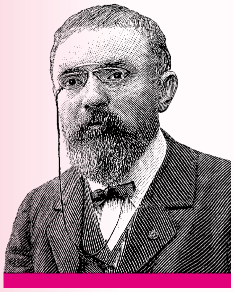

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Spatial Markov property in Brownian disks

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Abstract. We derive a new representation of the Brownian disk in terms of a forest of labeled trees, where labels correspond to distances from a subset of the boundary. We then use this representation to obtain a spatial Markov property showing that the complement of a hull centered at a boundary point of a Brownian disk is again a Brownian disk, with a random perimeter, and is independent of the hull conditionally on its perimeter. Our proofs rely in part on a study of the peeling process for triangulations with a boundary, which is of independent interest. The results of the present work will be applied to a continuous version of the peeling process for the Brownian half-plane in a companion paper.

Résumé. Nous donnons une nouvelle représentation du disque brownien en termes d'une forêt d'arbres étiquetés, où les étiquettes correspondent aux distances depuis une partie de la frontière. Nous utilisons cette représentation pour obtenir une propriété de Markov spatiale montrant que le complémentaire d'une boule complétée centrée en un point de la frontière est encore un disque brownien, avec un périmètre aléatoire, et est conditionnellement à ce périmètre indépendant de la boule complétée. Les preuves reposent en partie sur une étude du processus d'épluchage pour des triangulations avec frontière, qui est d'intérêt indépendant. Les résultats du présent travail seront appliqués dans un article suivant à une version continue du processus d'épluchage pour le demi-plan brownien.

MSC2020 subject classifications: Primary 60D05; secondary 05C80

Keywords: Brownian geometry; Brownian disk; spatial Markov property; hull

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On large 3/2-stable maps

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Abstract. We discuss asymptotics of large Boltzmann random planar maps such that every vertex of degree k has weight of order k^{-2}. Infinite maps of that kind were studied by Budd, Curien and Marzouk. These maps correspond to the dual of the discrete α-stable maps studied by Le Gall and Miermont for $\alpha = 3/2$ and to the gaskets of critical $O(2)$-decorated random planar maps. We compute the asymptotics of the graph and first passage percolation distances between two uniform vertices, which are respectively equivalent in probability to $(\log \ell)^2/\pi^2$ and $2(\log \ell)/(\pi^2 p_{\mathbf{q}})$ when the perimeter of the map ℓ goes to ∞, where $p_{\mathbf{q}}$ is a constant depending on the model. We also show that the diameter is of the same order as those distances for both metrics and establish the absence of scaling limits in the sense of Gromov–Prokhorov or Gromov–Hausdorff for lack of tightness. To study the peeling exploration of these maps, we prove local limit and scaling limit theorems for a class of random walks with heavy tails conditioned to remain positive until they die at $-\ell$ towards processes that we call stable Lévy processes conditioned to stay positive until they jump and die at -1.

Résumé. Nous traitons de l'asymptotique de grandes cartes aléatoires de Boltzmann telles qu'un sommet de degré k a un poids d'ordre k^{-2}. Les cartes infinies de ce type ont été étudiées par Budd, Curien et Marzouk. Ces cartes correspondent au dual des cartes α-stables discrètes étudiées par Le Gall et Miermont pour $\alpha = 3/2$ et aux ≪ gaskets ≫ des cartes décorées de boucles $O(2)$ critiques. Nous calculons l'asymptotique des distances de graphe et de percolation de premier passage entre deux sommets uniformes, qui sont équivalentes respectivement à $(\log \ell)^2/\pi^2$ et $2(\log \ell)/(\pi^2 p_{\mathbf{q}})$ quand le périmètre ℓ de la carte tend vers l'infini, où $p_{\mathbf{q}}$ est une constante dépendant du modèle. Nous montrons aussi que le diamètre est du même ordre de grandeur que ces distances pour les deux métriques et établissons l'absence de limite d'échelle au sens de Gromov–Prokhorov ou de Gromov–Hausdorff faute de tension. Pour étudier l'exploration par épluchage de ces cartes, nous démontrons des théorèmes de limite locale et de limite d'échelle pour une classe de marches aléatoires à queues lourdes conditionnées à rester positives jusqu'à ce qu'elles meurent en $-\ell$ vers des processus que nous appelons des processus de Lévy stables conditionnés à rester positifs jusqu'à ce qu'ils sautent et meurent en -1.

MSC2020 subject classifications: Primary 05C80; 60G52; 60F17; secondary 60K35

Keywords: Random planar map; Stable Lévy process; Scaling limit; Graph distance; First passage percolation; Peeling exploration

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Fractal geometry of the valleys of the parabolic Anderson equation

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Abstract. We study the macroscopic fractal properties of the *deep valleys* of the solution of the $(1+1)$-dimensional parabolic Anderson equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,x) \equiv u_0(x), & x\in\mathbb{R}, \end{cases}$$

where $\dot{W}$ is the time-space white noise and $0 < \inf_{x\in\mathbb{R}} u_0(x) \le \sup_{x\in\mathbb{R}} u_0(x) < \infty$. Unlike the macroscopic multifractality of the tall peaks, we show that valleys of the parabolic Anderson equation are macroscopically monofractal. In fact, the macroscopic Hausdorff dimension (introduced by Barlow and Taylor (*J. Phys. A: Math. Gen.* **22** (1989) 2621; *Proc. Lond. Math. Soc.* **3** (1992) 125–152)) of the valleys undergoes a *phase transition* as the level of the depth varies and this phenomenon does not depend on the initial data. The key tool of our proof is a lower bound to the lower tail probability of the parabolic Anderson equation. Such a lower bound is obtained for the first time in this paper and will be derived by utilizing the connection between the parabolic Anderson equation and the Kardar–Parisi–Zhang equation. Our techniques of proving this lower bound can be extended to other models in the KPZ universality class including the KPZ fixed point.

Résumé. Nous étudions les propriétés fractales macroscopiques des *vallées profondes* de la solution de l'équation parabolique d'Anderson en dimension $(1+1)$:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\dot{W}(t,x), & t>0, x\in\mathbb{R}, \\ u(0,x) \equiv u_0(x), & x\in\mathbb{R}, \end{cases}$$

où $\dot{W}$ désigne le bruit blanc espace-temps et $0 < \inf_{x\in\mathbb{R}} u_0(x) \le \sup_{x\in\mathbb{R}} u_0(x) < \infty$. Contrairement à la multifractalité macroscopique des pics élevés, nous montrons que les vallées de l'équation parabolique d'Anderson sont monofractales à l'échelle macroscopique. En effet, la dimension de Hausdorff macroscopique (introduite par Barlow et Taylor (*J. Phys. A: Math. Gen.* **22** (1989) 2621 ; *Proc. Lond. Math. Soc.* **3** (1992) 125–152)) des vallées subit une *transition de phase* lorsque le niveau de profondeur varie, et ce phénomène est indépendant des données initiales. L'outil clé de notre démonstration est une borne inférieure de la probabilité de la queue inférieure de la solution de l'équation parabolique d'Anderson. Une telle borne inférieure est obtenue pour la première fois dans cet article, et elle est dérivée en exploitant la connexion entre l'équation parabolique d'Anderson et l'équation de Kardar–Parisi–Zhang (KPZ). Nos techniques de preuve de cette borne inférieure peuvent être généralisées à d'autres modèles appartenant à la classe d'universalité KPZ, y compris le point fixe KPZ.

MSC2020 subject classifications: Primary 60H15; secondary 35R60; 60K37

Keywords: Parabolic Anderson model; KPZ equation; Macroscopic Hausdorff dimension

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A transfer principle for branched rough paths

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Abstract. A branched rough path $\boldsymbol{X}$ consists of a rough integral calculus for $X\colon [0,T] \to \mathbb{R}^d$ which may fail to satisfy integration by parts. Using Kelly's *bracket extension* (Kelly (2012)), we define a notion of pushforward of branched rough paths through smooth maps, which leads naturally to a definition of branched rough path on a smooth manifold. Once a covariant derivative is fixed, we are able to give a canonical, coordinate-free definition of integral against such rough paths. After characterising quasi-geometric rough paths in terms of their bracket extension, we use the same framework to define manifold-valued rough differential equations (RDEs) driven by quasi-geometric rough paths valued in a different manifold. These results extend previous work on $3 > p$-rough paths (*J. Lond. Math. Soc.* (2022)), itself a generalisation of the Itô calculus on manifolds developed by Schwartz, Meyer and Émery (*Sémin. Probab.* **16** (1982) 1–148; *Sémin. Probab.* **15** (1981) 44–102; *Stochastic Calculus in Manifolds* (1989) Springer; In *Séminaire de Probabilités, XXIV, 1988/89* (1990) 407–441 Springer), to the setting of non-geometric rough calculus of arbitrarily low regularity.

Résumé. Un chemin rugueux ramifié $\boldsymbol{X}$ consiste en un calcul intégral rugueux pour $X\colon [0,T] \to \mathbb{R}^d$ qui peut ne pas satisfaire à l'intégration par parties. En utilisant la *bracket extension* de Kelly (Kelly (2012)), nous définissons une notion de poussée vers l'avant des chemins rugueux ramifiés par des applications régulières, ce qui conduit naturellement à une définition de chemin rugueux ramifié sur une variété lisse. Une fois une dérivée covariante fixée, nous pouvons donner une définition canonique et sans coordonnées de l'intégrale par rapport à de tels chemins rugueux. Après avoir caractérisé les chemins rugueux quasi-géométriques en termes de *bracket extension*, nous utilisons le même cadre pour définir des équations différentielles rugueuses à valeurs de dans une variété pilotées par des chemins rugueux quasi-géométriques à valeurs dans une variété différente. Ces résultats étendent les travaux antérieurs sur les chemins $3 > p$-rugueux (*J. Lond. Math. Soc.* (2022)), lui-même une généralisation du calcul d'Itô sur les variétés développé par Schwartz, Meyer et Émery (*Sémin. Probab.* **16** (1982) 1–148 ; *Sémin. Probab.* **15** (1981) 44–102 ; *Stochastic Calculus in Manifolds* (1989) Springer ; In *Séminaire de Probabilités, XXIV, 1988/89* (1990) 407–441 Springer), au cadre du calcul rugueux non géométrique de régularité arbitrairement faible.

MSC2020 subject classifications: Primary 60L20; 60L70; secondary 16T05

Keywords: Branched rough paths; Manifolds

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Quantitative weak propagation of chaos for McKean–Vlasov SDEs driven by stable processes

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Abstract. We consider a general McKean–Vlasov stochastic differential equation driven by a rotationally invariant α-stable process on $\mathbb{R}^d$, with $\alpha \in (1,2)$. We assume that the diffusion coefficient is the identity matrix and that the drift is bounded and Hölder continuous with respect to both space and measure variables, in some precise sense. The main goal of this work is to prove new weak propagation of chaos estimates for the associated mean-field interacting particle system. We also establish a pointwise control on the difference between the density of one particle and the density of the limiting McKean–Vlasov stochastic differential equation. Our study relies on the regularizing properties and the dynamics of the semigroup associated with the McKean–Vlasov stochastic differential equation, which acts on functions defined on $\mathcal{P}_\beta(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$ having a finite moment of order $\beta \in (1,\alpha)$. More precisely, the dynamics of the semigroup is described by a backward Kolmogorov partial differential equation defined on the strip $[0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)$.

Résumé. On considère une équation différentielle stochastique de McKean–Vlasov dirigée par un processus α-stable rotationnellement invariant sur $\mathbb{R}^d$, avec $\alpha \in (1,2)$. On suppose que le coefficient de diffusion est la matrice identité et que la dérive est bornée et höldérienne en un sens précis par rapport aux variables d'espace et de mesure. L'objectif principal de ce travail est de prouver de nouvelles estimations de propagation du chaos faible pour le système de particules en interaction associé. On établit également un contrôle ponctuel entre la densité d'une particule et la densité du processus de McKean–Vlasov limite. Notre étude repose sur les propriétés régularisantes et la dynamique du semi-groupe associé à l'équation différentielle stochastique de McKean–Vlasov, qui agit sur les fonctions définies sur $\mathcal{P}_\beta(\mathbb{R}^d)$, l'espace des mesures de probabilités sur $\mathbb{R}^d$ ayant un moment d'ordre $\beta \in (1,\alpha)$ fini. Plus précisément, la dynamique du semi-groupe est décrite par une équation aux dérivées partielles de Kolmogorov rétrograde définie sur la bande $[0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)$.

MSC2020 subject classifications: Primary 60H10; 60G52; 60H30; secondary 35K40

Keywords: McKean–Vlasov stochastic differential equations; Stable processes; Martingale problem; Density estimates; Backward Kolmogorov partial differential equations; Propagation of chaos; Parametrix method

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On the fluctuations of an SDE system modelling grid cells

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Abstract. Several differential equation models have been proposed to explain the formation of patterns characteristic of the grid cell network. Understanding the effect of noise on these models is one of the key open questions in computational neuroscience. In the present work, we continue the analysis initiated in a previous paper of an SDE system commonly proposed for modelling. We show that the fluctuations of the empirical measure associated to the system around its mean field limit converge to the solution of a Langevin SPDE. The interaction between different columns of neurons along the cortex prescribes a peculiar scaling regime.

Résumé. Plusieurs modèles d'équations différentielles ont été proposés pour expliquer la formation de motifs caractéristiques du réseau de cellules en grille. Comprendre l'effet du bruit sur ces modèles est l'une des questions ouvertes clés en neurosciences computationnelles. Dans cette étude, nous poursuivons l'analyse initiée dans un article précédent d'un système EDS fréquemment proposé pour la modélisation. Nous montrons que les fluctuations de la mesure empirique associée au système autour de sa limite de champ moyen convergent vers la solution d'une EDPS de Langevin. L'interaction entre différentes colonnes de neurones le long du cortex induit un régime d'échelle particulier.

MSC2020 subject classifications: Primary 35Q70; 60F05; 92B20; secondary 35Q92; 35R60

Keywords: Grid cells; Fluctuations; Interacting particle system; Empirical measure; Fokker–Planck equation

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Solving the hyperbolic Anderson model 1: Skorohod setting

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Abstract. This paper is concerned with a wave equation in dimension $d \in \{1, 2, 3\}$, with a multiplicative space-time Gaussian noise which is fractional in time and homogeneous in space. We provide necessary and sufficient conditions on the space-time covariance of the Gaussian noise, allowing the existence and uniqueness of a mild Skorohod solution.

Résumé. Dans cet article, nous nous intéressons à un modèle d'équation des ondes, en dimension $d \in \{1, 2, 3\}$, perturbée par un bruit gaussien espace-temps multiplicatif, fractionnaire en temps et homogène en espace. Nous fournissons des conditions nécessaires et suffisantes sur la covariance du bruit qui garantissent l'existence et l'unicité d'une solution mild, interprétée au sens de Skorohod.

MSC2020 subject classifications: 60H15; 60H07; 60F10

Keywords: Stochastic wave equation; Malliavin calculus; Skorohod equation

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Reflected BSEs, reflected BSDEs and fixed-point problems

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Abstract. This paper gives a new approach to the reflected backward stochastic equations (RBSEs), which contain as a special case many interesting dynamics such as reflected backward stochastic differential equations (RBSDEs). Our method translates an RBSE into a fixed-point problem in the space of stochastic processes with bounded variations. This new approach works well in the setting of general discontinuous filtrations, general irregular obstacles, path dependent coefficients, as well as L^p-data for the terminal conditions and the generators. Consequently, we obtain a general existence and uniqueness result for RBSEs as well as RBSDEs.

Résumé. Cette étude propose une nouvelle approche sur les équations stochastiques rétrogrades réfléhies (ESRR), incluant notamment des dynamiques intéressantes telles que les équations différentielles stochastiques rétrogrades réfléchies (EDSRR). Notre méthode transforme l'ESRR en un problème de point fixe dans l'espace de processus aléatoires à variations bornées. Cette nouvelle approche s'applique bien dans le cadre de filtrations générales discontinues, d'obstacles irréguliers généraux, de coefficients dépendant du chemin, ainsi que de données dans L^p pour les conditions terminales et les générateurs. Nous obtenons ainsi des résultats d'existence et d'unicité généraux pour les EDSRR, de même que pour les ESRR.

MSC2020 subject classifications: 60H10; 60H30

Keywords: Reflected backward stochastic equation; Reflected backward stochastic differential equation; Fixed-point problems; Path-dependent coefficients; General filtration; L^p-data

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The conditioned Lyapunov spectrum for random dynamical systems

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Abstract. We establish the existence of a full spectrum of Lyapunov exponents for memoryless random dynamical systems with absorption. To this end, we crucially embed the process conditioned to never being absorbed, the Q-process, into the framework of random dynamical systems, allowing us to study multiplicative ergodic properties. We show that the finite-time Lyapunov exponents converge in conditioned probability and apply our results to iterated function systems and stochastic differential equations.

Résumé. Nous établissons l'existence d'un spectre complet d'exposants de Lyapunov pour les systèmes dynamiques aléatoires sans mémoire avec absorption. Pour celà, nous adaptons le processus conditionné à ne jamais être absorbé à la structure des systèmes dynamiques aléatoires, ce qui nous permet d'étudier ses propriétés multiplicatives ergodiques. Nous montrons que la convergence vers les exposants de Lyapunov se produit en probabilité conditionnelle, et nous appliquons nos résultats aux systèmes de fonctions itérées et aux équations différentielles stochastiques.

MSC2020 subject classifications: 37H05; 37H15; 47D07; 60J05; 60J25

Keywords: Absorbed Markov process; Quasi-stationary distribution; Quasi-ergodic distribution; Lyapunov spectrum; Lyapunov exponent; Q-process

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Slow-fast dynamics in stochastic Lotka–Volterra systems

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Abstract. We investigate the large population dynamics of a family of stochastic particle systems with three-state cyclic individual behaviour and parameter-dependent transition rates. On short time scales, the dynamics turns out to be approximated by an integrable Hamiltonian system whose phase space is foliated by periodic trajectories. This feature suggests to consider the effective dynamics of the long-term process that results from averaging over the rapid oscillations. We establish the convergence of this process in the large population limit to the solutions of an explicit stochastic differential equation. Remarkably, this averaging phenomenon is complemented by the convergence of stationary measures. The proof of averaging follows the Stroock-Varadhan approach to martingale problems and relies on a fine analysis of the system's dynamical features.

Résumé. Nous étudions la dynamique pour des populations de grande taille, d'une famille de systèmes de particules stochastiques à comportement individuel cyclique à trois états, dont les taux de transition dépendent d'un paramètre. Sur des échelles de temps courtes, la dynamique se trouve bien approchée par un système hamiltonien intégrable dont l'espace de phase est feuilleté par des trajectoires périodiques. Cette propriété suggère d'étudier la dynamique effective du processus à long terme qui résulte en effectuant une moyennisation sur les oscillations rapides. Nous prouvons la convergence de ce processus dans la limite des populations de grande taille, vers des solutions d'une équation différentielle stochastique explicite. De manière remarquable, ce phénomène de moyennisation est complété par la convergence des mesures stationnaires. La preuve de moyennisation s'inscrit dans l'approche de Stoock et Varadhan au problèmes martingales, et s'appuie sur une analyse fine des propriétés dynamiques du système.

MSC2020 subject classifications: Primary 60J27; secondary 92D25

Keywords: Interacting particle systems; Averaging

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Extreme local extrema of the sine-Gordon field

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Abstract. We prove that for $\beta < 6\pi$ the local extremal process of the massive sine-Gordon field on the unit torus in $d = 2$ converges to a Poisson point process with random intensity measure $Z^{SG}(dx) \otimes e^{-\alpha h}\, dh$ for some $\alpha > 0$. The proof combines existing methods for the extremal process associated with the Gaussian free field, which was introduced and studied by Biskup and Louidor, and a strong coupling between the sine-Gordon field and the Gaussian free field.

Résumé. Pour $\beta < 6\pi$, on démontre que le processus extrême local du modèle sine-Gordon sur le tore de longueur unité en dimension $d = 2$ converge vers un processus de Poisson d'intensité $Z^{SG}(dx) \otimes e^{-\alpha h}\, dh$, où $Z^{SG}(dx)$ est une mesure aléatoire et $\alpha > 0$. La preuve repose sur des méthodes existantes pour le processus extrême local du champ libre gaussien, introduit et étudié par Biskup et Louidor, et un couplage fort entre le modèle sine-Gordon et le champ libre gaussien.

MSC2020 subject classifications: Primary 60G60; secondary 82B41

Keywords: Local extrema; Sine-Gordon field; Log-correlated fields; Gaussian free field; Coupling

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Ergodicity results for the open KPZ equation

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Abstract. We give a new proof of existence as well as two proofs of uniqueness of the invariant measure of the open-boundary KPZ equation on [0, 1], for all possible choices of inhomogeneous Neumann boundary data. Both proofs yield an exponential convergence result in total variation when combined with the strong Feller property which was recently established in Knizel and Matetski (2022). An important ingredient in both proofs is the construction of a compact state space for the Markov operator to act on, which measurably contains all of the usual Hölder spaces modulo constants. Along the way, the strong Feller property is extended to this larger class of initial conditions. The arguments do not rely on exact descriptions of the invariant measures, and some of the results generalize to the case of a spatially colored noise and other boundary conditions.

Résumé. Nous proposons une nouvelle démonstration de l'existence ainsi que deux démonstrations de l'unicité de la mesure invariante de l'équation KPZ sur l'intervalle [0, 1] avec bord ouvert, pour tous les choix possibles de conditions au bord de Neumann inhomogènes. Les deux preuves conduisent à un résultat de convergence exponentielle en variation totale lorsqu'elles sont combinées avec la propriété de Feller forte récemment établie dans Knizel and Matetski (2022). Un ingrédient important dans les deux démonstrations est la construction d'un espace d'états compact sur lequel agit l'opérateur de Markov, contenant de manière mesurable tous les espaces de Hölder usuels modulo des constantes. Au fil de la démonstration, la propriété de Feller forte est étendue à cette classe élargie de conditions initiales. Les arguments ne reposent pas sur des descriptions exactes des mesures invariantes, et certains des résultats se généralisent au cas d'un bruit spatialement coloré ainsi qu'à d'autres conditions au bord.

MSC2020 subject classifications: 60H15; 60J25

Keywords: Stochastic partial differential equation; Spectral gap; Strong Feller property; Total variation norm; KPZ equation; Ergodicity

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Equivalence of metric gluing and conformal welding in γ-Liouville quantum gravity for $\gamma \in (0, 2)$

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Abstract. We consider the γ-Liouville quantum gravity (LQG) model for $\gamma \in (0, 2)$, formally described by $e^{\gamma h}$ where h is a Gaussian free field on a planar domain D. Sheffield showed that when a certain type of LQG surface, called a quantum wedge, is decorated by an appropriate independent SLE curve, the wedge is cut into two independent surfaces which are themselves quantum wedges, and that the original surface can be recovered as a unique conformal welding. We prove that the original surface can also be obtained as a metric space quotient of the two wedges, extending results of Gwynne and Miller in the special case $\gamma = \sqrt{8/3}$ to the whole subcritical regime $\gamma \in (0, 2)$.

Since the proof for $\gamma = \sqrt{8/3}$ used estimates for Brownian surfaces, which are equivalent to γ-LQG surfaces only when $\gamma = \sqrt{8/3}$, we instead use GFF techniques to establish estimates relating distances, areas and boundary lengths, as well as bi-Hölder continuity of the LQG metric w.r.t. the Euclidean metric at the boundary, which may be of independent interest.

Résumé. Nous considérons le modèle de gravité quantique Liouville (γ-LQG) de paramètre $\gamma \in (0, 2)$, formellement décrit par $e^{\gamma h}$ où h est un champ libre gaussien (GFF) sur un domaine planaire D. Sheffield a prouvé que lorsqu'un certain type de surface LQG, appelé coin quantique, est décoré par une courbe SLE indépendante appropriée, le coin est découpé en deux surfaces indépendantes qui sont elles-mêmes des coins quantiques, et que la surface d'origine peut être récupérée uniquement comme une soudure conforme. Nous prouvons que la surface d'origine peut également être obtenue comme un quotient métrique des deux coins, étendant les résultats de Gwynne et Miller dans le cas particulier $\gamma = \sqrt{8/3}$ à l'ensemble du régime sous-critique $\gamma \in (0, 2)$.

Puisque la preuve de $\gamma = \sqrt{8/3}$ utilisait des estimations pour les surfaces browniennes, qui sont équivalentes aux surfaces γ-LQG uniquement lorsque $\gamma = \sqrt{8/3}$, nous utilisons plutôt les techniques GFF pour établir des estimations concernant les distances, les volumes, et les longueurs des frontières, ainsi que la continuité bi-Hölder de la métrique LQG à l'égard de la métrique euclidienne à la frontière, qui peut présenter un intérêt indépendant.

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Fusion asymptotics for Liouville correlation functions

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Abstract. David, Kupiainen, Rhodes and Vargas introduced a probabilistic framework based on the Gaussian Free Field and Gaussian Multiplicative Chaos in order to make sense rigorously of the path integral approach to Liouville conformal field theory. We use this setting to compute fusion estimates for the four-point correlation function on the Riemann sphere, and find that it is consistent with predictions from the framework of theoretical physics known as the conformal bootstrap. This result fits naturally into the famous KPZ conjecture which relates the four-point function to the expected density of points around the root of a large random planar map weighted by some statistical mechanics model.

From a purely probabilistic point of view, we give non-trivial results on negative moments of GMC. We give exact formulae based on the DOZZ formula in the Liouville case and asymptotic behaviours in the other cases, with a probabilistic representation of the limit.

Finally, we show how to extend our results to boundary LCFT, treating the cases of the fusion of two boundary or bulk insertions as well as the absorption of a bulk insertion on the boundary.

Résumé. David, Kupiainen, Rhodes et Vargas ont introduit un cadre probabiliste fondé sur le champ libre gaussien et le chaos multiplicatif gaussien pour donner une construction rigoureuse de l'intégrale de chemin de la théorie conforme des champs de Liouville. Nous utilisons cette approche pour calculer des estimées de fusion pour la fonction de corrélation à quatre points sur la sphère de Riemann, en accord avec les prédictions de la physique théorique venant du "bootstrap conforme". Ce résultat s'inscrit naturellement dans le cadre de la célèbre conjecture KPZ, laquelle fait le lien entre la corrélation à quatre points et la densité moyenne de points autour de la racine d'une grande carte planaire aléatoire pondérée par un modèle de physique statistique.

D'un point de vue purement probabiliste, nous établissons des résultats non triviaux sur les moments négatifs du chaos multiplicatif gaussien. Nous donnons des formules exactes liées à la formule DOZZ pour les corrélations de la théorie de Liouville. Dans le cas général, nous donnons des estimées asymptotiques avec une représentation probabiliste de la limite.

Enfin, nous étendons nos résultats à la théorie de Liouville à bord, en traitant la fusion de deux insertions au bord ou dans l'intérieur de la surface, ainsi que l'absorption d'une insertion de l'intérieur par le bord.

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Keywords: log-correlated fields; Gaussian multiplicative chaos; Liouville conformal field theory

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Reducing metastable continuous-space Markov chains to Markov chains on a finite set

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Abstract. We consider continuous-space, discrete-time Markov chains on $\mathbb{R}^d$, that admit a finite number N of metastable states. Our main motivation for investigating these processes is to analyse random Poincaré maps, which describe random perturbations of ordinary differential equations admitting several periodic orbits. We show that under a few general assumptions, which hold in many examples of interest, the kernels of these Markov chains admit N eigenvalues exponentially close to 1, which are separated from the remainder of the spectrum by a spectral gap that can be quantified. Our main result states that these Markov chains can be approximated, uniformly in time, by a finite Markov chain with N states. The transition probabilities of the finite chain are exponentially close to first-passage probabilities at neighbourhoods of metastable states, when starting in suitable quasistationary distributions.

Résumé. Nous considérons des chaînes de Markov à espace continu et temps discret sur $\mathbb{R}^d$, admettant un nombre fini N d'états métastables. Notre principale motivation pour étudier ces processus est l'analyse d'applications de Poincaré aléatoires, décrivant des perturbations aléatoires d'équations différentielles ordinaires admettant plusieurs orbites périodiques. Nous montrons que sous certaines hypothèses générales, qui sont satisfaites dans de nombreuses applications intéressantes, les noyaux de ces chaînes de Markov admettent N valeurs propres exponentiellement proches de 1, qui sont séparées du reste du spectre par un trou spectral qui peut être quantifié. Notre résultat principal affirme que ces chaînes de Markov peuvent être approchées, uniformément en temps, par une chaîne de Markov finie à N états. Les probabilités de transition de la chaîne finie sont exponentiellement proches de probabilités de premier passage dans des voisinages d'états métastables, en partant de distributions quasistationnaires adéquates.

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Keywords: Random Poincaré map; Metastability; Quasistationary distribution; Trace process; Large deviations

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Large deviations asymptotics for unbounded additive functionals of diffusion processes

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Abstract. We study large deviations asymptotics for a class of unbounded additive functionals, interpreted as normalized accumulated areas, of one-dimensional Langevin diffusions with sub-linear gradient drifts. Our results provide parametric insights on the speed and the rate functions in terms of the growth rate of the drift and the growth rate of the additive functional. We find a critical value in terms of these growth parameters that dictates regions of sub-linear speed for our large deviations asymptotics. Our approach is based upon various constructions of independent interest, including a decomposition of the diffusion process in terms of alternating renewal cycles and a detailed analysis of the paths during a cycle using suitable time and spatial scales. The key to the sub-linear behavior is a heavy-tailed large deviations phenomenon arising from the principle of a single big jump coupled with the result that at each regeneration cycle the upper-tail asymptotic behavior of the accumulated area of the diffusion process is proven to be semi-exponential (i.e., of heavy-tailed Weibull type).

Résumé. Nous étudions les asymptotiques de grandes déviations pour une classe de fonctionnelles additives non bornées, interprétées comme des zones accumulées normalisées, de diffusions de Langevin unidimensionnelles avec des dérives de gradient sous-linéaires. Nos résultats fournissent des informations paramétriques sur les fonctions de vitesse et de taux en termes de taux de croissance de la dérive et de taux de croissance de la fonctionnelle additive. Nous trouvons une valeur critique en termes de ces paramètres de croissance qui identifie les régions de vitesse sous-linéaire pour nos asymptotiques de grandes déviations. Notre approche s'appuie sur diverses constructions d'intérêt indépendant, dont une décomposition du processus de diffusion en termes de cycles de renouvellement alternés et une analyse détaillée des chemins au cours d'un cycle à l'aide d'échelles temporelles et spatiales appropriées. La clé du comportement sous-linéaire est un phénomène de grandes déviations à queue lourde découlant du principe d'un seul grand saut couplé au résultat qu'à chaque cycle de régénération le comportement asymptotique de la queue supérieure de la zone accumulée du processus de diffusion est semi-exponentiel (c'est-à-dire à queue lourde de type Weibull).

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Robust density estimation with the $\mathbb{L}_1$-loss. Applications to the estimation of a density on the line satisfying a shape constraint

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Abstract. We tackle the problem of estimating the distribution of presumed i.i.d. observations for the total variation loss. Our approach is based on density models and is versatile enough to cope with many different ones, including some for which the Maximum Likelihood Estimator (MLE for short) does not exist. We mainly illustrate the properties of our estimator on models of densities on the line that satisfy a shape constraint. We show that it possesses some similar optimality properties, with regard to some global rates of convergence, as the MLE does when it exists. It also enjoys some adaptation properties with respect to some specific densities in the model for which our estimator is proven to converge at the parametric rate $1/\sqrt{n}$. More important is the fact that our estimator is robust, not only with respect to model misspecification, but also to contamination, the presence of outliers among the dataset and the equidistribution assumption we started from. This means that the estimator performs almost as well as if the data were i.i.d. with density p in a situation where these data are only independent and most of their marginals are close enough in total variation to a distribution with density p. We also show that our estimator converges to the average density of the data, when this density belongs to the model, even when none of the marginal densities does. Our main result on the risk of the estimator takes the form of an exponential deviation inequality which is nonasymptotic and involves explicit numerical constants. We deduce from it several global rates of convergence, including some bounds for the minimax $\mathbb{L}_1$-risks over the sets of concave, log-concave or more generally s-concave densities with $s > -1$. These bounds derive from some approximation results of densities which are monotone, convex, concave or more generally s-concave. These results may be of independent interest.

Résumé. Nous abordons le problème d'estimation de la distribution d'observations présumées i.i.d. pour la perte en variation totale. Notre procédure, développée sur des modèles de densité, se révèle suffisamment versatile pour s'appliquer à certains d'entre eux pour lesquels l'estimateur du maximum de vraisemblance (MLE) n'existe pas. Nous illustrons principalement les propriétés de notre estimateur sur des modèles de densités univariées qui satisfont une certaine contrainte de forme. Nous montrons qu'il partage avec le MLE – lorsqu'il existe – certaines propriétés remarquables d'optimalité et d'adaptation. En particulier, notre estimateur atteint des vitesses de convergence globales optimales dans tous les modèles que nous considérons. Il bénéficie également d'intéressantes propriétés d'adaptation vis à vis de certaines densités spécifiques du modèle pour lesquelles il converge à la vitesse paramétrique $1/\sqrt{n}$. De façon plus remarquable encore, notre estimateur est robuste, non seulement par rapport à une mauvaise spécification du modèle, mais également à une possible contamination des données, à la présence de valeurs aberrantes dans l'échantillon et à l'hypothèse d'équidistribution des données que nous avions supposée vraie dans notre cadre initial. En d'autres termes, dans une situation où ces données ne sont que partiellement indépendantes et où la plupart de leurs lois marginales sont suffisamment proches en variation totale d'une certaine distribution avec une densité p, notre estimateur s'avère presque aussi performant que si les données étaient vraiment i.i.d. avec cette même densité p. Nous montrons également que dans ce cas de non-équidistribution des données, notre estimateur converge vers leur densité moyenne lorsque celle-ci appartient au modèle, et ce, même lorsqu'aucune des densités marginales n'y appartient. Notre principal résultat sur le risque de l'estimateur s'exprime sous forme d'une inégalité de déviation exponentielle non asymptotique et pour laquelle nous donnons des constantes numériques explicites. Nous en déduisons plusieurs vitesses de convergence globales, dont des bornes pour les risques minimax pour la norme $\mathbb{L}_1$ sur les ensembles de densités concaves, log-concaves ou plus généralement s-concaves avec $s > -1$. Ces bornes dérivent de résultats d'approximation nouveaux et d'intérêt indépendant pour des densités monotones, convexes, concaves ou plus généralement s-concaves.

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Keywords: Density estimation; Robust estimation; Shape constraint; Total variation loss; Minimax theory; Approximation theory

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Gibbs measures for the repulsive Bose gas

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Abstract. We prove the existence of Gibbs measures for the Feynman representation of the Bose gas with non-negative interaction in the grand-canonical ensemble. Our results are valid for all negative chemical potentials as well as slightly positive chemical potentials. We consider both the Gibbs property of marked points as well as a Markov–Gibbs property of paths.

Résumé. On montre l'existence de la distribution de Gibbs pour la représentation de Feynman du gaz de Bose avec une interaction non négative dans l'ensemble grand-canonique. Nos résultats sont valables pour tous les potentiels chimiques négatifs ainsi que pour les potentiels chimiques légèrement positifs. On considère à la fois la propriété de Gibbs des points marqués et la propriété de Markov–Gibbs des chemins.

MSC2020 subject classifications: Primary 60K35; secondary 82B21; 82B41

Keywords: Gibbs measures; Bose gas; Feynman representation

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Invariant measures for multilane exclusion process

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Abstract. We consider the simple exclusion process on $\mathbb{Z} \times \{0, 1\}$ (that is, an "horizontal ladder" composed of 2 lanes) depending on 6 parameters. Particles can jump according to a lane-dependent translation-invariant nearest neighbour jump kernel, i.e. "horizontally" along each lane, and "vertically" along the scales of the ladder. We prove that generically, the set of extremal invariant measures consists of *(i)* horizontally translation-invariant Bernoulli measures; and, modulo translations along $\mathbb{Z}$: *(ii)* at most two shock measures (i.e. asymptotic to distinct Bernoulli measures at $\pm\infty$) with asymptotic densities 0 and 2; *(iii)* at most one (outside degenerate cases) shock measure with a density jump of magnitude 1. We fully determine this set for a range of parameter values. Our results can be generalized in several directions using the same approach and answer certain open questions formulated in (*Ann. Probab.* **33** (2005) 2255–2313) as a step towards the process on $\mathbb{Z}^2$.

Résumé. Nous considérons le processus d'exclusion simple sur $\mathbb{Z} \times \{0, 1\}$ (soit une échelle horizontale constituée de deux routes) dépendant de 6 paramètres. Les particules sautent horizontalement sur chaque route suivant un noyau de transition dépendant de la route, et verticalement d'une route à l'autre. Nous démontrons que l'ensemble des mesures invariantes extrémales se compose génériquement de *(i)* des mesures de Bernoulli invariantes par translations horizontales ; et, modulo des translations le long de $\mathbb{Z}$: *(ii)* au plus deux mesures de choc (i.e. asymptotiques à des mesures de Bernoulli distinctes en $\pm\infty$) de densités asymptotiques 0 and 2 ; *(iii)* au plus une (en dehors de cas dégénérés) mesure de choc avec un saut d'amplitude 1. Nous déterminons entièrement cet ensemble dans un domaine de valeurs des paramètres. Nos résultats se généralisent dans différentes directions suivant la même approche, et répondent à des questions ouvertes formulées dans (*Ann. Probab.* **33** (2005) 2255–2313) comme une étape vers la compréhension du processus sur $\mathbb{Z}^2$.

MSC2020 subject classifications: Primary 60K35; secondary 82C22

Keywords: Multilane exclusion process; Invariant measures; Blocking measures; Shock measures

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Homological percolation on a torus: Plaquettes and permutohedra

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Abstract. We study higher-dimensional homological analogues of bond percolation on a square lattice and site percolation on a triangular lattice.

By taking a quotient of certain infinite cell complexes by growing sublattices, we obtain finite cell complexes with a high degree of symmetry and with the topology of the torus $\mathbb{T}^d$. When random subcomplexes induce nontrivial i-dimensional cycles in the homology of the ambient torus, we call such cycles *giant*. We show that for every i and d there is a sharp transition from nonexistence of giant cycles to giant cycles spanning the homology of the torus.

We also prove convergence of the threshold function to a constant in certain cases. In particular, we prove that $p_c = 1/2$ in the case of middle dimension $i = d/2$ for both models. This gives finite-volume high-dimensional analogues of Kesten's theorems that $p_c = 1/2$ for bond percolation on a square lattice and site percolation on a triangular lattice.

Résumé. Nous étudions des analogues homologiques de dimension supérieure de la percolation de lien sur un réseau carré et la percolation de site sur un réseau triangulaire.

Nous obtenons des complexes cellulaires finies hautement symétriques homéomorphe au tore $\mathbb{T}^d$ en prenant un quotient de certains réseaux infinis. Si des sous-complexes aléatoires induisent des cycles i-dimensionnels non triviaux dans l'homologie du tore ambiant, nous appelons ces cycles *géants*. Nous démontrons que pour tout i et d il y a une transition de phase nette de l'inexistence de cycles géants et l'existence d'une base de cycles géants par l'homologie du tore.

Nous prouvons aussi la convergence de la fonction de seuil vers une constante dans certains cas. En particulier, nous prouvons que $p_c = 1/2$ dans le cas de la dimension moyenne $i = d/2$ pour les deux modèles. Cela donne des analogues, en dimension supérieure et en volume fini, des théorèmes de Kesten que $p_c = 1/2$ pour la percolation de lien sur un réseau carré et la percolation de site sur un réseau triangulaire.

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Improved rates of bootstrap approximation for the operator norm: A coordinate-free approach

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Abstract. Let $\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} X_i \otimes X_i$ denote the sample covariance operator of centered i.i.d. observations $X_1, \dots, X_n$ in a real separable Hilbert space, and let $\Sigma = \mathbf{E}(X_1 \otimes X_1)$. The focus of this paper is to understand how well the bootstrap can approximate the distribution of the operator norm error $\sqrt{n}\|\widehat{\Sigma} - \Sigma\|_{\text{op}}$, in settings where the eigenvalues of Σ decay as $\lambda_j(\Sigma) \asymp j^{-2\beta}$ for some fixed parameter $\beta > 1/2$. Our main result shows that the bootstrap can approximate the distribution of $\sqrt{n}\|\widehat{\Sigma} - \Sigma\|_{\text{op}}$ at a rate of order $n^{-\frac{\beta-1/2}{2\beta+4+\epsilon}}$ with respect to the Kolmogorov metric, for any fixed $\epsilon > 0$. In particular, this shows that the bootstrap can achieve near $n^{-1/2}$ rates in the regime of large β—which substantially improves on previous near $n^{-1/6}$ rates in the same regime. In addition to obtaining faster rates, our analysis leverages a fundamentally different perspective based on coordinate-free techniques. Moreover, our result holds in greater generality, and we propose a model that is compatible with both elliptical and Marčenko–Pastur models in high-dimensional Euclidean spaces, which may be of independent interest.

Résumé. Soit $\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} X_i \otimes X_i$ l'opérateur de covariance empirique d'observations i.i.d. centrées $X_1, \dots, X_n$ dans un espace de Hilbert réel, séparable, et soit $\Sigma = \mathbf{E}(X_1 \otimes X_1)$. L'objectif de cet article est de comprendre dans quelle mesure le bootstrap peut approcher la distribution de l'erreur en norme d'opérateur $\sqrt{n}\|\widehat{\Sigma} - \Sigma\|_{\text{op}}$, dans des contextes où les valeurs propres de Σ décroissent en $\lambda_j(\Sigma) \asymp j^{-2\beta}$, pour un paramètre fixe $\beta > 1/2$. Notre résultat principal montre que le bootstrap peut approcher la distribution de $\sqrt{n}\|\widehat{\Sigma} - \Sigma\|_{\text{op}}$ à une vitesse de l'ordre de $n^{-\frac{\beta-1/2}{2\beta+4+\epsilon}}$ pour la métrique de Kolmogorov, pour tout $\epsilon > 0$ fixe. En particulier, ceci montre que le bootstrap peut atteindre des taux proches de $n^{-1/2}$ dans le régime β grand — ce qui améliore considérablement les taux proches de $n^{-1/6}$ établis jusqu'à présent dans le même régime. En plus d'obtenir des taux plus rapides, notre analyse exploite une perspective fondamentalement différente reposant sur des techniques indépendantes du choix d'un système de coordoonnées. De plus, notre résultat est plus général et nous proposons un modèle compatible avec les modèles elliptiques et les modèles de Marčenko–Pastur dans les espaces euclidiens de grande dimension, ce qui présente un intérêt indépendant.

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Errata for: Limit law for the cover time of a random walk on a binary tree

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Abstract. We correct an error in (*Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 830–855), Lemma 1.4.

Résumé. Nous corrigeons une erreur dans (*Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 830–855), Lemma 1.4.

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Keywords: Cover time; Binary tree; Barrier estimate

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