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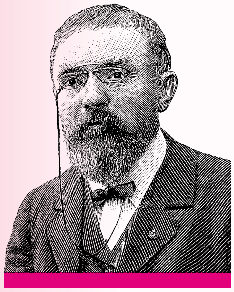

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Deviation inequalities and moderate deviations for the symmetric exclusion process

Fuqing Gao[1,a] and Jeremy Quastel[2,b]

[1]*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China,* [a]*fqgao@whu.edu.cn*
[2]*Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, ON M5S 3G3, Canada,* [b]*quastel@math.toronto.edu*

Abstract. We consider deviation inequalities and moderate deviations for the symmetric exclusion process on $\mathbb{Z}^d$. For any $d \geq 1$, we establish a Gaussian type deviation inequality with speed dependent on the dimension d for local additive functionals by the local average method, and obtain moderate deviation principles with the speed dependent on d and with the rate function dependent on the variance of the initial distribution for the occupation times via the martingale method. In the $d = 1$ case, by using the local average method, we show that any local additive functional is exponentially equivalent to a linear function of the occupation time, and establish the moderate deviation principle for any local additive functional.

Résumé. Nous considérons des inégalités de déviation et des déviations modérées pour le processus d'exclusion symétrique sur $\mathbb{Z}^d$. Pour tout $d \geq 1$, nous établissons une inégalité de déviation de type gaussien avec une vitesse dépendant de la dimension d pour les fonctionnels additifs locaux par la méthode de la moyenne locale, et obtenons des principes de déviation modérée avec une vitesse dépendant de la dimension d et avec la fonction de taux dépendant de la variance de la distribution initiale pour les temps d'occupation via la méthode de la martingale. Dans le cas $d = 1$, en utilisant la méthode de la moyenne locale, nous montrons que tout fonctionnel additif local est exponentiellement équivalent à une fonction linéaire du temps d'occupation, et établissons le principe de déviation modérée pour tout fonctionnel additif local.

MSC2020 subject classifications: Primary 60K35; 60J55; 60F10; secondary 82C22

Keywords: Additive functional; Occupation time; Moderate deviation; Deviation inequality; Exclusion process

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Hydrodynamics of a d-dimensional long jumps symmetric exclusion with a slow barrier

Pedro Cardoso[1,a], Patrícia Gonçalves[2,b] and Byron Jiménez-Oviedo[3,c]

[1] *Institute for Applied Mathematics, University of Bonn, Endenicher Allee, no. 60, 53115 Bonn, Germany,* [a]*pgondimc@uni-bonn.de*
[2] *Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, no. 1, 1049-001 Lisboa, Portugal,* [b]*pgoncalves@tecnico.ulisboa.pt*
[3] *Escuela de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Costa Rica, Heredia, Costa Rica,* [c]*byron.jimenez.oviedo@una.cr*

Abstract. We obtain the hydrodynamic limit of symmetric long-jumps exclusion in $\mathbb{Z}^d$ (for $d \geq 1$), where the jump rate is inversely proportional to a power of the jump's length with exponent $\gamma + 1$, where $\gamma \geq 2$. Moreover, movements between $\mathbb{Z}^{d-1} \times \mathbb{Z}_-^*$ and $\mathbb{Z}^{d-1} \times \mathbb{N}$ are slowed down by a factor $\alpha n^{-\beta}$ (with $\alpha > 0$ and $\beta \geq 0$). In the hydrodynamic limit we obtain the heat equation in $\mathbb{R}^d$ without boundary conditions or with homogeneous Neumann boundary conditions at both sides of the hyperplane $\mathcal{H} := \mathbb{R}^{d-1} \times \{0\}$, depending on the values of β and γ. The homogeneous Neumann boundary conditions correspond to a decoupling across $\mathcal{H}$. The (rather restrictive) condition in (*Ann. Inst. Henri Poincaré Probab. Stat.* (2023)) (for $d = 1$) about the initial distribution satisfying an entropy bound with respect to a Bernoulli product measure with constant parameter is weakened or completely dropped.

Résumé. Nous obtenons la limite hydrodynamique du processus d'exclusion à sauts longs symétriques dans $\mathbb{Z}^d$ (pour $d \geq 1$), où le taux de saut est inversement proportionnel à une puissance de la longueur du saut avec un exposant $\gamma + 1$, où $\gamma \geq 2$. De plus, les mouvements entre $\mathbb{Z}^{d-1} \times \mathbb{Z}_-^*$ et $\mathbb{Z}^{d-1} \times \mathbb{N}$ sont ralentis par un facteur $\alpha n^{-\beta}$ (avec $\alpha > 0$ et $\beta \geq 0$). Dans la limite hydrodynamique, nous obtenons l'équation de la chaleur dans $\mathbb{R}^d$ sans conditions au bord ou avec des conditions de Neumann homogènes des deux côtés de l'hyperplan $\mathcal{H} := \mathbb{R}^{d-1} \times 0$, en fonction des valeurs de β et γ. Les conditions au bord de Neumann homogènes correspondent à un découplage à travers $\mathcal{H}$. La condition (plutôt restrictive) dans (*Ann. Inst. Henri Poincaré Probab. Stat.* (2023)) (pour $d = 1$) sur la loi initiale satisfaisant une borne d'entropie par rapport à une mesure produit de Bernoulli avec un paramètre constant est assouplie ou complètement supprimée.

MSC2020 subject classifications: 60K35; 35R11; 35S15

Keywords: Hydrodynamic limit; Exclusion process; Long jumps

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Uniform-in-time propagation of chaos for mean field Langevin dynamics

Fan Chen[1,a], Zhenjie Ren[2,b] and Songbo Wang[3,c]

[1]*School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, China,* [a]*alexnwish@sjtu.edu.cn*
[2]*CEREMADE, Université Paris-Dauphine, PSL, Paris, France,* [b]*ren@ceremade.dauphine.fr*
[3]*CMAP, École polytechnique, IP Paris, Palaiseau, France,* [c]*songbo.wang@polytechnique.edu*

Abstract. We study the mean field Langevin dynamics and the associated particle system. By assuming the functional convexity of the energy, we obtain the L^p-convergence of the marginal distributions toward the unique invariant measure for the mean field dynamics. Furthermore, we prove the uniform-in-time propagation of chaos in both the L^2-Wasserstein metric and relative entropy.

Résumé. Nous étudions la dynamique de Langevin à champ moyen et le système de particules correspondant. En supposant la convexité fonctionnelle de l'énergie, nous obtenons la convergence dans L^p des distributions marginales vers l'unique mesure invariante pour la dynamique à champ moyen. De plus, nous montrons la propagation du chaos uniforme en temps à la fois dans la métrique de Wasserstein d'ordre 2 et dans l'entropie relative.

MSC2020 subject classifications: Primary 60J60; 60K35; secondary 35B40; 35Q83; 35Q84

Keywords: Langevin diffusion; Fokker–Planck equation; Mean field interaction; Convergence to equilibrium; Uniform-in-time propagation of chaos; Logarithmic Sobolev inequality; Hypercontractivity; Wasserstein distance; Relative entropy

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On the stability of the invariant probability measures of McKean–Vlasov equations

Quentin Cormier[a]

Inria, CMAP, CNRS, École polytechnique, Institut Polytechnique de Paris, 91200 Palaiseau, France, [a]*quentin.cormier@inria.fr*

Abstract. We study the long-time behavior of some McKean–Vlasov stochastic differential equations used to model the evolution of large populations of interacting agents. We give conditions ensuring the local stability of an invariant probability measure. Lions derivatives are used in a novel way to obtain our stability criteria. We obtain results for non-local McKean–Vlasov equations on $\mathbb{R}^d$ and for McKean–Vlasov equations on the torus where the interaction kernel is given by a convolution. On $\mathbb{R}^d$, we prove that the location of the roots of an analytic function determines the stability. On the torus, our stability criterion involves the Fourier coefficients of the interaction kernel. In both cases, we prove the convergence in the Wasserstein metric W_1 with an exponential rate of convergence.

Résumé. On étudie le comportement en temps long d'équations non-linéaires au sens de McKean–Vlasov. Ces équations sont utilisées pour modéliser les comportements de grandes populations d'agents en interaction. On donne des critères garantissant la stabilité locale d'une mesure de probabilité invariante. Pour ce faire, on utilise de manière novatrice les dérivées de Lions. On obtient des résultats pour des équations de McKean–Vlasov non locales sur $\mathbb{R}^d$, ainsi que pour des équations de McKean–Vlasov sur le tore pour lesquelles le noyau d'interaction est donné par une convolution. Sur $\mathbb{R}^d$, on montre que la stabilité est déterminée par l'emplacement des zéros d'une fonction holomorphe. Sur le tore, notre critère de stabilité fait intervenir les coefficients de Fourier du noyau d'interaction. Dans les deux cas, on montre la convergence vers la mesure de probabilité invariante à vitesse exponentielle pour la métrique de Wasserstein W_1.

MSC2020 subject classifications: Primary 60H10; secondary 60K35; 45D05; 37A30

Keywords: McKean–Vlasov SDE; Long-time behavior; Mean-field interaction; Lions derivative

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Periodicity and longtime diffusion for mean field systems in $\mathbb{R}^d$

Eric Luçon[1,2,a] and Christophe Poquet[3,b]

[1]Université Paris Cité, CNRS, MAP5, F-75006 Paris, France, [a]eric.lucon@u-paris.fr
[2]FP2M, CNRS FR 2036, Paris, France
[3]Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, F-69622 Villeurbanne, France, [b]poquet@math.univ-lyon1.fr

Abstract. We study in this paper the longtime behavior of some large but finite populations of interacting stochastic differential equations whose (infinite population) limit Fokker–Planck PDE admits a stable periodic solution. We show that the empirical measure for the population of size N stays close to the periodic solution, but with a random dephasing on a timescale proportional to N that converges weakly to a Brownian motion with constant drift.

Résumé. Nous étudions dans ce papier le comportement en temps long d'une population (grande mais finie) de diffusions en interaction pour laquelle la limite en population infinie, décrite par une équation de Fokker–Planck non linéaire, admet une solution périodique stable. Nous montrons que la mesure empirique de la population de taille N reste proche de la solution périodique, avec l'apparition d'un déphasage aléatoire sur une échelle de temps d'ordre N qui converge faiblement vers un mouvement brownien de drift constant.

MSC2020 subject classifications: Primary 60K35; 35K55; 35Q84; 37N25; secondary 82C31; 92B20

Keywords: Mean-field systems; McKean–Vlasov process; Longtime behavior of Markov processes; Nonlinear Fokker–Planck equation

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The extremal position of a branching random walk on the general linear group

Ion Grama[1,a], Sebastian Mentemeier[2,b] and Hui Xiao[3,c]

[1]*Univ Bretagne Sud, CNRS UMR 6205 LMBA, Campus de Tohannic 56017, Vannes, France,* [a]*ion.grama@univ-ubs.fr*
[2]*Universität Hildesheim, Institut für Mathematik und Angewandte Informatik, Hildesheim, Germany,* [b]*mentemeier@uni-hildesheim.de*
[3]*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China,* [c]*xiaohui@amss.ac.cn*

Abstract. Consider a branching random walk $(G_u)_{u\in\mathbb{T}}$ on the general linear group $\mathrm{GL}(V)$ of a finite dimensional space V, where $\mathbb{T}$ is the associated genealogical tree with nodes u. For any starting point $v \in V \setminus \{0\}$ with $\|v\| = 1$ and $x = \mathbb{R}v \in \mathbb{P}(V)$, let $M_n^x = \max_{|u|=n} \log \|G_u v\|$ denote the maximal position of the walk $\log \|G_u v\|$ in the generation n. We first show that under suitable conditions, $\lim_{n\to\infty} \frac{M_n^x}{n} = \gamma$ almost surely, where $\gamma \in \mathbb{R}$ is a constant. Then, in the case when $\gamma = 0$, under appropriate *boundary conditions*, we refine the last statement by determining the rate of convergence at which M_n^x converges to $-\infty$. We prove in particular that $\lim_{n\to\infty} \frac{M_n^x}{\log n} = -\frac{3}{2\alpha}$ in probability, where $\alpha > 0$ is a constant determined by the boundary conditions. Analogous properties are established for the minimal position. As a consequence we derive the asymptotic speed of the maximal and minimal positions for the coefficients, the operator norm and the spectral radius of G_u.

Résumé. Considérons une marche aléatoire branchante $(G_u)_{u\in\mathbb{T}}$ sur le groupe linéaire général $\mathrm{GL}(V)$ d'un espace V de dimension finie, où $\mathbb{T}$ est l'arbre généalogique associé, dont les sommets sont notés par u. Pour tout point de départ $v \in V \setminus 0$ avec $\|v\| = 1$ et $x = \mathbb{R}v \in \mathbb{P}(V)$, soit $M_n^x = \max_{|u|=n} \log \|G_u v\|$ la position maximale de la marche $\log \|G_u v\|$ à la génération n. Nous montrons d'abord que sous des conditions appropriées, $\lim_{n\to\infty} \frac{M_n^x}{n} = \gamma$ presque sûrement, où $\gamma \in \mathbb{R}$ est une constante. Ensuite, dans le cas où $\gamma = 0$, sous des *conditions frontières* appropriées, nous affinons la dernière affirmation en déterminant la vitesse de convergence à laquelle M_n^x converge vers $-\infty$. Nous prouvons en particulier que $\lim_{n\to\infty} \frac{M_n^x}{\log n} = -\frac{3}{2\alpha}$ en probabilité, où $\alpha > 0$ est une constante déterminée par les conditions frontières. Des propriétés analogues sont établies pour la position minimale. Comme conséquence, nous dérivons la vitesse asymptotique des positions maximale et minimale pour les coefficients, la norme opérateur et le rayon spectral de G_u.

MSC2020 subject classifications: Primary 60J80; secondary 60B20; 60J05

Keywords: Branching random walk; Maximal and minimal positions; Products of random matrices; Conditioned local limit theorem

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Prudent walk in dimension six and higher

Markus Heydenreich[1,a], Lorenzo Taggi[2,b] and Niccolò Torri[3,c]

[1] *Universität Augsburg, Institut für Mathematik, 86135 Augsburg, Germany,* [a]*markus.heydenreich@uni-a.de*
[2] *Sapienza Università di Roma, Dipartimento di Matematica. Piazzale Aldo Moro 5, 00186, Roma, Italy.,* [b]*lorenzo.taggi@uniroma1.it*
[3] *MODAL'X, UMR 9023, UPL, Univ. Paris Nanterre, F92000 Nanterre France,* [c]*ntorri@parisnanterre.fr*

Abstract. We study the high-dimensional uniform prudent self-avoiding walk, which assigns equal probability to all nearest-neighbor self-avoiding paths of a fixed length that respect the prudent condition, namely, the path cannot take any step in the direction of a previously visited site. We prove that the prudent self-avoiding walk converges to Brownian motion under diffusive scaling if the dimension is large enough. The same result is true for weakly prudent walk in dimension $d > 5$. A challenging property of the high-dimensional prudent walk is the presence of an infinite-range self-avoidance constraint. Interestingly, as a consequence of such a strong self-avoidance constraint, the upper critical dimension of the prudent walk is five, and thus greater than for the classical self-avoiding walk.

Résumé. Nous étudions la marche prudente en grande dimension. Nous nous intéressons au cas uniforme, c'est-à-dire celui où tous les chemins auto-évitants d'une longueur fixée ont la même probabilité. Pour être prudente, une trajectoire de la marche ne peut jamais se diriger vers un site déjà visité auparavant par la marche. Nous prouvons que la marche prudente converge vers le mouvement brownien après une renormalisation diffusive si la dimension est suffisamment grande. Le même résultat est vrai pour la marche faiblement prudente en dimension $d > 5$. Une propriété intéressante qui rend la marche prudente en haute dimension un objet mathématiquement complexe est la présence d'une contrainte d'auto-évitement à portée infinie. Il est intéressant de noter que la conséquence de cette contrainte est que la dimension critique de la marche prudente est au moins cinq, et donc strictement plus grande de celle de la marche auto-évitante classique.

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Keywords: Prudent walk; Self-avoiding random walk; Lace Expansio; Scaling limit; Critical dimension

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Recurrence, transience and degree distribution for the Tree Builder Random Walk

János Engländer[1,a], Giulio Iacobelli[2,c] and Rodrigo Ribeiro[1,b]

[1]*Department of Mathematics, University of Colorado, Boulder, CO-80309-0395, USA,* [a]*janos.englander@colorado.edu,* [b]*Rodrigo.Ribeiro@colorado.edu*
[2]*Mathematical Institute, Federal University of Rio de Janeiro (UFRJ), Caixa Postal 68530, 21945-970, Rio de Janeiro, Brazil,* [c]*giulio@im.ufrj.br*

Abstract. We investigate a self-interacting random walk, whose dynamically evolving environment is a random tree built by the walker itself, as it walks around. At time $n = 1, 2, \ldots$, right before stepping, the walker adds a random number (possibly zero) Z_n of leaves to its current position. We assume that the Z_n's are independent, but, importantly, we do *not* assume that they are identically distributed.

We obtain non-trivial conditions on their distributions under which the random walk is recurrent. This result is in contrast with some previous work in which, under the assumption that $Z_n \sim \mathsf{Ber}(p)$ (thus i.i.d.), the random walk was shown to be ballistic for every $p \in (0, 1]$. We also obtain results on the transience of the walk, and the possibility that it "gets stuck."

From the perspective of the environment, we provide structural information about the sequence of random trees generated by the model when $Z_n \sim \mathsf{Ber}(p_n)$, with $p_n = \Theta(n^{-\gamma})$ and $\gamma \in (2/3, 1]$. We prove that the empirical degree distribution of this random tree sequence converges almost surely to a power-law distribution of exponent 3, thus revealing a connection to the well known preferential attachment model.

Résumé. Nous étudions une marche aléatoire sur un arbre aléatoire construit par le marcheur lui-même, alors qu'il se promène. Au temps $n = 1, 2, \ldots$, juste avant de faire un pas, le marcheur aléatoire ajoute un nombre aléatoire (éventuellement zéro) Z_n de feuilles à sa position actuelle. Nous supposons que les Z_n sont indépendants, mais pas nécessairement identiquement distribuées.

Nous obtenons des conditions sur leurs distributions sous lesquelles la marche aléatoire est récurrente. Ce résultat contraste avec certains travaux antérieurs dans lesquels, sous l'hypothèse que $Z_n \sim \mathsf{Ber}(p)$ (donc i.i.d.), la marche aléatoire s'est avérée balistique pour chaque $p \in (0, 1]$. Nous obtenons également des résultats sur la *transience* de la marche aléatoire, et sur la possibilité qu'elle "reste bloquée".

Du point de vue de l'environnement, nous fournissons des informations structurelles sur la séquence d'arbres aléatoires générés par le modèle lorsque $Z_n \sim \mathsf{Ber}(p_n)$, avec $p_n = \Theta(n^{-\gamma})$ et $\gamma \in (2/3, 1]$. Nous prouvons que la distribution de degré empirique de cette séquence d'arbres aléatoires converge presque sûrement vers une loi de puissance d'exposant 3, révélant ainsi un lien avec le modèle de Barabási–Albert.

MSC2020 subject classifications: 60K37

Keywords: Random walk; Dynamic random environment; Random tree; Recurrence; Transience; Power-law degree distribution; Barabási-Albert model; Preferential attachment

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Massive SLE_4 and the scaling limit of the massive harmonic explorer

Léonie Papon[a]

Department of Mathematical Sciences, Durham University, Durham, United Kingdom, [a]*leonie.b.papon@durham.ac.uk*

Abstract. The massive harmonic explorer is a model of random discrete path on the hexagonal lattice that was proposed by Makarov and Smirnov as a massive perturbation of the harmonic explorer. They argued that the scaling limit of the massive harmonic explorer in a bounded domain is a massive version of chordal SLE_4, called massive SLE_4, which is conformally covariant and absolutely continuous with respect to chordal SLE_4. In this paper, we provide a full and rigorous proof of this statement. Moreover, we show that a massive SLE_4 curve can be coupled with a massive Gaussian free field as its level line, when the field has appropriate boundary conditions.

Résumé. L'explorateur harmonique massif est un modèle de chemin aléatoire sur la grille hexagonale introduit par Makarov et Smirnov. Ce modèle est l'analogue de l'explorateur harmonique en présence d'une masse. Makarov et Smirnov ont suggéré que la limite de ce chemin aléatoire lorsque la taille des arêtes de la grille tend vers 0 est une version massive du processus SLE_4, appelée processus SLE_4 massif. Ce processus est covariant sous l'action d'une application conforme et sa loi est absolument continue par rapport à celle du processus SLE_4. Dans cet article, nous donnons une preuve complète et rigoureuse de ces assertions. Nous montrons aussi qu'un processus SLE_4 massif peut être couplé à un champ libre gaussien massif lorsque le champ satisfait à des conditions au bord particulières. Dans ce couplage, le processus SLE_4 massif est une « ligne de niveau » du champ.

MSC2020 subject classifications: Primary 60J67; secondary 60G15; 60B10

Keywords: Massive harmonic explorer; Massive SLE_4; Massive Gaussian free field

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Strong rate of convergence of the Euler scheme for SDEs with irregular drift driven by Lévy noise

Oleg Butkovsky[1,a], Konstantinos Dareiotis[2,b] and Máté Gerencsér[3,c]

[1] *Weierstrass Institute, Mohrenstraße 39, 10117 Berlin, Germany,* [a]*oleg.butkovskiy@gmail.com*
[2] *University of Leeds, Woodhouse, LS2 9JT Leeds, United Kingdom,* [b]*k.dareiotis@leeds.ac.uk*
[3] *TU Wien, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria,* [c]*mate.gerencser@tuwien.ac.at*

Abstract. We study the strong rate of convergence of the Euler–Maruyama scheme for a multidimensional stochastic differential equation (SDE)

$$dX_t = b(X_t)\,dt + dL_t,$$

with irregular β-Hölder drift, $\beta > 0$, driven by a Lévy process with exponent $\alpha \in (0,2]$. For $\alpha \in [2/3, 2]$, we obtain strong L_p and almost sure convergence rates in the entire range $\beta > 1 - \alpha/2$, where the SDE is known to be strongly well-posed. This significantly improves the current state of the art, both in terms of convergence rate and the range of α. Notably, the obtained convergence rate does not depend on p, which is a novelty even in the case of smooth drifts. As a corollary of the obtained moment-independent error rate, we show that the Euler–Maruyama scheme for such SDEs converges almost surely and obtain an explicit convergence rate. Additionally, as a byproduct of our results, we derive strong L_p convergence rates for approximations of nonsmooth additive functionals of a Lévy process. Our technique is based on a new extension of stochastic sewing arguments and Lê's quantitative John–Nirenberg inequality.

Résumé. Nous étudions le taux de convergence fort du schéma d'Euler–Maruyama pour une équation différentielle stochastique (EDS) multidimensionnelle

$$dX_t = b(X_t)\,dt + dL_t,$$

avec une dérive irrégulière de type β-Hölderienne, $\beta > 0$, dirigée par un processus de Lévy avec un exposant $\alpha \in (0,2]$. Pour $\alpha \in [2/3, 2]$, nous obtenons des taux de convergence forts dans L_p et presque sûrs dans toute la plage $\beta > 1 - \alpha/2$, plage pour laquelle il est connu que l'EDS est bien posée. Cela améliore significativement l'état de l'art actuel concernant le taux de convergence et la plage de α. Notamment, le taux de convergence obtenu ne dépend pas de p, ce qui constitue une nouveauté, même dans le cas de dérives régulières. Grâce à l'indépendance du taux d'erreur par rapport au moment, nous montrons que le schéma d'Euler–Maruyama pour de telles EDS converge presque sûrement et obtenons un taux de convergence explicite. De plus, à partir de nos résultats, nous dérivons facilement des taux de convergence forts dans L_p pour des approximations de fonctionnelles additives non lisses d'un processus de Lévy. Notre technique est basée sur de nouvelles extensions des arguments de couture stochastique et sur l'inégalité quantitative de John–Nirenberg démontrée par Lê.

MSC2020 subject classifications: Primary 60H10; 60H35; 60H50; secondary 60J76

Keywords: Euler–Maruyama approximation; Stochastic differential equations; Regularization by noise; Irregular drift; Lévy noise; Stochastic sewing

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A converse to Pitman's theorem for a space-time Brownian motion in a type A_1^1 Weyl chamber

Manon Defosseux[1,a] and Charlie Hérent[1,2,b]

[1] *Université Paris Cité, CNRS, MAP5, F-75006 Paris, France,* [a]*manon.defosseux@parisdescartes.fr*
[2] *Université Paris-Est, CNRS, Institut Gaspard Monge, Champs-Sur-Marne, France,* [b]*charlie.herent@ens-rennes.fr*

Abstract. We prove an inverse Pitman's theorem for a space-time Brownian motion conditioned in Doob's sense to remain in an affine Weyl chamber. Our theorem provides a way to recover an unconditioned space-time Brownian motion from a conditioned one by applying a sequence of path transformations.

Résumé. Nous démontrons un théorème de Pitman inverse pour le brownien espace-temps conditionné au sens de Doob à rester dans une chambre de Weyl de type affine. Le théorème permet de construire un brownien espace-temps non conditionné à partir d'un brownien conditionné, en appliquant une suite de transformations trajectorielles.

MSC2020 subject classifications: Primary 60J65; secondary 17B65

Keywords: Conditioned space-time Brownian motion; Affine Weyl Chamber; Pitman's Theorem

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Uniform attachment with freezing: Scaling limits

Étienne Bellin[1,a], Arthur Blanc-Renaudie[2,c], Emmanuel Kammerer[1,b] and Igor Kortchemski[3,d]

[1]CMAP, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France, [a]etienne.bellin@polytechnique.edu, [b]emmanuel.kammerer@polytechnique.edu
[2]Tel Aviv University, Tel Aviv, Israel, [c]ablancrenaudiepro@gmail.com
[3]CNRS & CMAP, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France, [d]igor.kortchemski@math.cnrs.fr

Abstract. We investigate scaling limits of trees built by uniform attachment with freezing, which is a variant of the classical model of random recursive trees introduced in a companion paper. Here vertices are allowed to freeze, and arriving vertices cannot be attached to already frozen ones. We identify a phase transition when the number of non-frozen vertices roughly evolves as the total number of vertices to a given power. In particular, we observe a critical regime where the scaling limit is a random compact real tree, closely related to a time non-homogenous Kingman coalescent process identified by Aldous. Interestingly, in this critical regime, a condensation phenomenon can occur.

Résumé. Nous étudions les limites d'échelle d'arbres construits par attachement uniforme avec gel, dont le modèle, une variante du modèle classique des arbres récursifs aléatoires, a été introduit dans un autre article. Ici, les sommets peuvent geler, et les nouveaux sommets ne peuvent être reliés à des sommets déjà gelés. Nous identifions une transition de phase quand le nombre de sommets non-gelés évolue comme le nombre de sommets total à une certaine puissance. En particulier, on observe un régime critique où la limite d'échelle est un arbre réel compact aléatoire, étroitement lié à un processus coalescent de Kingman inhomogène en temps identifié par Aldous. Curieusement, dans ce régime critique, un phénomène de condensation peut se produire.

MSC2020 subject classifications: 60D05; 60F05

Keywords: Random trees; Uniform attachment; Scaling limits; Freezing; Coalescent processes

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Un-inverting the Parisi formula

Jean-Christophe Mourrat[a]

Department of Mathematics, ENS Lyon and CNRS, Lyon, France, [a]*jean-christophe.mourrat@ens-lyon.fr*

Abstract. The free energy of any system can be written as the supremum of a functional involving an energy term and an entropy term. Surprisingly, the limit free energy of mean-field spin glasses is expressed as an infimum instead, a phenomenon sometimes called an inverted variational principle. Using a stochastic-control representation of the Parisi functional and convex duality arguments, we rewrite this limit free energy as a supremum over martingales of a Wiener space.

Résumé. L'énergie libre de n'importe quel système peut être écrite comme le supremum d'une fonctionnelle impliquant un terme d'énergie et un terme d'entropie. Au lieu de cela, l'énergie libre limite des verres de spin à champ moyen est exprimée comme un infimum, un phénomène parfois appelé un principe variationnel inversé. En utilisant une représentation de contrôle stochastique de la fonctionnelle de Parisi et des arguments de dualité convexe, nous réécrivons cette énergie libre limite comme un supremum sur les martingales d'un espace de Wiener.

MSC2020 subject classifications: 82B44; 82D30

Keywords: Spin glasses; Parisi formula

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On the hard edge limit of the zero temperature Laguerre β-corners process

Matthew Lerner-Brecher[a]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA, [a]*melb@mit.edu*

Abstract. We study the hard edge limit of a multilevel extension of the Laguerre β-ensemble at zero temperature. In particular, we show that asymptotically the ensemble is given by Gaussians with covariance matrix expressible in terms of the Fourier–Bessel series. These Gaussians also have an explicit representation as the partition functions of additive polymers arising from a random walk on roots of the Bessel functions. Our approach builds off of techniques introduced by Gorin and Kleptsyn and is rooted in using the theory of dual and associated polynomials to diagonalize transition matrices relating levels of the ensemble.

Résumé. Nous étudions la limite bord dur (hard edge) d'une extension multiniveau de l'ensemble β de Laguerre à température nulle. En particulier, nous montrons qu'asymptotiquement, l'ensemble est donné par des variables gaussiennes avec une matrice de covariance que nous exprimons en termes de série de Fourier–Bessel. Ces gaussiennes ont également une représentation explicite comme fonctions de partition de polymères additifs résultants d'une marche aléatoire sur les racines des fonctions de Bessel. Notre approche s'appuie sur les techniques introduites par Gorin et Kleptsyn et repose sur l'utilisation de la théorie des polynômes duaux et associés pour diagonaliser les matrices de transition reliant les niveaux de l'ensemble.

MSC2020 subject classifications: Primary 60B20; secondary 60G15; 34B24

Keywords: Random matrices; Gaussian processes; Additive polymers

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Limiting eigenvalue distribution of heavy-tailed Toeplitz matrices

Ratul Biswas[a] and Arnab Sen[b]

School of Mathematics, University of Minnesota, Minneapolis, MN, USA, [a]*biswa087@umn.edu,* [b]*arnab@umn.edu*

Abstract. We consider an $N \times N$ random symmetric Toeplitz matrix with an i.i.d. input sequence drawn from a distribution that lies in the domain of attraction of an α-stable law for $0 < \alpha < 2$. We show that under an appropriate scaling, its empirical eigenvalue distribution, as $N \to \infty$, converges weakly to a random symmetric probability distribution on $\mathbb{R}$, which can be described as the expected spectral measure of a certain random unbounded self-adjoint operator on $\ell^2(\mathbb{Z})$. The limiting distribution turns out to be almost surely subgaussian. Furthermore, the support of the limiting distribution is bounded almost surely if $0 < \alpha < 1$ and is unbounded almost surely if $1 \leq \alpha < 2$.

Résumé. Nous considérons une matrice aléatoire $N \times N$ symétrique de Toeplitz avec une suite de coefficients i.i.d. dont la loi est dans le domaine d'attraction d'une loi α-stable avec $0 < \alpha < 2$. Nous montrons qu'après renormalisation appropriée, quand $N \to \infty$, la mesure spectrale empirique converge faiblement vers une loi symétrique aléatoire sur $\mathbb{R}$, reliée à la mesure spectrale d'un certain opérateur auto-adjoint non borné aléatoire sur $\ell^2(\mathbb{Z})$. La loi limite s'avère être presque sûrement sous-gaussienne. De plus, presque sûrement, son support est borné si $0 < \alpha < 1$ et non borné si $1 \leq \alpha < 2$.

MSC2020 subject classifications: Primary 15B05; 60E07; secondary 60B20

Keywords: Random Toeplitz matrix; Stable distribution; Moment method

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Local statistics and shuffling for dimers on a square-hexagon lattice

Matthew Nicoletti[a]

Department of Mathematics, MIT, Cambridge, MA, USA, [a]*mnicolet@mit.edu*

Abstract. We study the dimer model on special subgraphs of the square hexagon lattice called "tower graphs" of size N. Using integrable probability techniques, we confirm that as $N \to \infty$, the local statistics are translation invariant Gibbs measures, as conjectured by Kenyon–Okounkov–Sheffield. We also present a $2+1$-dimensional discrete time growth process, whose time N distribution is exactly the dimer model on the size N tower, and we compute the current of this growth process and confirm that the model belongs to the Anisotropic KPZ universality class.

Résumé. Nous étudions le modèle de dimères sur des sous-graphes spéciaux du réseau hexagonal carré appelés "tower graphs", de taille N. En utilisant des techniques de probabilités intégrables, nous confirmons que lorsque $N \to \infty$, les statistiques locales sont des mesures de Gibbs invariantes par translation, comme conjecturé par Kenyon–Okounkov–Sheffield. Nous présentons également un processus de croissance à temps discret en dimensions $2+1$, dont la distribution au temps N est exactement le modèle de dimères sur la tour de taille N. Nous calculons le courant de ce processus de croissance et nous confirmons que le modèle appartient à la classe d'universalité anisotrope KPZ.

MSC2020 subject classifications: 82C22; 60K35; 60G55

Keywords: Probability; Combinatorics; Statistical Mechanics; Dimer model

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Asymptotics of cross-validation

Morgane Austern[1,a] and Wenda Zhou[2,b]

[1]*Department of Statistics, Harvard University, Cambridge MA, USA,* [a]*morgane.austern@gmail.com*
[2]*Center for Data Science, New York University, and Center for Computational Mathematics, Flatiron Institute, New York, NY, USA,* [b]*wzhou@flatironinstitute.org*

Abstract. Cross-validation is a central tool in evaluating the performance of machine learning and statistical models. However, despite its ubiquitous role, its theoretical properties are still not well understood. We study the asymptotic properties of the cross-validated risk for a large class of models. Under stability conditions, we establish a central limit theorem and Berry–Esseen bounds, which enable us to compute asymptotically accurate confidence intervals. Using our results, we paint a big picture for the statistical speed-up of cross-validation compared to a train-test split procedure. A corollary of our results is that parametric M-estimators (or empirical risk minimizers) benefit from the "full" speed-up when performing cross-validation under the training loss. In other common cases, such as when the training is performed using a surrogate loss or a regularizer, we show that the behavior of the cross-validated risk is complex with a variance reduction which may be smaller or larger than the "full" speed-up, depending on the model and the underlying distribution. We allow the number of folds to grow with the number of observations at any rate.

Résumé. La validation-croisée est au cœur des méthodes d'évaluation de la fiabilité des modèles statistiques et d'apprentissage automatique. Cependant, malgré son rôle omniprésent, ses propriétés théoriques sont encore mal comprises. Dans cet article, nous étudions, pour une large classe de modèles, les propriétés asymptotiques de l'estimateur par validation croisée du risque. Sous des conditions sur la stabilité des estimateurs considérés, nous établissons un théorème limite central et une inégalité de type Berry–Esseen, qui nous permettent de calculer des intervalles de confiance asymptotiquement corrects. À l'aide de nos résultats, nous comparons la vitesse de convergence de l'estimateur par validation croisée du risque avec celle d'une procédure de découpage des données en deux (entraînement et test). Un corollaire de nos résultats est que les M-estimateurs paramétriques (ou minimiseurs du risque empirique) bénéficient d'une accélération «maximale» lorsque l'on compare la validation croisée à la validation simple. Dans d'autres cas courants, par exemple lorsque l'estimateur est entraîné en utilisant une fonction objectif auxiliaire ou lorsque l'estimateur est régularisé, nous observons que le comportement de l'estimateur par validation croisée du risque est complexe, avec une réduction de la variance qui peut être inférieure ou supérieure à l'accélération « maximale », selon la distribution des données et le modèle. Enfin, dans notre analyse, le nombre de blocs de validation croisée est autorisé à croître à une vitesse arbitraire.

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Keywords: Cross-validation; Asymptotics; Variance reduction; Stein method; Central-limit theorem; Berry–Esseen bound; Leave-one-out

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Malliavin calculus for the optimal estimation of the invariant density of discretely observed diffusions in intermediate regime

Chiara Amorino[1,a] and Arnaud Gloter[2,b]

[1]*Department of Mathematics, University of Luxembourg, Esch-sur Alzette, Luxembourg,* [a]*chiara.amorino@uni.lu*
[2]*LaMME, Université Paris-Saclay, CNRS, Univ Evry, 91037, Evry-Courcouronnes, France,* [b]*arnaud.gloter@univ-evry.fr*

Abstract. Let $(X_t)_{t\geq 0}$ be solution of a one-dimensional stochastic differential equation. Assuming that a discrete observation of the process $(X_t)_{t\in[0,T]}$ is available, when T tends to ∞, our aim is to study the convergence rate for the estimation of the invariant density in cases where the effect of sampling is non-negligible. This scenario is referred to as the 'intermediate regime'. We find the convergence rates associated to the kernel density estimator we proposed and a condition on the discretization step Δ_n which plays the role of threshold between the intermediate regime and the continuous case. In intermediate regime the convergence rate is $n^{-\frac{2\beta}{2\beta+1}}$, where β is the smoothness of the invariant density. After that, we complement the upper bounds previously found with a lower bound over the set of all the possible estimator, which provides the same convergence rate: it means it is not possible to propose a different estimator which achieves better convergence rates. This is obtained by the two hypotheses method; the most challenging part consists in bounding the Hellinger distance between the laws of the two models. The key point is a Malliavin representation for a score function, which allows us to bound the Hellinger distance through a quantity depending on the Malliavin weight.

Résumé. Soit $(X_t)_{t\geq 0}$ la solution d'une équation différentielle stochastique unidimensionnelle. En supposant qu'une observation discrète du processus $(X_t)_{t\in[0,T]}$ est disponible, avec $T\to\infty$, notre objectif est d'étudier la vitesse de convergence pour l'estimation de la densité invariante dans les cas où l'effet de l'échantillonnage n'est pas négligeable. Cette situation est désignée sous le nom de « régime intermédiaire ». Nous déterminons la vitesse de convergence associée à l'estimateur à noyau que nous proposons, ainsi que le seuil sur le pas de discrétisation Δ_n qui sépare le régime intermédiaire de celui de l'observation continue. Dans le régime intermédiaire la vitesse d'estimation est $n^{-\frac{2\beta}{2\beta+1}}$, où β est l'indice de régularité de la densité invariante. Ensuite, nous complétons les bornes supérieures précédemment obtenues par une borne inférieure sur l'ensemble de tous les estimateurs possibles et qui fournit la même vitesse de convergence : cela signifie qu'il n'est pas possible de proposer un estimateur différent avec une vitesse meilleure. Ce résultat est obtenu par la méthode des deux hypothèses ; la partie la plus difficile consiste à majorer la distance de Hellinger entre les lois des deux hypothèses. Le point crucial est la représentation par calcul de Malliavin d'une fonction de score, qui permet de majorer la distance de Hellinger au travers d'une quantité dépendant du poids de Malliavin.

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Keywords: Non-parametric estimation; Malliavin calculus; Invariant density; Discrete observations; Convergence rate; Local time; Ergodic diffusion

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A Gaussian correlation inequality for plurisubharmonic functions

Franck Barthe[1,a] and Dario Cordero-Erausquin[2,b]

[1]*Institut de Mathématiques de Toulouse. Université de Toulouse & CNRS. F-31062 Toulouse Cedex 9, France,* [a]*franck.barthe@math.univ-toulouse.fr*
[2]*Institut de Mathématiques de Jussieu (UMR 7586), Sorbonne Université & CNRS, 75252 Paris Cedex 5, France,* [b]*dario.cordero@imj-prg.fr*

Abstract. A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein–Uhlenbeck semigroup and another natural semigroup associated to the Gaussian $\overline{\partial}$-Laplacian.

Résumé. Nous établissons une inégalité de correlation pour les fonction plurisubharmoniques circulaires, par rapport aux mesures gaussiennes (complexes) centrées. Les preuves utilisent le semi-groupe d'Ornstein–Uhlenbeck ainsi qu'un autre semi-groupe naturel associé au Laplacien $\overline{\partial}$ gaussien.

MSC2020 subject classifications: Primary 60G15; secondary 31C10

Keywords: Correlation inequalities; Complex Gaussian measures; Gaussian moments; Circular-invariant psh functions

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Central limit theorem over non-linear functionals of empirical measures: Beyond the iid setting

Roberta Flenghi[a] and Benjamin Jourdain[b]

Cermics, École des Ponts, INRIA, Marne-la-Vallée, France, [a]*roberta.flenghi@enpc.fr,* [b]*benjamin.jourdain@enpc.fr*

Abstract. The central limit theorem is, with the strong law of large numbers, one of the two fundamental limit theorems in probability theory. Benjamin Jourdain and Alvin Tse have extended to non-linear functionals of the empirical measure of independent and identically distributed random vectors the central limit theorem which is well known for linear functionals. The main tool permitting this extension is the linear functional derivative, one of the notions of derivation on the Wasserstein space of probability measures that have recently been developed. The purpose of this work is to relax first the equal distribution assumption made by Jourdain and Tse and then the independence property to be able to deal with the successive values of an ergodic Markov chain.

Résumé. Le théorème de la limite centrale est, après la loi forte des grands nombres, l'un des deux résultats asymptotiques fondamentaux de la théorie des probabilités. Benjamin Jourdain et Alvin Tse ont étendu à des fonctionnelles non linéaires de la mesure empirique de vecteurs aléatoires indépendants et identiquement distribués, le résultat qui porte sur les fonctionnelles linéaires données par l'intégrale d'une fonction contre la mesure. Cette extension repose sur la dérivée fonctionnelle linéaire, l'une des notions récentes de dérivation sur l'espace de Wasserstein des mesures de probabilités. Dans cet article, nous relâchons d'abord l'hypothèse d'équidistribution puis celle d'indépendance pour généraliser le résultat à la mesure empirique des valeurs successives d'une chaîne de Markov ergodique.

MSC2020 subject classifications: 60F05; 60F17

Keywords: Central limit theorem; Derivation on the Wasserstein space; Ergodic Markov chain

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A family of natural equilibrium measures for Sinai billiard flows

Jérôme Carrand[1,2,a]

[1]*Laboratoire de Probabilités, Statistiques et Modélisation (LPSM), CNRS, Sorbonne Université, Université de Paris, 75005 Paris, France*
[2]*Centro di Ricerca Matematica Ennio De Giorgi, SNS, 56100 Pisa, Italy,* [a]*jerome.carrand@sns.it*

Abstract. The Sinai billiard flow on the two-torus, i.e., the periodic Lorentz gas, is a continuous flow, but it is not everywhere differentiable. Assuming finite horizon, we relate the equilibrium states of the flow with those of the Sinai billiard map T – which is a discontinuous map. We propose a definition for the topological pressure $P_*(T, g)$ associated to a potential g. We prove that for any piecewise Hölder potential g satisfying a mild assumption, $P_*(T, g)$ is equal to the definitions of Bowen using spanning or separating sets. We give sufficient conditions under which a potential gives rise to equilibrium states for the Sinai billiard map. We prove that in this case the equilibrium state μ_g is unique, Bernoulli, adapted and gives positive measure to all nonempty open sets. For this, we make use of a well chosen transfer operator acting on anisotropic Banach spaces, and construct the measure by pairing its maximal eigenvectors. Last, we prove that the flow invariant probability measure $\bar{\mu}_g$, obtained by taking the product of μ_g with the Lebesgue measure along orbits, is Bernoulli and flow adapted. We give examples of billiard tables for which there exists an open set of potentials satisfying those sufficient conditions.

Résumé. Le flot billard d'un billard de Sinai sur le tore bidimensionnel, ou gaz de Lorentz périodique, est un flot continu, mais qui n'est pas partout différentiable. En supposant l'horizon fini, nous relions les états d'équilibre du flot à ceux de l'application de collisions T – qui est une application discontinue. Nous proposons une définition de la pression topologique $P_*(T, g)$ associée à un potentiel g. Nous prouvons que pour tout potentiel g Hölder par morceaux satisfaisant une hypothèse faible, $P_*(T, g)$ est égale aux définitions de Bowen utilisant des ensembles couvrants ou séparés. Nous donnons des conditions suffisantes pour qu'un potentiel admette des états d'équilibre pour l'application de collisions. Nous prouvons que dans ce cas l'état d'équilibre μ_g est unique, Bernoulli, adapté et donne une mesure positive à tous les ensembles ouverts non vides. Pour cela, nous utilisons un opérateur de transfert pondéré agissant sur des espaces de Banach anisotropes, et construisons la mesure en combinant ses vecteurs propres maximaux. Enfin, nous prouvons que la mesure de probabilité $\bar{\mu}_g$ invariante par le flot, obtenue en prenant le produit de μ_g avec la mesure de Lebesgue le long des orbites, est Bernoulli et adaptée au flot. Nous donnons des exemples de tables de billard pour lesquelles il existe un ensemble ouvert de potentiels satisfaisant ces conditions suffisantes.

MSC2020 subject classifications: Primary 37C83; secondary 37C30; 37D35; 37A25; 46E35; 47B38

Keywords: Dynamical systems; Dispersive billiard; Anisotropic Banach space

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On genericity of non-uniform Dvoretzky coverings of the circle

Michihiro Hirayama[1,a] and Davit Karagulyan[2,b]

[1]*Department of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Ibaraki, Japan,* [a]*hirayama@math.tsukuba.ac.jp*
[2]*KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden,* [b]*davitk@kth.se*

Abstract. The classical Dvoretzky covering problem asks for conditions on the sequence of lengths $(\ell_n)_{n\in\mathbb{N}}$ so that the random intervals $I_n = (\omega_n - (\ell_n/2), \omega_n + (\ell_n/2))$, where ω_n is a sequence of i.i.d. uniformly distributed random variables, cover any point on the circle $\mathbb{T}$ infinitely often. We consider the Dvoretzky covering problem when the distribution of ω_n is absolutely continuous with a density function f. When the essential infimum of f is positive, we determine the critical value that separates covering and non-covering phenomena. Several scenarios in the critical case are discussed from the viewpoint of the "size" of the set $K(f)$ of its essential infimum points. We give, among other things, a necessary and sufficient condition for $\mathbb{T}$ to be Dvoretzky covered when the essential infimum of f is positive and $\overline{\dim}_\mathrm{B} K(f) < 1$, where $\overline{\dim}_\mathrm{B}$ is the upper box-counting dimension. Next we show that if $K(f)$ has zero Lebesgue measure, then a Menshov type genericity result holds, i.e., Dvoretzky covering can be achieved by changing f on a set of arbitrarily small Lebesgue measure.

Résumé. Le problème classique du recouvrement de Dvoretzky pose des conditions sur la suite des longueurs $(\ell_n)_{n\in\mathbb{N}}$ telle que les intervalles aléatoires $I_n = (\omega_n - (\ell_n/2), \omega_n + (\ell_n/2))$, où ω_n est une suite de i.i.d. variables aléatoires uniformes, couvrent le cercle $\mathbb{T}$ une infinité de fois. Nous considérons le problème du recouvrement de Dvoretzky lorsque la distribution de ω_n est absolument continue avec une fonction de densité f. Quand l'infimum essentiel de f est positif, nous déterminons la valeur critique qui sépare les phénomènes de recouvrement et de non-recouvrement. Plusieurs scénarios dans le cas critique sont discutés du point de vue de la "taille" de l'ensemble $K(f)$ de ses points infimum essentiels. Nous donnons, entre autres, une condition nécessaire et suffisante pour que $\mathbb{T}$ soit recouvert au sense de Dvoretzky lorsque l'infimum essentiel de f est positif et que l'ensemble $K(f)$ satisfait $\overline{\dim}_\mathrm{B} K(f) < 1$, où $\overline{\dim}_\mathrm{B}$ est la dimension de Minkowski-Bouligand (ou de box-counting) supérieure. Nous montrons ensuite que si $K(f)$ a une mesure de Lebesgue nulle, alors un résultat de type Menshov est vérifié, c'est-à-dire que le recouvrement de Dvoretzky peut être obtenu en modifiant f sur un ensemble de mesure de Lebesgue arbitrairement petite.

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