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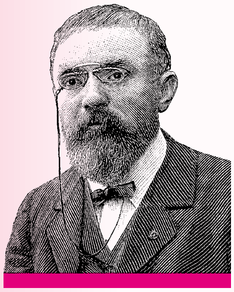

ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES

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Hydrodynamic limit and cutoff for the biased adjacent walk on the simplex

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Abstract. We investigate the asymptotic in N of the mixing times of a Markov dynamics on $N-1$ ordered particles in an interval. This dynamics consists in resampling at independent Poisson times each particle according to a probability measure on the segment formed by its nearest neighbours. In the setting where the resampling probability measures are symmetric, the asymptotic of the mixing times were obtained and a cutoff phenomenon holds. In the present work, we focus on an asymmetric version of the model and we establish a cutoff phenomenon. An important part of our analysis consists in the derivation of a hydrodynamic limit, which is given by a non-linear Hamilton–Jacobi equation with degenerate boundary conditions.

Résumé. Nous étudions l'asymptotique en N des temps de mélange d'une dynamique de Markov portant sur un ensemble de $N-1$ particules ordonnées sur un intervalle. Dans cette dynamique, on ré-échantillonne à des temps Poissonniens indépendants chaque particule selon une mesure de probabilité sur le segment formé par ses plus proches voisins. Lorsque la loi de ré-échantillonnage est symétrique, l'asymptotique des temps de mélange a déjà été obtenue et un phénomène de cutoff se produit. Dans ce travail, nous nous intéressons à une version asymétrique du modèle et nous établissons un phénomène de cutoff. Une part importante de notre analyse porte sur l'obtention d'une limite hydrodynamique, qui est donnée par une équation d'Hamilton–Jacobi nonlinéaire avec des conditions de bord dégénérées.

MSC2020 subject classifications: Primary 60J25; secondary 37A25; 70H20

Keywords: Mixing time; Cutoff; Adjacent walk; Hydrodynamic limit; Hamilton–Jacobi equation

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Eyring–Kramers type formulas for some piecewise deterministic Markov processes

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Abstract. In this work, we give sharp asymptotic equivalents in the small temperature regime of the smallest eigenvalues of the generator of some piecewise deterministic Markov processes (including the Zig-Zag process and the Bouncy Particle Sampler process) with refreshment rate α on the one-dimensional torus $\mathbb{T}$. These asymptotic equivalents are usually called Eyring–Kramers type formulas in the literature. The case when the refreshment rate α vanishes on $\mathbb{T}$ is also considered.

Résumé. Dans ce travail, nous donnons des équivalents précis à basse température des petites valeurs propres du générateur de certains processus de Markov déterministes par morceaux (comprenant le générateur du processus Zig-Zag et le générateur du processus BPS) avec un taux de rafraîchissement α sur le tore $\mathbb{T}$ en dimension un. Dans la littérature, de tels équivalents sont appelés des formules d'Eyring–Kramers. Nous considérons aussi le cas où le rafraîchissement α s'annule sur le tore $\mathbb{T}$.

MSC2020 subject classifications: 35P15; 47F05; 35P20; 35Q82; 35Q92

Keywords: Piecewise Deterministic Markov Processes; Small temperature regime; Eyring–Kramers formulas; Metastability; Spectral theory; Semiclassical analysis

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Law of large numbers for ballistic random walks in dynamic random environments under lateral decoupling

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Abstract. We establish a strong law of large numbers for one-dimensional continuous-time random walks in dynamic random environments under two main assumptions: the environment is required to satisfy a decoupling inequality that can be interpreted as a bound on the speed of dependence propagation, while the random walk is assumed to move ballistically with a speed larger than this bound. Applications include environments with strong space-time correlations such as the zero-range process and the asymmetric exclusion process.

Résumé. Nous obtenons une loi forte des grands nombres pour marches aléatoires en milieux aléatoires dynamiques sous deux hypothèses principales : le milieu doit satisfaire une inégalité de découplage qui peut être interprétée comme une borne sur la vitesse de propagation de la dépendance, et la marche doit être ballistique à vitesse plus grande que cette borne. Les applications incluent des milieux à corrélations spatio-temporelles fortes comme le processus d'exclusion asymmétrique et le processus «zero-range».

MSC2020 subject classifications: 60K35; 82B43

Keywords: Random walks; Dynamical random environment; Zero-range process; Asymmetric exclusion process

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Gaussian, stable, tempered stable and mixed limit laws for random walks in cooling random environments

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Abstract. Random Walks in Cooling Random Environments (RWCRE) is a model of random walks in dynamic random environments where the entire environment is resampled along a fixed sequence of times, called the "cooling sequence", and is kept fixed in between those times. This model interpolates between that of a homogeneous random walk, where the environment is reset at every step, and Random Walks in (static) Random Environments (RWRE), where the environment is never resampled. In this work we focus on the limiting distributions of one-dimensional RWCRE in the regime where the fluctuations of the corresponding (static) RWRE is given by a s-stable random variable with $s \in (1,2)$. In this regime, due to the two extreme cases (resampling every step and never resampling, respectively), a crossover from Gaussian to stable limits for sufficiently regular cooling sequence was previously conjectured. Our first result answers affirmatively this conjecture by making clear critical exponent, norming sequences and limiting laws associated with the crossover which demonstrates a change from Gaussian to s-stable limits, passing at criticality through a certain generalized tempered stable distribution which have not appeared as limits of random walks in dynamic random environments previously. We then explore the resulting RWCRE scaling limits for general cooling sequences. On the one hand, we offer sets of operative sufficient conditions that guarantee asymptotic emergence of either Gaussian, s-stable or generalized tempered distributions from a certain class. On the other hand, we give explicit examples and describe how to construct irregular cooling sequences for which the corresponding limit law is characterized by mixtures of the three above mentioned laws. To obtain these results, we need and derive a number of refined asymptotic results for the static RWRE with $s \in (1,2)$ which are of independent interest.

Résumé. Les marches aléatoires dans des environnements aléatoires refroidissants (RWCRE) constituent un modèle de marches aléatoires dans des environnements aléatoires dynamiques, dans lequel l'ensemble de l'environnement est rééchantilloné selon une suite fixe de temps, appelée « séquence de refroidissement », et reste fixe entre chacun de ces instants. Il s'agit d'un modèle intermédiaire entre les marche aléatoires homogènes, où l'environnement est réinitialisé à chaque étape, et les marches aléatoires dans des environnements aléatoires (statiques) (RWRE) où l'environnement n'est jamais rééchantilloné. Dans ce travail, nous nous concentrons sur les distributions limites du RWCRE unidimensionnel dans le régime où les fluctuations du RWRE (statique) correspondant sont données par une variable aléatoire s-stable avec $s \in (1,2)$. Dans ce régime, en raison des deux cas extrêmes (respectivement rééchantillonnage à chaque étape ou'a aucune), un passage de limites gaussiennes aux lois stables pour une séquence de refroidissement suffisamment régulière a été précédemment conjecturé. Notre premier résultat répond positivement à cette conjecture en faisant apparaître l'exposant critique, les suites de normalisation et les lois limites associées à la transition entre les régimes gaussien et s-stable, cette transition passant par la criticité d'une certaine loi stable tempérée généralisée. Nous explorons ensuite les limites d'échelle des RWCRE résultantes pour des séquences de refroidissement générales. D'une part, nous proposons des ensembles de conditions opérationnelles suffisantes qui garantissent l'émergence asymptotique de distributions gaussiennes, s-stables ou tempérées généralisées d'une certaine classe. D'autre part, nous donnons des exemples explicites et décrivons comment construire des séquences de refroidissement irrégulières pour lesquelles la loi limite correspondante est caractérisée par des mélanges des trois lois mentionnées ci-dessus. Pour obtenir ces résultats, nous avons besoin de plusieurs résultats asymptotiques raffinés pour le RWRE statique avec $s \in (1,2)$ que nous démontrons et qui peuvent présenter un intérêt indépendant.

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Keywords: Random walk; Dynamic random environment; Resetting dynamics; Stable laws; Anomalous diffusion

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The supremum of Brownian local times on Hölder curves revisited

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Abstract. For $f : [0, 1] \to \mathbb{R}$, we consider L_t^f, the local time of space-time Brownian motion on the curve f. Let $\mathcal{S}_\alpha$ be the class of all functions whose Hölder norm of order α is less than or equal to 1. We show that the supremum of L_1^f over f in $\mathcal{S}_\alpha$ is finite if $\alpha > \frac{1}{2}$.

Résumé. Soit W_t un mouvement brownien, et L_t^f le temps local du processus (t, W_t) pour le courbe $f : [0, 1] \to \mathbb{R}$. On note $\mathcal{S}_\alpha$ la classe de toutes fonctions telles que la norme holderienne du ordre α soit inférieure à 1. Nous démontrons que $\sup_{f \in \mathcal{S}_\alpha} L_1^f < \infty$ p.s. si $\alpha > \frac{1}{2}$.

MSC2020 subject classifications: Primary 60J65; secondary 60J55

Keywords: Brownian motion; Local time; Hölder function

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Pruning, cut trees, and the reconstruction problem

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Abstract. We consider a pruning of the inhomogeneous continuum random trees, as well as the cut trees that encode the genealogies of the fragmentations that come with the pruning. We propose a new approach to the reconstruction problem, which has been treated for the Brownian CRT in (*Electron. J. Probab.* **22** (2017) 1–23) and for the stable trees in (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1349–1376). Our approach does not rely upon self-similarity and can potentially apply to general Lévy trees as well.

Résumé. Nous considérons l'élagage d'arbres continus inhomogènes, et les arbres des coupes qui encodent les généalogies des fragmentations associées. Une nouvelle approche pour le problème de reconstruction est proposée, qui ne repose pas sur l'auto-similarité contrairement aux solutions pour l'arbre continu Brownien dans (*Electron. J. Probab.* **22** (2017) 1–23) ou la famille des arbres stables (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1349–1376). Cela permettrait potentiellement de traiter les arbres de Lévy.

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Keywords: Random cutting of random trees; Inhomogeneous continuum random tree; Cut tree

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Binary branching processes with Moran type interactions

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Abstract. The aim of this paper is to study the large population limit of a new collection of interacting particle systems (IPS) that encompasses branching models and fixed size Moran type IPS (see (*J. Phys. A* **29** (1996) 2633–2642); (*Comm. Math. Phys.* **214** (2000) 679–703); (Feynman–Kac Formulae: Genealogical and Interacting Particle Systems with Applications (2004) Springer-Verlag); (Mean Field Simulation for Monte Carlo Integration (2013) CRC Press); (*ESAIM Probab. Stat.* **18** (2014) 441–467)). We define an IPS where particles evolve, reproduce and die independently and, with a probability that may depend on the configuration of the whole system, the death of a particle may trigger the reproduction of another particle, while a branching event may trigger the death of an other one. We study the occupation measure of the new model, explicitly relating it to the Feynman–Kac semigroup of the underlying Markov evolution and quantifying the L^2 distance between their normalisations. We show that our model outperforms the fixed size Moran type IPS when used to approximate a birth and death process. We discuss several other applications of our model including the neutron transport equation (Cox et al. (2020); Stochastic Neutron Transport and Non-local Branching Markov Processes (2023) Birkhäuser) and population size dynamics.

Résumé. L'objectif de cet article est d'étudier la limite en grande population d'une nouvelle famille de systèmes de particules en interaction (IPS) qui englobe les modèles de branchement et les IPS de type Moran de taille fixe (voir (*J. Phys. A* **29** (1996) 2633–2642) ; (*Comm. Math. Phys.* **214** (2000) 679–703) ; (Feynman–Kac Formulae : Genealogical and Interacting Particle Systems with Applications (2004) Springer-Verlag) ; (Mean Field Simulation for Monte Carlo Integration (2013) CRC Press) ; (*ESAIM Probab. Stat.* **18** (2014) 441–467)). Nous définissons un IPS où les particules évoluent, se reproduisent et meurent indépendamment et, avec une probabilité qui peut dépendre de la configuration du système entier, la mort d'une particule peut déclencher la reproduction d'une autre particule, tandis qu'un événement de branchement peut déclencher la mort d'une autre particule. Nous étudions la mesure d'occupation du nouveau modèle, en la reliant explicitement au semigroupe de Feynman–Kac de l'évolution markovienne sous-jacente et en quantifiant la distance L^2 entre leurs normalisations. Nous montrons que notre modèle est plus performant que l'IPS de Moran de taille fixe lorsqu'il est utilisé pour approximer un processus de naissance et mort. Nous discutons de plusieurs autres applications de notre modèle, notamment l'équation de transport des neutrons (Cox et al. (2020) ; Stochastic Neutron Transport and Non-local Branching Markov Processes (2023) Birkhäuser) et la dynamique de la taille des populations.

MSC2020 subject classifications: 82C22; 82C80; 65C05; 60J25; 92D25; 60J80

Keywords: Interacting particle systems; Branching processes; Many-to-one; Markov processes; Birth-and-death process; Moran model

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Emergence of fractional Gaussian free field correlations in subcritical long-range Ising models

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Abstract. We study corrections to the scaling limit of subcritical long-range Ising models with (super)-summable interactions on $\mathbb{Z}^d$. For a wide class of models, the scaling limit is known to be white noise, as shown by Newman (1980). In the specific case of couplings $J_{x,y} = |x-y|^{-d-\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} > 0$ and $|\cdot|$ is the Euclidean norm, we find an emergence of fractional Gaussian free field correlations in appropriately renormalised and rescaled observables. The proof exploits the exact asymptotics of the two-point function, first established by Newman and Spohn (1998), together with the rotational symmetry of the interaction.

Résumé. On étudie les corrections à la limite d'échelle de modèles d'Ising sous-critiques de longue portée et à interactions (super)-sommables. Pour une large classe de modèles la limite d'échelle est le bruit blanc, comme démontré par Newman (1980). Dans le cas particulier de constantes de couplage $J_{x,y} = |x-y|^{-d-\boldsymbol{\alpha}}$, où $\boldsymbol{\alpha} > 0$ et $|\cdot|$ est la norme Euclidienne, on observe l'émergence des fonctions de corrélation du champ libre Gaussien fractionnaire pour certaines observables proprement renormalisées. La preuve repose sur les asymptotiques exactes de la fonction à deux points dans ce régime d'abord obtenues par Newman et Spohn (1998), ainsi que sur l'invariance par rotation de l'interaction.

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Keywords: Long-range Ising models; Fractional Gaussian free fields; Scaling limits

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Representations of Hecke algebras and Markov dualities for interacting particle systems

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Abstract. Many continuous reaction-diffusion models on $\mathbb{Z}$ (annihilating or coalescing random walks, exclusion processes, voter models) admit a rich set of Markov duality functions which determine the single time distribution. A common feature of these models is that their generators are given by sums of two-site idempotent operators. In this paper, we classify all continuous time Markov processes on $\{0, 1\}^{\mathbb{Z}}$ whose generators have this property, although to simplify the calculations we only consider models with equal left and right jumping rates. The classification leads to six familiar models and three exceptional models. The generators of all but the exceptional models turn out to belong to an infinite dimensional Hecke algebra, and the duality functions appear as spanning vectors for small-dimensional irreducible representations of this Hecke algebra. A second classification explores generators built from two site operators satisfying the Hecke algebra relations. The duality functions are intertwiners between configuration and co-ordinate representations of Hecke algebras, which results in a novel co-ordinate representations of the Hecke algebra. The standard Baxterisation procedure leads to new solutions of the Young-Baxter equation corresponding to particle systems which do not preserve the number of particles.

Résumé. De nombreux modèles de réaction-diffusion continus sur $\mathbb{Z}$ (marches aléatoires annihilantes ou coalescentes, processus d'exclusion, modèles de votants) admettent un ensemble riche de fonctions de dualité markoviennes qui déterminent la loi à un instant donné. Une caractéristique commune de ces modèles est que leurs générateurs sont donnés par des sommes d'opérateurs idempotents à deux sites. Dans cet article, nous classifions tous les processus de Markov en temps continu sur $\{0, 1\}^{\mathbb{Z}}$ dont les générateurs ont cette propriété, bien que pour simplifier les calculs, nous ne considérons que des modèles avec des taux de saut égaux à gauche et à droite. La classification conduit à six modèles familiers et trois modèles exceptionnels. Les générateurs de tous sauf les modèles exceptionnels appartiennent à une algèbre de Hecke de dimension infinie, et les fonctions de dualité apparaissent comme des vecteurs générateurs des représentations irréductibles de petite dimension de cette algèbre de Hecke. Une deuxième classification explore les générateurs construits à partir d'opérateurs à deux sites satisfaisant aux relations de l'algèbre de Hecke. Les fonctions de dualité sont des entrelacs entre les représentations en configurations et en coordonnées des algèbres de Hecke, ce qui donne lieu à une nouvelle représentation en coordonnées de l'algèbre de Hecke. La procédure standard de Baxtérisation conduit à de nouvelles solutions de l'équation de Young-Baxter correspondant à des systèmes de particules qui ne préservent pas le nombre de particules.

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Keywords: Interacting particle systems; Markov duality; Hecke Algebra; Yang–Baxter equation

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Uniform in time weak propagation of chaos on the torus

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Abstract. We address the long time behaviour of weakly interacting diffusive particle systems on the d-dimensional torus. Our main result is to show that, under certain mild regularity conditions, the weak error between the empirical distribution of the particle system and the limiting theoretical law (governed by a Fokker–Planck equation) is of the order $\mathcal{O}(1/N)$, uniform in time on $[0, \infty)$, where N is the number of particles. This comprises Fokker–Planck equations with a globally attracting invariant measure for which the linearisation at the invariant measure enjoys appropriate ergodic properties. Our approach relies on a systematic analysis of the long-time behaviour of the derivatives of the semigroup generated by the Fokker–Planck equation. This strategy is flexible enough to cover a wider broad of situations, including the super-critical Kuramoto model, for which the corresponding Fokker–Planck equation has several invariant measures.

Résumé. Nous étudions le comportement en temps long de systèmes de particules en interaction faible à valeurs dans le tore de dimension d. Le résultat principal montre que, sous des hypothèses de régularité raisonnables, l'erreur faible entre la distribution empirique marginale des particules et la loi limite théorique (gouvernée par une équation de Fokker–Planck) est de l'ordre de $\mathcal{O}(1/N)$, uniformément en temps sur $[0, \infty)$, où N est le nombre de particules. Ce résultat s'applique en particulier aux équations de Fokker–Planck possédant une mesure invariante globalement attractive dont le linéarisé en la mesure invariante satisfait des propriétés ergodiques appropriées. Notre approche s'appuie sur une étude systématique du comportement en temps long des dérivées du semi-groupe généré par l'équation de Fokker–Planck. Cette démarche est suffisamment flexible pour couvrir des situations variées, dont le modèle sur-critique de Kuramoto, pour lequel l'équation de Fokker–Planck correspondante admet plusieurs mesures invariantes.

MSC2020 subject classifications: Primary 60F99; 60K35; secondary 35Q84; 82C31

Keywords: Uniform in time propagation of chaos; Weakly interacting particle system; Weak error; McKean Vlasov equation; Fokker–Planck equation

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Infinite geodesics, competition interfaces and the second class particle in the scaling limit

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Abstract. We establish fundamental properties of infinite geodesics and competition interfaces in the directed landscape. We construct infinite geodesics in the directed landscape, establish their uniqueness and coalescence, and define Busemann functions. We then define competition interfaces in the directed landscape. We prove the second class particle in tasep converges under KPZ scaling to a competition interface. Under suitable conditions, we show the competition interface has an asymptotic direction, analogous to the speed of a second class particle, and determine its law. Moreover, we prove the competition interface has an absolutely continuous law on compact sets with respect to infinite geodesics.

Résumé. Nous établissons les propriétés fondamentales des géodésiques infinies et des interfaces de compétition dans le paysage dirigé. Nous construisons des géodésiques infinies dans le paysage dirigé, établissons leur unicité et leur coalescence, et définissons les fonctions de Busemann. Nous définissons ensuite les interfaces de compétition dans le paysage dirigé et démontrons que la particule de deuxième classe dans le modèle TASEP converge, sous la renormalisation de KPZ, vers une interface de compétition. Sous des conditions appropriées, nous montrons que l'interface de compétition a une direction asymptotique, analogue à la vitesse d'une particule de deuxième classe, et déterminons sa loi. De plus, nous prouvons que l'interface de compétition a une loi absolument continue sur des ensembles compacts par rapport aux géodésiques infinies.

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Steady state large deviations for one-dimensional, symmetric exclusion processes in weak contact with reservoirs

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Abstract. Consider the symmetric exclusion process evolving on an interval and weakly interacting at the end-points with reservoirs. Denote by $I_{[0,T]}(\cdot)$ its dynamical large deviations functional and by $V(\cdot)$ the associated quasi-potential, defined as $V(\gamma) = \inf_{T>0} \inf_u I_{[0,T]}(u)$, where the infimum is carried over all trajectories u such that $u(0) = \bar{\rho}$, $u(T) = \gamma$, and $\bar{\rho}$ is the stationary density profile. We derive the partial differential equation which describes the evolution of the optimal trajectory, and deduce from this result the formula obtained by Derrida, Hirschberg and Sadhu (*J. Stat. Phys.* (2021)) for the quasi-potential through the representation of the steady state as a product of matrices.

Résumé. Considérons le processus d'exclusion simple symétrique sur un intervalle et en interaction faible avec des réservoirs à ses extremités. On définit sa fonctionelle de grandes déviations dynamiques $I_{[0,T]}(\cdot)$ et le quasi-potentiel associé $V(\cdot)$, défini comme $V(\gamma) = \inf_{T>0} \inf_u I_{[0,T]}(u)$, où l'infimum est réalisé sur toutes les trajectoires u satisfaisant $u(0) = \bar{\rho}$, $u(T) = \gamma$, et $\bar{\rho}$ et le profil de densité stationnaire. Nous obtenons l'équation aux dérivées partielles qui décrit l'évolution de la trajectoire optimale, et on en déduit la formule obtenue par Derrida, Hirschberg et Sadhu (*J. Stat. Phys.* (2021)) pour le quasi-potentiel a travers la représentation de l'état stationnaire comme produit de matrices.

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Keywords: Statistical mechanics; Large deviations; Boundary-driven systems; Non-equilibrium

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Hypergeometric SLE with $\kappa = 8$: Convergence of UST and LERW in topological rectangles

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Abstract. We consider uniform spanning tree (UST) in topological rectangles with alternating boundary conditions. The Peano curves associated to the UST converge weakly to hypergeometric SLE_8, denoted by hSLE_8. From the convergence result, we obtain the continuity and reversibility of hSLE_8 as well as an interesting connection between SLE_8 and hSLE_8. The loop-erased random walk (LERW) branch in the UST converges weakly to $\mathrm{SLE}_2(-1,-1;-1,-1)$. We also obtain the limiting joint distribution of the two end points of the LERW branch.

Résumé. Nous considérons l'arbre couvrant uniforme (UST) en rectangles topologiques avec conditions au bord alternées. Les courbes de Peano associées à l'UST convergent vers le SLE_8 hypergéométrique, noté hSLE_8. À partir du résultat de convergence, nous obtenons la continuité et la réversibilité de hSLE_8 ainsi qu'une connexion intéressante entre SLE_8 et hSLE_8. La branche de marche aléatoire à boucles effacées (LERW) dans l'UST converge vers $\mathrm{SLE}_2(-1,-1;-1,-1)$. Nous obtenons la distribution conjointe limite des deux extrémités de la branche LERW.

MSC2020 subject classifications: 60J67

Keywords: Uniform spanning tree; Loop-erased random walk; Schramm Loewner evolution

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$\mathrm{SLE}_\kappa(\rho)$ bubble measures

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Abstract. For $\kappa > 0$ and $\rho > -2$, we construct a σ-finite measure, called a rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure, on the space of curves in the upper half-plane $\mathbb{H}$ started and ended at the same boundary point, which satisfies some $\mathrm{SLE}_\kappa(\rho)$-related domain Markov property, and is the weak limit of $\mathrm{SLE}_\kappa(\rho)$ curves in $\mathbb{H}$ with the two endpoints both tending to the root. For $\kappa \in (0,8)$ and $\rho \in ((-2) \vee (\frac{\kappa}{2}-4), \frac{\kappa}{2}-2)$, we derive decomposition theorems for the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble with respect to the Minkowski content measure of the intersection of the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble with $\mathbb{R}$, and construct unrooted $\mathrm{SLE}_\kappa(\rho)$ bubble measures.

Résumé. Pour $\kappa > 0$ et $\rho > -2$, nous construisons une mesure σ-finie, appelée mesure de bulle $\mathrm{SLE}\kappa(\rho)$ enracinée, sur l'espace des courbes dans le demi-plan supérieur $\mathbb{H}$ commençant et se terminant au même point de la frontière, qui satisfait une certaine propriété de Markov de domaine liée à $\mathrm{SLE}\kappa(\rho)$, et qui est la limite faible des courbes $\mathrm{SLE}\kappa(\rho)$ dans $\mathbb{H}$ avec les deux extrémités tendant vers la racine. Pour $\kappa \in (0,8)$ et $\rho \in ((-2) \vee (\frac{\kappa}{2}-4), \frac{\kappa}{2}-2)$, nous dérivons des théorèmes de décomposition pour la bulle $\mathrm{SLE}\kappa(\rho)$ enracinée par rapport à la mesure de contenu de Minkowski de l'intersection de la bulle $\mathrm{SLE}\kappa(\rho)$ enracinée avec $\mathbb{R}$, et construisons des mesures de bulle $\mathrm{SLE}\kappa(\rho)$ non enracinées.

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Keywords: SLE; Loop measure; Minkowski content

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Functional limits for "tied down" occupation time processes of infinite ergodic transformations

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Abstract. We prove functional, distributional limit theorems for the occupation times of pointwise dual ergodic transformations at "tied-down" times (immediately after "excursions"). The limiting processes are the tied down Mittag-Leffler processes and the transformations involved exhibit functional versions of the tied-down renewal properties in (Aaronson and Sera (2019)).

Résumé. Nous prouvons des théorèmes limites distributionnels fonctionnels pour les temps d'occupation des transformations ponctuellement dualement ergodiques à des instants « contraints » (immédiatement après des « excursions »). Les processus limites sont les processus « contraints » de Mittag-Leffler. Les transformations concernées présentent des versions fonctionnelles des propriétés de renouvellement « contraints » données dans (Aaronson and Sera (2019)).

MSC2020 subject classifications: 37A40; 60F05

Keywords: Infinite ergodic theory; Pointwise dual ergodic; Darling–Kac theorem; Mittag Leffler processes; Stable subordinator; Rational weak mixing; Local time process; Conditional limit theorem; Tied-down renewal theory

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The contact process on dynamical scale-free networks

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Abstract. We investigate the contact process on four different types of scale-free inhomogeneous random graphs evolving according to a stationary dynamics, where each potential edge is updated with a rate depending on the strength of the adjacent vertices. Depending on the type of graph, the tail exponent of the degree distribution and the updating rate, we find parameter regimes of fast and slow extinction and in the latter case identify metastable exponents that undergo first order phase transitions.

Résumé. Nous étudions le processus de contact sur quatre types différents de graphes aléatoires inhomogènes invariants d'échelle évoluant selon une dynamique stationnaire, où chaque arête potentielle est rafraîchie à un taux dépendant de la force des sommets adjacents. En fonction du type de graphe, de l'exposant de la queue de distribution des degrés et du taux de rafraîchissement, nous trouvons des régimes d'extinction rapide ou lente et, dans ce dernier cas, nous identifions des exposants métastables qui subissent des transitions de phase de premier ordre.

MSC2020 subject classifications: Primary 05C82; secondary 82C22; 82C43

Keywords: Phase transitions; Metastable density; Evolving network; Temporal network; Dynamic network; Inhomogeneous random graph; Preferential attachment network; Network dynamics; SIS infection

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Absence of weak disorder for directed polymers on supercritical percolation clusters

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Abstract. We study the directed polymer model on infinite clusters of supercritical Bernoulli percolation containing the origin in dimensions $d \geq 3$, and prove that for almost every realization of the cluster and every strictly positive value of the inverse temperature β, the polymer is in a strong disorder phase, answering a question from Cosco, Seroussi and Zeitouni (*Comm. Math. Phys.* **386** (2021) 395–432).

Résumé. Nous étudions le modèle de polymère dirigé sur le cluster infini de percolation de Bernoulli surcritique contenant l'origine dans les dimensions $d \geq 3$, et prouvons que, pour presque toute réalisation du cluster et pour chaque valeur strictement positive de l'inverse de la température β, le polymère se trouve dans une phase de désordre fort, répondant ainsi à une question de Cosco, Seroussi et Zeitouni (*Comm. Math. Phys.* **386** (2021) 395–432).

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Keywords: Directed polymers; Percolation; Weak disorder phase

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A lower-tail limit in the weak noise theory

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Abstract. We consider the variational problem associated with the Freidlin–Wentzell Large Deviation Principle for the Stochastic Heat Equation (SHE). The logarithm of the minimizer of the variational problem gives the most probable shape of the solution of the Kardar–Parisi–Zhang equation conditioned on achieving certain unlikely values. Taking the SHE with the delta initial condition and conditioning the value of its solution at the origin at a later time, under suitable scaling, we prove that the logarithm of the minimizer converges to an explicit function as we tune the value of the conditioning to 0. Our result confirms the physics predictions of (*Phys. Rev. E* **80** (2009) 031107; *Phys. Rev. Lett.* **116** (2016) 070601; *Phys. Rev. E* **94** (2016) 032108).

Résumé. Nous considérons le problème variationnel associé au principe de grandes déviations de Freidlin–Wentzell pour l'équation de la chaleur stochastique (SHE). Le logarithme du minimiseur du problème variationnel donne la forme la plus probable de la solution de l'équation de Kardar–Parisi–Zhang conditionnée à l'obtention de certaines valeurs peu probables. En prenant la SHE avec la condition initiale de type delta et en conditionnant la valeur de sa solution à l'origine à un temps ultérieur, sous une mise à l'échelle appropriée, nous prouvons que le logarithme du minimiseur converge vers une fonction explicite lorsque nous ajustons la valeur du conditionnement à 0. Notre résultat confirme les prédictions physiques de (*Phys. Rev. E* **80** (2009) 031107 ; *Phys. Rev. Lett.* **116** (2016) 070601 ; *Phys. Rev. E* **94** (2016) 032108).

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Keywords: Freidlin–Wentzell; Kardar–Parisi–Zhang equation; Large deviations; Most probable shapes; Weak noise theory

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A unified approach to gradient type formulas for decoupled FBSDEs and some applications

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Abstract. In this paper we present a unified approach to establish gradient type formulas and Bismut type formulas for decoupled forward-backward stochastic differential equations (FBSDEs). This approach relies on a mix of derivative formulas with respect to the conditional probability of forward SDEs and the expression of the solution of BSDEs. Some concrete examples are given to illustrate the results. As applications, we provide representation formulas for the control solutions to McKean–Vlasov decoupled FBSDEs and derive gradient estimates with respect to the position argument for related PDEs.

Résumé. Dans cet article, nous présentons une approche unifiée pour établir des formules de type gradient et de type Bismut pour des équations différentielles stochastiques progressives et rétrogrades découplées (FBSDEs). Cette approche repose sur un mélange de formules dérivées par rapport à la probabilité conditionnelle des EDS progressives et sur l'expression de la solution des EDSRs. Des exemples concrets sont donnés pour illustrer les résultats. Comme applications, nous fournissons des formules de représentation pour les solutions de contrôle des FBSDEs découplées de McKean–Vlasov et obtenons des estimations de gradient par rapport à l'argument de position pour des EDPs liées.

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Keywords: Gradient type formula; Bismut type formula; BSDEs; McKean–Vlasov BSDEs; Gradient estimate

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Eigenvalues and spectral gap in sparse random simplicial complexes

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Abstract. We consider the adjacency operator A of the Linial–Meshulam model $X(d,n,p)$ for random d-dimensional simplicial complexes on n vertices, where each d-cell is added independently with probability $p \in [0,1]$ to the complete $(d-1)$-skeleton. We consider sparse random matrices H, which are generalizations of the centered and normalized adjacency matrix $\mathcal{A} := (np(1-p))^{-1/2} \cdot (A - \mathbb{E}[A])$, obtained by replacing the Bernoulli(p) random variables used to construct A with arbitrary bounded distribution Z. We obtain bounds on the expected Schatten norm of H, which allow us to prove results on eigenvalue confinement and in particular that $\|H\|_2$ converges to $2\sqrt{d}$ both in expectation and $\mathbb{P}$-almost surely as $n \to \infty$, provided that $\mathrm{Var}(Z) \gg \frac{\log n}{n}$. The main ingredient in the proof is a generalization of (*Invent. Math.* **214** (2018) 1031–1080, Theorem 4.8) to the context of high-dimensional simplicial complexes, which may be regarded as sparse random matrix models with dependent entries.

Résumé. Nous étudions l'opérateur d'adjacence A du modèle Linial–Meshulam $X(d,n,p)$ pour les complexes simpliciaux aléatoires d-dimensionnels sur n sommets, où chaque d-cellule est ajoutée indépendamment avec une probabilité $p \in [0,1]$ au squelette complet $(d-1)$-dimensionnel. Nous considérons des matrices aléatoires éparses H, qui sont des généralisations de la matrice d'adjacence centrée et normalisée $\mathcal{A} := (np(1-p))^{-1/2} \cdot (A - \mathbb{E}[A])$, obtenue en remplaçant les variables aléatoires Bernoulli(p) utilisées pour construire A par une distribution bornée arbitraire Z. Nous obtenons des bornes sur la norme de Schatten attendue de H, ce qui nous permet de démontrer des résultats sur le confinement des valeurs propres et en particulier que la norme $\|H\|_2$ converge vers $2\sqrt{d}$ à la fois en espérance et presque sûrement selon $\mathbb{P}$ lorsque $n \to \infty$, à condition que la variance de $\mathrm{Var}(Z)$ soit significativement supérieure à $\frac{\log n}{n}$. L'ingrédient principal de la preuve est une généralisation du (*Invent. Math.* **214** (2018) 1031–1080, Theorem 4.8) au contexte des complexes simpliciaux de grande dimension, pouvant être envisagés comme des modèles de matrices aléatoires éparses avec des entrées dépendantes.

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Keywords: Random matrices; Random simplicial complexes

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Singularity degree of structured random matrices

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Abstract. We consider the density of states of structured Hermitian random matrices with a variance profile. As the dimension tends to infinity the associated eigenvalue density can develop a singularity at the origin. The severity of this singularity depends on the relative positions of the zero submatrices. We provide a classification of all possible singularities and determine the exponent in the density blow-up, which we label the singularity degree.

Résumé. Nous considérons la densité d'états de matrices aléatoires hermitiennes structurées avec un profil de variance. Lorsque la dimension tend vers l'infini, la densité de valeurs propres associée peut développer une singularité à l'origine. La sévérité de cette singularité dépend des positions relatives des sous-matrices nulles. Nous fournissons une classification de toutes les singularités possibles et déterminons l'exposant d'explosion de la densité, que nous appelons le degré de singularité.

MSC2020 subject classifications: Primary 60B20; secondary 15B52

Keywords: Wigner-type matrix; Eigenvalue distribution; Condition number

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Large deviations for the volume of hyperbolic k-nearest neighbor balls

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Abstract. We prove a large deviation principle for the point process of large Poisson k-nearest neighbor balls in hyperbolic space. More precisely, we consider a stationary Poisson point process of unit intensity in a growing sampling window in hyperbolic space. We further take a growing sequence of thresholds such that there is a diverging expected number of Poisson points whose k-nearest neighbor ball has a volume exceeding this threshold. Then, the point process of exceedances satisfies a large deviation principle whose rate function is described in terms of a relative entropy. The proof relies on a fine coarse-graining technique such that inside the resulting blocks the exceedances are approximated by independent Poisson point processes.

Résumé. Nous prouvons un principe de grande déviation pour le processus ponctuel des boules de Poisson à k plus proches voisins dans l'espace hyperbolique. Plus précisément, nous considérons un processus ponctuel de Poisson stationnaire d'intensité unitaire dans une fenêtre d'échantillonnage croissante dans l'espace hyperbolique. Nous prenons en outre une séquence croissante de seuils telle qu'il existe en moyenne un nombre divergent de points de Poisson dont la boule des k-voisins les plus proches a un volume supérieur à ce seuil. Le processus ponctuel des dépassements satisfait alors un principe de grande déviation dont la fonction de taux est décrite en termes d'entropie relative. La preuve s'appuie sur une technique de granulométrie fine de sorte qu'à l'intérieur des blocs résultants, les dépassements sont approximés par des processus ponctuels de Poisson indépendants.

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Keywords: Hyperbolic space; Large deviation principle; Nearest neighbor balls; Poisson point process; Stochastic geometry

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Inducing techniques for quantitative recurrence and applications to Misiurewicz maps and doubly intermittent maps

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Abstract. We prove an abstract result establishing that one can obtain the convergence of Rare Events Point Processes counting the number of orbital visits to a sequence of shrinking target sets from the convergence of corresponding point processes for some induced system and matching shadowing shrinking sets inside the base of the inducing scheme. We apply this result to prove a dichotomy for two classes of non-uniformly hyperbolic interval maps: Misiurewicz quadratic maps and doubly intermittent maps. The dichotomy holds in the sense that the shrinking target sets may accumulate in any individual point ζ chosen in the phase space and then one either obtains a limiting homogeneous Poisson process at every non-periodic point ζ or a limiting compound Poisson process with geometric multiplicity distribution at every periodic point. We also highlight the reconstruction performed in order to recover the multiplicity distribution for a periodic orbit sitting outside the base of the induced map.

Résumé. Nous prouvons un résultat abstrait établissant que l'on peut obtenir la convergence des processus ponctuels d'événements rares (REPP), qui comptent le nombre de visites le long d'une orbite à une suite décroissante d'ensembles cibles, à partir de la convergence de processus ponctuels correspondants pour un certain système induit et des ensembles rares correspondants dans la base du schéma d'induction. Nous appliquons ce résultat pour prouver une dichotomie pour deux classes de fonctions de l'intervalle non uniformément hyperboliques : les fonctions quadratiques de Misiurewicz et les fonctions doublement intermittentes. La dichotomie est valable dans le sens où les ensembles cibles peuvent s'accumuler en n'importe quel point ζ choisi dans l'espace des phases et alors la limite obtenue est soit un processus de Poisson homogène à chaque point non périodique ζ, soit un processus de Poisson composé avec une multiplicité suivant une loi géométrique à chaque point périodique. Nous mettons également en évidence la reconstruction effectuée afin de retrouver la distribution de la multiplicité pour une orbite périodique située en dehors de la base d'induction.

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Scaling limit of the directional conductivity of random resistor networks on point processes

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Abstract. We consider random resistor networks with nodes given by a point process on $\mathbb{R}^d$ and with random conductances. The length range of the electrical filaments can be unbounded. We assume that the randomness is stationary and ergodic w.r.t. the action of the group $\mathbb{G}$, given by $\mathbb{R}^d$ or $\mathbb{Z}^d$. This action is covariant w.r.t. translations on the Euclidean space. Under minimal assumptions we prove that a.s. the suitably rescaled directional conductivity of the resistor network along the principal directions of the effective homogenized matrix D converges to the corresponding eigenvalue of D times the intensity of the point process. More generally, we prove a quenched scaling limit of the directional conductivity along any vector $e \in \mathrm{Ker}(D) \cup \mathrm{Ker}(D)^{\perp}$. Our results cover plenty of models including e.g. the standard conductance model on $\mathbb{Z}^d$ (also with long filaments), the Miller–Abrahams resistor network for conduction in amorphous solids (to which we can now extend the bounds in agreement with Mott's law previously obtained in (*Ann. Appl. Probab.* **19** (2009) 1459–1494, *Comm. Math. Phys.* **281** (2008) 263–286, *Comm. Math. Phys.* **263** (2006) 21–64) for Mott's random walk), resistor networks on the supercritical cluster in lattice and continuum percolations, resistor networks on crystal lattices and on Delaunay triangulations.

Résumé. Nous considérons des réseaux de résistances aléatoires avec des nœuds donnés par un processus ponctuel sur $\mathbb{R}^d$ et avec des conductances aléatoires. La plage de longueur des filaments électriques peut être non bornée. Nous supposons que l'aléa est stationnaire et ergodique par rapport à l'action du groupe $\mathbb{G}$, donné par $\mathbb{R}^d$ ou $\mathbb{Z}^d$. Cette action est covariante par rapport aux translations sur l'espace euclidien. Sous des hypothèses minimales, nous prouvons que p.s. la conductivité directionnelle du réseau de résistances le long des directions principales de la matrice homogénéisée effective D, convenablement remise à l'échelle, converge vers la valeur propre correspondante de D multipliée par l'intensité du processus ponctuel. Plus généralement, à environnement fixé, nous prouvons une limite d'échelle de la conductivité directionnelle le long de tout vecteur $e \in \mathrm{Ker}(D) \cup \mathrm{Ker}(D)^{\perp}$. Nos résultats couvrent de nombreux modèles, comme par exemple le modèle standard de conductance sur $\mathbb{Z}^d$ (même avec de longs filaments), le réseau de résistances de Miller–Abrahams pour la conduction dans les solides amorphes (dont on peut maintenant étendre les bornes en accord avec la loi de Mott précédemment obtenue dans (*Ann. Appl. Probab.* **19** (2009) 1459–1494, *Comm. Math. Phys.* **281** (2008) 263–286, *Comm. Math. Phys.* **263** (2006) 21–64) pour la marche aléatoire de Mott), les réseaux de résistances sur l'amas surcritique de la percolation discrète ou continue, ou les réseaux de résistances sur les réseaux cristallins et sur les triangulations de Delaunay.

MSC2020 subject classifications: Primary 82D30; 74Q05; secondary 60G55

Keywords: Simple point process; Resistor network; Miller–Abrahams random resistor network; Random conductance model; Discrete and continuum supercritical percolation; Stochastic homogenization; 2-scale convergence

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