extrêmes des matrices aléatoires éparses, une classe de matrices aléatoires comprenant les matrices d'adjacence des graphes d'Erdős–Rényi $\mathcal{G}(N, p)$. Récemment, il a été démontré que les fluctuations d'ordre principal des valeurs propres extrêmes sont données par une seule variable aléatoire associée au degré total du graphe (*Ann. Probab.* **48** (2020) 916–962 ; *Probab. Theory Related Fields* **180** (2021) 985–1056). Nous construisons une suite de termes de correction aléatoires pour capter les fluctuations d'ordre supérieur (secondaire) des valeurs propres extrêmes dans le régime $N^\epsilon < pN < N^{1/3-\epsilon}$. En utilisant ces termes de correction aléatoires, nous prouvons une loi locale jusqu'à un bord décalé et retrouvons la rigidité des valeurs propres extrêmes sous certaines corrections pour $pN > N^\epsilon$.

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Keywords: Sparse random matrices; Extremal eigenvalues; Edge rigidity; Random correction terms

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Overlaps, eigenvalue gaps, and pseudospectrum under real Ginibre and absolutely continuous perturbations

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Abstract. Let G_n be an $n \times n$ matrix with real i.i.d. $N(0, 1/n)$ entries, let A be a real $n \times n$ matrix with $\|A\| \leq 1$, and let $\gamma \in (0, 1)$. We show that with probability 0.99, $A + \gamma G_n$ has all of its eigenvalue condition numbers bounded by $O(n^{5/2}/\gamma^{3/2})$ and eigenvector condition number bounded by $O(n^3/\gamma^{3/2})$. Furthermore, we show that for any $s > 0$, the probability that $A + \gamma G_n$ has two eigenvalues within distance at most s of each other is $O(n^4 s^{2/7}/\gamma^{5/2})$. In fact, we show the above statements hold in the more general setting of non-Gaussian perturbations with real, independent, absolutely continuous entries with a finite moment assumption and appropriate normalization.

This extends the previous work (Banks et al. (2019)) which proved an eigenvector condition number bound of $O(n^{3/2}/\gamma)$ for the simpler case of *complex* i.i.d. Gaussian matrix perturbations. The case of real perturbations introduces several challenges stemming from the weaker anticoncentration properties of real vs. complex random variables. A key ingredient in our proof is new lower tail bounds on the small singular values of the complex shifts $z - (A + \gamma G_n)$ which recover the tail behavior of the complex Ginibre ensemble when $\Im z \neq 0$. This yields sharp control on the area of the pseudospectrum $\Lambda_\varepsilon(A + \gamma G_n)$ in terms of the pseudospectral parameter $\varepsilon > 0$, which is sufficient to bound the overlaps and eigenvector condition number via a limiting argument.

Résumé. Soit G_n une matrice $n \times n$ avec des entrées réelles i.i.d. $N(0, 1/n)$, soit A une matrice réelle $n \times n$ avec $\|A\| \leq 1$ et $\gamma \in (0, 1)$. Nous montrons qu'avec une probabilité de 0,99, $A + \gamma G_n$ a tous ses conditionnements de valeurs propres bornés par $O(n^{5/2}/\gamma^{3/2})$ et son conditionnement de vecteurs propres borné par $O(n^3/\gamma^{3/2})$. De plus, nous montrons que pour tout $s > 0$, la probabilité que $A + \gamma G_n$ ait deux valeurs propres à une distance maximale de s l'une de l'autre est de $O(n^4 s^{2/7}/\gamma^{5/2})$. En fait, nous montrons que les énoncés ci-dessus sont valables dans le cadre plus général des perturbations non gaussiennes avec des entrées réelles, indépendantes et absolument continues, sous l'hypothèse de moment fini et d'une normalisation appropriée.

Ceci étend le travail précédent (Banks et al. (2019)) qui a prouvé une borne du conditionnement de vecteur propre de $O(n^{3/2}/\gamma)$ pour le cas plus simple de perturbations matricielles gaussiennes i.i.d. complexes. Le cas des perturbations réelles introduit plusieurs défis découlant des propriétés d'anti-concentration plus faibles des variables aléatoires réelles par rapport aux variables aléatoires complexes. Un ingrédient clé de notre preuve est une nouvelle borne inférieure de la queue des petites valeurs singulières des déplacements complexes $z - (A + \gamma G_n)$, qui caractérise le comportement de la queue de l'ensemble de Ginibre complexe lorsque $\Im(z) \neq 0$. Cela permet un contrôle précis de l'aire du pseudospectre $\Lambda_\epsilon(A + \gamma G_n)$ en termes de paramètre pseudospectral $\epsilon > 0$, ce qui est suffisant pour borner les chevauchements et le conditionnement du vecteur propre grâce à un argument de limite.

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Keywords: Pseudospectrum; Non-Hermitian random matrix theory; Ginibre ensemble

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Linear spectral statistics of eigenvectors of anisotropic sample covariance matrices

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Abstract. Consider sample covariance matrices of the form $Q := \Sigma^{1/2} X X^\top \Sigma^{1/2}$, where $X = (x_{ij})$ is an $n \times N$ random matrix whose entries are independent random variables with mean zero and variance N^{-1}, and Σ is a deterministic positive-definite covariance matrix. We study the limiting behavior of the eigenvectors of Q through the so-called eigenvector empirical spectral distribution $F_{\mathbf{v}}$, which is an alternative form of empirical spectral distribution with weights given by $|\mathbf{v}^\top \xi_k|^2$, where $\mathbf{v}$ is a deterministic unit vector and ξ_k are the eigenvectors of Q. We prove a functional central limit theorem for the linear spectral statistics of $F_{\mathbf{v}}$, indexed by functions with Hölder continuous derivatives. We show that the linear spectral statistics converge to some Gaussian processes both on global scales of order 1 and on local scales that are much smaller than 1 but much larger than the typical eigenvalue spacing N^{-1}. Moreover, we give explicit expressions for the covariance functions of the Gaussian processes, where the exact dependence on Σ and $\mathbf{v}$ is identified for the first time in the literature.

Résumé. Considérons des matrices de covariance empiriques de la forme $Q := \Sigma^{1/2} X X^\top \Sigma^{1/2}$, où $X = (x_{ij})$ est une matrice aléatoire $n \times N$ dont les entrées sont des variables aléatoires indépendantes centrées et de variance N^{-1}, et Σ est une matrice de covariance déterministe définie positive. Nous étudions le comportement limite des vecteurs propres de Q par le biais de ce que l'on appelle la loi spectrale empirique des vecteurs propres $F_{\mathbf{v}}$, qui est une forme alternative de la loi spectrale empirique avec des poids donnés par $|\mathbf{v}^\top \xi_k|^2$, où $\mathbf{v}$ est un vecteur unitaire déterministe et ξ_k sont les vecteurs propres de Q. Nous prouvons un théorème central limite fonctionnel pour les statistiques spectrales linéaires de $F_{\mathbf{v}}$, indexées par des fonctions de dérivées Höldériennes. Nous montrons que les statistiques spectrales linéaires convergent vers certains processus gaussiens à la fois sur des échelles globales d'ordre 1 et sur des échelles locales qui sont beaucoup plus petites que 1 mais beaucoup plus grandes que l'espacement typique des valeurs propres N^{-1}. De plus, nous donnons des expressions explicites pour les fonctions de covariance des processus gaussiens, où la dépendance exacte de Σ et $\mathbf{v}$ est identifiée pour la première fois dans la littérature.

MSC2020 subject classifications: Primary 15B52; 62E20; secondary 62H99

Keywords: Sample covariance matrix; Linear spectral statistics; Eigenvector empirical spectral distribution; Marčenko–Pastur distribution

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Phase transitions for infinite products of large non-Hermitian random matrices

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Abstract. Products of M i.i.d. non-Hermitian random matrices of size $N \times N$ relate Gaussian fluctuation of Lyapunov and stability exponents in dynamical systems (finite N and large M) to local eigenvalue universality in random matrix theory (finite M and large N). The remaining task is to study local eigenvalue statistics as M and N tend to infinity simultaneously, which lies at the heart of understanding two kinds of universal patterns. For products of i.i.d. complex Ginibre matrices, truncated unitary matrices and spherical ensembles, as $M + N \to \infty$ we prove that local statistics undergoes a transition when the relative ratio M/N changes from 0 to ∞: Ginibre statistics when $M/N \to 0$, normality when $M/N \to \infty$, and new critical phenomena when $M/N \to \gamma \in (0, \infty)$.

Résumé. Les produits de M matrices aléatoires non-Hermitiennes i.i.d. de taille $N \times N$ permettent de relier les fluctuations gaussiennes des exposants de Lyapunov et de stabilité dans les systèmes dynamiques (pour N fini et M grand) à l'universalité locale des valeurs propres en théorie des matrices aléatoires (pour M fini et N grand). La tâche restante consiste à étudier les statistiques locales des valeurs propres lorsque M et N tendent simultanément vers l'infini, ce qui est au cœur de la compréhension de deux types de modèles universels. Pour les produits de matrices de Ginibre complexes i.i.d., de matrices unitaires tronquées, et d'ensembles sphériques, quand $M + N \to \infty$, nous démontrons que les statistiques locales subissent une transition lorsque le rapport relatif M/N passe de 0 à ∞ : des statistiques de Ginibre quand $M/N \to 0$, une normalité quand $M/N \to \infty$, et de nouveaux phénomènes critiques quand $M/N \to \gamma \in (0, \infty)$.

MSC2020 subject classifications: 60B20

Keywords: Ginibre matrices; Ginibre statistics; Phase transition; Products of random matrices; Spherical ensembles; Truncated unitary matrices

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The contact process over a dynamical d-regular graph

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Abstract. We consider the contact process on a dynamic graph defined as a random d-regular graph with a stationary edge-switching dynamics. In this graph dynamics, independently of the contact process state, each pair $\{e_1, e_2\}$ of edges of the graph is replaced by new edges $\{e_1', e_2'\}$ in a crossing fashion: each of e_1', e_2' contains one vertex of e_1 and one vertex of e_2. As the number of vertices of the graph is taken to infinity, we scale the rate of switching in a way that any fixed edge is involved in a switching with a rate that approaches a limiting value v, so that locally the switching is seen in the same time scale as that of the contact process. We prove that if the infection rate of the contact process is above a threshold value $\bar{\lambda}$ (depending on d and v), then the infection survives for a time that grows exponentially with the size of the graph. By proving that $\bar{\lambda}$ is strictly smaller than the lower critical infection rate of the contact process on the infinite d-regular tree, we show that there are values of λ for which the infection dies out in logarithmic time in the static graph but survives exponentially long in the dynamic graph.

Résumé. Nous considérons le processus de contact sur un graphe dynamique défini comme un graphe aléatoire d-régulier avec une dynamique stationnaire de changement des arêtes. Dans cette dynamique de graphe, indépendamment de l'état du processus de contact, chaque paire $\{e_1, e_2\}$ d'arêtes du graphe est remplacée par de nouvelles arêtes $\{e_1', e_2'\}$ de manière croisée : chacun des e_1', e_2' contient un sommet de e_1 et un sommet de e_2. Lorsque le nombre de sommets du graphe est porté à l'infini, nous échelonnons le taux de commutation de manière à ce que toute arête fixe soit impliquée dans une commutation avec un taux qui se rapproche d'une valeur limite v, de sorte que localement la commutation est vue dans la même échelle de temps que celle du processus de contact. Nous prouvons que si le taux d'infection du processus de contact est supérieur à une valeur seuil $\bar{\lambda}$ (dépendant de d et v), alors l'infection survit pendant un temps qui croît de manière exponentielle avec la taille du graphe. En prouvant que $\bar{\lambda}$ est strictement inférieur au taux d'infection critique inférieur du processus de contact sur l'arbre infini d-régulier, nous montrons qu'il existe des valeurs de λ pour lesquelles l'infection s'éteint en temps logarithmique dans le graphe statique mais survit pendant un temps exponentiellement long dans le graphe dynamique.

MSC2020 subject classifications: Primary 60J85; 60K35; secondary 82C22

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From flip processes to dynamical systems on graphons

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Abstract. We introduce a class of random graph processes, which we call *flip processes*. Each such process is given by a function $\mathcal{R}:\mathcal{H}_k\to\mathcal{H}_k$ from all labeled k-vertex graphs into itself (k is fixed). The process starts with a given n-vertex graph G_0. In each step, the graph G_i is obtained by sampling k random vertices $v_1,\dots,v_k$ of G_{i-1} and replacing the induced graph $F:=G_{i-1}[v_1,\dots,v_k]$ by $\mathcal{R}(F)$. Actually, our definition is more general, in that $\mathcal{R}(F)$ is a probability distribution on $\mathcal{H}_k$, thus allowing randomised replacements.

Given a flip process $\mathcal{R}$, we construct time-indexed trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\to\mathcal{W}_0$ in the space of graphons. We prove that for any $T>0$ starting with a large n-vertex graph G_0 which is close to a graphon W_0, with high probability the flip process stays in a thin sausage around $(\Phi(W_0,t))_{t=0}^T$ (after rescaling the time by n^2).

Among others, we study continuity properties of these trajectories with respect to time and initial graphon, existence and stability of fixed points and speed of convergence. We give an example of a flip process with a periodic trajectory.

Résumé. Nous introduisons une classe de processus aléatoires sur graphes, que nous appelons *processus de retournement* (*flip processes*). Chacun de ces processus est donné par une *règle* qui est une fonction $\mathcal{R}:\mathcal{H}_k\to\mathcal{H}_k$ de tous les graphes à k sommets étiquetés vers lui-même (k est fixe). Le processus commence avec un graphe G_0 à n sommets. À chaque étape, le graphe G_i est obtenu en échantillonnant k sommets aléatoires $v_1,\dots,v_k$ de G_{i-1} et en remplaçant le graphe induit $F:=G_{i-1}[v_1,\dots,v_k]$ par $\mathcal{R}(F)$. Cette classe contient plusieurs processus connus comme le graphe aléatoire Erdős–Rényi et le processus de suppression de triangles. En fait, notre définition est plus générale, dans le sens que $\mathcal{R}(F)$ est une distribution de probabilité sur $\mathcal{H}_k$, permettant ainsi des remplacements aléatoires.

Étant donné un processus de retournement de règle $\mathcal{R}$, on construit des trajectoires indexées en temps $\Phi:\mathcal{W}_0\times[0,\infty)\to\mathcal{W}_0$ dans l'espace des graphons. Nous prouvons que pour chaque $T>0$ et chaque (grand) graphe fini G_0 qui est proche d'un graphon W_0 dans la norme de coupe, avec une forte probabilité le processus de retournement initié à G_0 restera dans une fine région autour de la trajectoire $(\Phi(W_0,t))_{t=0}^T$ (après remise à l'échelle du temps par le carré de l'ordre du graphe).

Ces trajectoires de graphons sont ensuite étudiées du point de vue des systèmes dynamiques. Entre autres sujets, nous étudions *(i)* les propriétés de continuité de ces trajectoires par rapport au temps et au graphon initial, *(ii)* l'existence et la stabilité des points fixes et *(iii)* la vitesse de convergence (lorsque la limite de temps infinie existe). Nous donnons un exemple de processus de retournement avec une trajectoire périodique.

MSC2020 subject classifications: 05C80

Keywords: Random graph processes; Evolving networks; Graph limits

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Extremal independence in discrete random systems

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Abstract. Let $\boldsymbol{X}(n) \in \mathbb{R}^d$ be a sequence of random vectors, where $n \in \mathbb{N}$ and $d = d(n)$. Under certain weakly dependence conditions, we prove that the distribution of the maximal component of $\boldsymbol{X}$ and the distribution of the maximum of their independent copies are asymptotically equivalent. Our result on extremal independence relies on new lower and upper bounds for the probability that none of a given finite set of events occurs. As applications, we obtain the distribution of various extremal characteristics of random discrete structures such as maximum codegree in binomial random hypergraphs and the maximum number of cliques sharing a given vertex in binomial random graphs. We also generalise Berman-type conditions for a sequence of Gaussian random vectors to possess the extremal independence property.

Résumé. Soit $\boldsymbol{X}(n) \in \mathbb{R}^d$ une suite de vecteurs aléatoires, où $n \in \mathbb{N}$ et $d = d(n)$. Sous certaines conditions de faible dépendance, nous prouvons que la loi de la composante maximale de $\boldsymbol{X}$ et la loi du maximum de leurs copies indépendantes sont asymptotiquement équivalents. Notre résultat sur l'indépendance extrémale repose sur de nouvelles bornes inférieures et supérieures pour la probabilité qu'aucun événement d'un ensemble fini donné ne se produise. Comme applications, nous obtenons la distribution de diverses caractéristiques extrémales de structures discrètes aléatoires telles que le codegré maximum dans les hypergraphes aléatoires binomiaux et le nombre maximum de cliques partageant un sommet donné dans les graphes aléatoires binomiaux. Nous généralisons également les conditions de type Berman pour qu'une séquence de vecteurs aléatoires gaussiens possède la propriété d'indépendance extrémale.

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Keywords: Extreme value theory; Limit theorems; Random graphs; Gaussian vectors; Berman condition

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Random walks in the high-dimensional limit I: The Wiener spiral

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Abstract. We prove limit theorems for random walks with n steps in the d-dimensional Euclidean space as both n and d tend to infinity. One of our results states that the path of such a random walk, viewed as a compact subset of the infinite-dimensional Hilbert space ℓ^2, converges in probability in the Hausdorff distance up to isometry and also in the Gromov–Hausdorff sense to the Wiener spiral, as $d, n \to \infty$. Another group of results describes various possible limit distributions for the squared distance between the random walker at time n and the origin.

Résumé. Nous démontrons des théorèmes limite pour des marches aléatoires de longueur n dans l'espace euclidien d-dimensionnel, lorsque n et d tendent vers l'infini. Nous établissons notamment que la trajectoire de telles marches aléatoires, vue comme un sous-ensemble compact de l'espace de Hilbert ℓ^2 de dimension infinie, converge en probabilité vers la spirale de Wiener quand n et d tendent vers l'infini, à la fois pour la distance de Hausdorff à isométries près et la distance de Gromov–Hausdorff. Nous décrivons également les limites en loi possibles pour le carré de la distance entre la marche aléatoire au temps n et l'origine.

MSC2020 subject classifications: Primary 60F05; 60G50; secondary 60D05

Keywords: Central limit theorem; Crinkled arc; Gromov–Hausdorff convergence; Hausdorff distance up to isometry; High-dimensional limit; Random metric space; Random walk; Wiener spiral

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The multi-type bisexual Galton–Watson branching process

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Abstract. In this work we study the bisexual Galton–Watson process with a finite number of types, where females and males mate according to a "mating function" and form couples of different types. We assume that this function is superadditive, which in simple words implies that two groups of females and males will form a larger number of couples together rather than separate. Leveraging on concave Perron-Frobenius theory, we prove a necessary and sufficient condition for almost sure extinction as well as a law of large numbers. Finally, we study the almost sure long-time convergence of the rescaled process through the identification of a supermartingale, and we give sufficient conditions to ensure a convergence in L^1 to a non-degenerate limit.

Résumé. Dans cet article, nous nous intéressons au processus de Galton–Watson bisexué avec un nombre fini de types, où des femelles et des mâles s'accouplent selon une "fonction d'accouplement" pour former des couples de différents types. Nous supposons que cette fonction est superadditive, ce qui, en termes simples, implique que deux groupes de femelles et de mâles forment un plus grand nombre de couples ensemble que séparés. À l'aide d'une théorie de Perron-Frobenius pour les opérateurs concaves, nous démontrons une condition nécessaire et suffisante d'extinction presque sûre ainsi qu'une loi des grands nombres. Enfin, nous étudions la convergence presque sûre en temps long du processus par identification d'une surmartingale et nous donnons des conditions suffisantes assurant la convergence dans L^1 du processus vers une limite non dégénérée.

MSC2020 subject classifications: 60F15; 60J10; 60J80; 60K35; 92D25

Keywords: Branching processes with interaction; Extinction criterion; Law of large numbers; Malthusian behaviour; Concave operators

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Corrigendum: Global martingale solutions for quasilinear SPDEs via the boundedness-by-entropy method

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Abstract. We correct the proof of Theorem 3 in the paper (G. Dhariwal, F. Huber, A. Jüngel, C. Kuehn, and A. Neamţu, *Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 577–602). The correction is based on a new regularization of quasilinear stochastic PDEs, which was suggested in (M. Braukhoff, F. Huber, and A. Jüngel, *Stoch. Partial Differ. Eqs.: Anal. Comput.* **43** (2023) 3839–3861. https://doi.org/10.1007/s40072-023-00289-7).

Résumé. Nous corrigeons la preuve du théorème 3 dans l'article (G. Dhariwal, F. Huber, A. Jüngel, C. Kuehn, and A. Neamţu, *Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 577–602). La correction est basée sur une nouvelle régularisation des EDP stochastiques quasi-linéaires, qui a été suggérée dans (M. Braukhoff, F. Huber, et A. Jüngel, *Stoch. Partial Differ. Eqs.: Anal. Comput.* **43** (2023) 3839–3861. https://doi.org/10.1007/s40072-023-00289-7).

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