ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES

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Lace expansion and mean-field behavior for the random connection model

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Abstract. We study the random connection model driven by a stationary Poisson process. In the first part of the paper, we derive a lace expansion with remainder term in the continuum and bound the coefficients using a new version of the BK inequality. For our main results, we consider three versions of the connection function: a finite-variance version (including the Boolean model), a spread-out version, and a long-range version. For sufficiently large dimension (resp., large spread-out parameter and dimension $d > 6$), we then prove the convergence of the lace expansion, derive the triangle condition, and establish an infra-red bound. From this, mean-field behavior of the model can be deduced. As an example, we show that the critical exponent γ takes its mean-field value $\gamma = 1$ and that the percolation function is continuous.

Résumé. Nous étudions le modèle de connexion aléatoire dirigé par un processus de Poisson stationnaire. Dans la première partie de l'article, nous établissons un développement en lacets avec un terme de reste dans le cadre continu, et nous majorons les coefficients à l'aide d'une nouvelle version de l'inégalité de BK. Pour nos principaux résultats, nous considérons trois versions de la fonction de connexion : une version à variance finie (incluant le modèle booléen), une version spread-out, et une version à longue portée. Lorsque la dimension (ou respectivement le paramètre spread-out et $d > 6$) est suffisamment grande, nous prouvons alors la convergence du développement en lacet, établissons la condition du triangle, et obtenons une borne infra-rouge. On peut en déduire des comportements de champ moyen pour le modèle. À titre d'exemple, nous montrons que l'exposant critique γ prend sa valeur de champ moyen $\gamma = 1$ et que la fonction de percolation est continue.

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Gaussian deconvolution and the lace expansion for spread-out models

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Abstract. We present a new proof of $|x|^{-(d-2)}$ decay of critical two-point functions for spread-out statistical mechanical models on $\mathbb{Z}^d$ above the upper critical dimension, based on the lace expansion and assuming appropriate diagrammatic estimates. Applications include spread-out models of the Ising model and self-avoiding walk in dimensions $d > 4$, and spread-out percolation for $d > 6$. The proof is based on an extension of the new Gaussian deconvolution theorem we obtained in a recent paper. It provides a technically simpler and conceptually more transparent approach than the method of Hara, van der Hofstad and Slade (*Ann. Probab.* **31** (2003) 349–408).

Résumé. Nous présentons une nouvelle preuve de la décroissance en $|x|^{-(d-2)}$ des fonctions à deux points critiques pour les modèles mécaniques statistiques étendus sur $\mathbb{Z}^d$ au-dessus de la dimension critique supérieure, basée sur le développement en lacets et en supposant des estimations diagrammatiques appropriées. Les applications incluent des modèles étendus du modèle d'Ising et de la marche auto-évitante en dimensions $d > 4$, ainsi que la percolation étendue pour $d > 6$. La preuve est basée sur une extension du nouveau théorème de déconvolution gaussienne que nous avons obtenu dans un article récent. Elle fournit une approche techniquement plus simple et conceptuellement plus transparente que la méthode de Hara, van der Hofstad et Slade (*Ann. Probab.* **31** (2003) 349–408).

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Universal 'winner-takes-it-all' phenomenon in scale-free random graphs

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Abstract. We study competition on scale-free configuration model random graphs, where the degree distribution satisfies an asymptotic power-law with infinite variance. Our competition process is such that the two types attempt at occupying vertices incident to the presently occupied sets, and the passage times are independent and identically distributed, possibly with different distributions for the two types. Once vertices are occupied by a type, they remain on being occupied by that type forever.

We focus on the explosive setting, where our main result shows that the winning type occupies all but a *finite* number of vertices. This universal 'winner-takes-it-all' phenomenon for the edge-weight distributions significantly generalises previous work with Deijfen for exponential edge-weights, and considerably simplifies its proof.

Résumé. Nous étudions la compétition sur des graphes aléatoires de type modèle de configuration invariant d'échelle, dont les degrés suivent une loi de puissance asymptotique à variance infinie. Notre processus de compétition est tel que les deux types tentent d'occuper les sommets adjacents aux ensembles actuellement occupés, et les temps de passage sont indépendants et identiquement distribués, éventuellement avec des lois différentes pour les deux types. Une fois qu'un sommet est occupé par un type, il reste occupé par ce type pour toujours.

Nous nous concentrons sur le cadre explosif, où notre principal résultat montre que le type gagnant occupe tous les sommets sauf un nombre fini. Ce phénomène universel du « le gagnant rafle tout » pour les lois de poids des arêtes généralise de manière significative le travail précédent avec Deijfen pour les poids d'arêtes exponentiels et simplifie considérablement la démonstration.

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Mixing of the symmetric beta-binomial splitting process on arbitrary graphs

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Abstract. We study the mixing time of the symmetric beta-binomial splitting process on finite weighted connected graphs $G = (V, E, \{r_e\}_{e \in E})$ with vertex set V, edge set E and positive edge-weights $r_e > 0$ for $e \in E$. This is an interacting particle system with a fixed number of particles that updates through vertex-pairwise interactions which redistribute particles. We show that the mixing time of this process can be upper-bounded in terms of the maximal expected meeting time of two independent random walks on G. Our techniques involve using a process similar to the chameleon process invented in (*Ann. Appl. Probab.* **16** (2006) 615–635) to bound the mixing time of the exclusion process.

Résumé. Nous étudions le temps de mélange du processus de séparation bêta-binomiale symétrique sur des graphes finis connexes $G = (V, E, \{r_e\}_{e \in E})$, où V est l'ensemble de sommets, E l'ensemble d'arêtes et $r_e > 0$ les poids associés aux arêtes $e \in E$. Il s'agit d'un système de particules en interaction avec un nombre fixe de particules, qui évolue par interactions entre paires de sommets redistribuant les particules. Nous montrons que le temps de mélange de ce processus peut être borné à l'aide de l'espérance maximale du temps de rencontre de deux marches aléatoires indépendantes sur G. Notre approche utilise un processus similaire au processus caméléon inventé dans (*Ann. Appl. Probab.* **16** (2006) 615–635) pour borner le temps de mélange du processus d'exclusion.

MSC2020 subject classifications: Primary 60J27; 60K35; secondary 82C22

Keywords: Mixing times; Chameleon process; Interacting particle systems

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Sharp convergence to equilibrium for the SSEP with reservoirs

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Abstract. We consider the symmetric simple exclusion process evolving on the interval of length $n - 1$ in contact with boundary reservoirs of density $\rho \in (0, 1)$ and whose initial measure is associated with a profile $u_0 : [0, 1] \to (0, 1)$. We prove that the distance to equilibrium, in total variation, converges to a profile, in the following sense: there is an explicit function $t^n : \mathbb{R} \to \mathbb{R}_+$, which depends on u_0, such that the distance to equilibrium at time $t^n(b)$ converges, as $n \to \infty$, to $\mathcal{G}(\gamma e^{-b})$, where γ also depends on the initial profile u_0 and $\mathcal{G}(m) := \|\mathcal{N}(m, 1) - \mathcal{N}(0, 1)\|_{\mathrm{TV}}$. The proof is based on Yau's relative entropy method.

Résumé. Nous considérons le processus d'exclusion simple symétrique évoluant sur l'intervalle de longueur $n - 1$ en contact avec des réservoirs de densité $\rho \in (0, 1)$ et dont la mesure initiale est associée à un profil $u_0 : [0, 1] \to (0, 1)$. Nous prouvons que la distance à l'équilibre, en variation totale, converge vers un profil, dans le sens suivant : il existe une fonction explicite $t^n : \mathbb{R} \to \mathbb{R}_+$, que dépend de u_0, telle que la distance à l'équilibre au temps $t^n(b)$ converge, lorsque $n \to \infty$, vers $\mathcal{G}(\gamma e^{-b})$, où γ dépend aussi du profil initial u_0 et $\mathcal{G}(m) := \|\mathcal{N}(m, 1) - \mathcal{N}(0, 1)\|_{\mathrm{TV}}$. La preuve est basée sur la méthode d'entropie relative de Yau.

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Sausage volume of the random string and survival in a medium of Poisson traps

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Abstract. We provide asymptotic bounds on the survival probability of a moving polymer in an environment of Poisson traps. Our model for the polymer is the vector-valued solution of a stochastic heat equation driven by additive spacetime white noise; solutions take values in $\mathbb{R}^d$, $d \geq 1$. We give upper and lower bounds for the survival probability in the cases of hard and soft obstacles. Our bounds decay exponentially with rate proportional to $T^{d/(d+2)}$, the same exponent that occurs in the case of Brownian motion. The exponents also depend on the length J of the polymer, but here our upper and lower bounds involve different powers of J.

Secondly, our main theorems imply upper and lower bounds for the growth of the Wiener sausage around our string. The Wiener sausage is the union of balls of a given radius centered at points of our random string, with time less than or equal to a given value.

Résumé. Nous fournissons des bornes asymptotiques sur la probabilité de survie d'un polymère dans un environnement constitué de pièges de Poisson. Notre modèle pour le polymère est la solution à valeur vectorielle d'une équation de la chaleur stochastique pilotée par un bruit blanc spatio-temporel additif ; les solutions prennent des valeurs dans $\mathbb{R}^d$, $d \geq 1$. Nous donnons des bornes supérieures et inférieures pour la probabilité de survie dans les cas d'obstacles durs et mous. Nos bornes décroissent de façon exponentielle avec un taux proportionnel à $T^{d/(d+2)}$, le même exposant que dans le cas du mouvement brownien. Les exposants dépendent aussi de la longueur J du polymère, mais nos bornes supérieure et inférieure font apparaître différentes puissances de J.

Deuxièmement, nos principaux théorèmes impliquent des bornes supérieure et inférieure pour la croissance de la saucisse de Wiener autour de notre chaîne. La saucisse de Wiener est l'union de boules d'un rayon donné centré en points de notre chaîne aléatoire, avec un temps inférieur ou égal à une valeur donnée.

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A Gaussian convexity for logarithmic moment generating functions with applications in spin glasses

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Abstract. For any convex function F of n-dimensional Gaussian vector g with $\mathbb{E}e^{\lambda F(g)} < \infty$ for any $\lambda > 0$, we show that $\lambda^{-1} \ln \mathbb{E}e^{\lambda F(g)}$ is convex in $\lambda \in \mathbb{R}$. Based on this convexity, we draw three major consequences. The first recovers a version of the Paouris–Valettas lower deviation inequality for Gaussian convex functions with an improved exponent. The second establishes a quantitative bound for the Dotsenko–Franz–Mézard conjecture arising from the study of the Sherrington–Kirkpatrick (SK) mean-field spin glass model, which states that in the absence of external field, the annealed free energy of negative replica is asymptotically equal to the free energy. The last further establishes the differentiability for this annealed free energy with respect to the negative replica variable at any temperature and external field.

Résumé. Pour toute fonction convexe F d'un vecteur gaussien g de dimension n tel que $\mathbb{E}e^{\lambda F(g)} < \infty$ pour tout $\lambda > 0$, nous montrons que $\lambda^{-1} \ln \mathbb{E}e^{\lambda F(g)}$ est convexe en $\lambda \in \mathbb{R}$. En nous basant sur cette convexité, nous en tirons trois principales conséquences. La première retrouve une version de l'inégalité de déviation inférieure de Paouris–Valettas pour les fonctions convexes gaussiennes avec un exposant amélioré. La deuxième établit une borne quantitative pour la conjecture de Dotsenko–Franz–Mézard, issue de l'étude du modèle de verre de spin de Sherrington–Kirkpatrick (SK) en champ moyen, qui affirme qu'en l'absence de champ externe, l'énergie libre recuite des répliques négatives est asymptotiquement égale à l'énergie libre. La dernière établit en outre la différentiabilité pour cette énergie libre recuite par rapport à la variable de réplique négative, à toute température et champ externe.

MSC2020 subject classifications: 60D05; 60B20; 60K35; 82B44

Keywords: Sherrington–Kirkpatrick Model; Replica Method; Stochastic Optimal Control

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Edge universality of sparse random matrices

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Abstract. We consider the statistics of the extreme eigenvalues of sparse random matrices, a class of random matrices that includes the normalized adjacency matrices of the Erdős-Rényi graph $G(N, p)$. Recently, it was shown in (Lee (2021)), up to an explicit random shift, the optimal rigidity of extreme eigenvalues holds, provided the averaged degree grows with the size of the graph, $pN > N^\varepsilon$. We prove the following statements in the same regime.

1. Optimal rigidity holds for all eigenvalues with respect to an explicit random measure.
2. Up to an explicit random shift, the fluctuations of the extreme eigenvalues are given the Tracy–Widom distribution.

Résumé. Nous considérons les statistiques des valeurs propres extrêmes des matrices aléatoires clairsemées, une classe de matrices aléatoires qui comprend les matrices d'adjacence normalisées du graphe d'Erdős-Rényi $G(N, p)$. Récemment, il a été montré dans (Lee (2021)) que, à un décalage aléatoire explicite près, la rigidité optimale des valeurs propres extrêmes est vérifiée, pourvu que le degré moyen croisse avec la taille du graphe, $pN > N^\varepsilon$. Nous prouvons les énoncés suivants dans le même régime.

1. La rigidité optimale est vérifiée pour toutes les valeurs propres par rapport à une mesure aléatoire explicite.
2. À un décalage aléatoire explicite près, les fluctuations des valeurs propres extrêmes suivent la loi de Tracy–Widom.

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What is a p-adic Dyson Brownian motion?

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Abstract. We consider the singular numbers of a certain explicit continuous-time Markov jump process on $\mathrm{GL}_N(\mathbb{Q}_p)$, which we argue gives the closest p-adic analogue of multiplicative Dyson Brownian motion. We do so by explicitly classifying the possible dynamics of singular numbers of processes on $\mathrm{GL}_N(\mathbb{Q}_p)$ satisfying natural properties possessed by Brownian motion on $\mathrm{GL}_N(\mathbb{C})$. Computing the evolution of singular numbers explicitly, we find that the N-tuple of singular numbers in decreasing order evolves as a Poisson jump process on $\mathbb{Z}^N$, with ordering enforced by reflection off the walls of the positive type A Weyl chamber. This contrasts with – and provides a p-adic analogue to – the behavior of classical Dyson Brownian motion, where ordering is enforced by conditioning to avoid the Weyl chamber walls.

Résumé. Nous considérons les valeurs singulières d'un certain processus de saut de Markov en temps continu explicite sur $\mathrm{GL}_N(\mathbb{Q}_p)$. Nous affirmons qu'il constitue l'analogue p-adique le plus proche du mouvement brownien multiplicatif de Dyson. Pour cela, nous classons explicitement les dynamiques possibles des valeurs singulières de processus sur $\mathrm{GL}_N(\mathbb{Q}_p)$ satisfaisant les propriétés naturelles du mouvement brownien sur $\mathrm{GL}_N(\mathbb{C})$. En calculant explicitement l'évolution des valeurs singulières, nous trouvons que le N-uplet des valeurs singulières, dans l'ordre décroissant, évolue comme un processus de saut de Poisson sur $\mathbb{Z}^N$, avec un ordre imposé par réflexion sur les murs de la chambre de Weyl de type A positive. Cela contraste avec – et fournit un analogue p-adique au – comportement du mouvement brownien classique de Dyson, où l'ordre est imposé par la condition d'éviter les murs de la chambre de Weyl.

MSC2020 subject classifications: 15B52

Keywords: Dyson Brownian motion; Multiplicative Brownian motion; Poisson random walks; p-adic random matrices

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Biorthogonal measures, polymer partition functions, and random matrices

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Abstract. We develop the study of a particular class of biorthogonal measures, encompassing at the same time several random matrix models and partition functions of polymers. This general framework allows us to characterize the partition functions of the Log Gamma polymer and the mixed polymer in terms of explicit biorthogonal measures, as it was previously done for the homogeneous O'Connell–Yor polymer by Imamura and Sasamoto. In addition, we show that the biorthogonal measures associated to these three polymer models (Log Gamma, O'Connell–Yor, and mixed polymer) converge to random matrix eigenvalue distributions in small temperature limits. We also clarify the connection between different Fredholm determinant representations and explain how our results might be useful for asymptotic analysis and large deviation estimates.

Résumé. Nous développons l'étude d'une classe particulière de mesures biorthogonales, englobant à la fois plusieurs modèles de matrices aléatoires et des fonctions de partition de polymères. Ce cadre général nous permet de caractériser les fonctions de partition du polymère Log Gamma et du polymère mixte en termes de mesures biorthogonales explicites, comme cela a été fait précédemment pour le polymère de O'Connell–Yor, dans le cas homogène, par Imamura et Sasamoto. Nous montrons aussi que les mesures biorthogonales associées à ces trois modèles de polymères (Log Gamma, O'Connell–Yor et polymère mixte) convergent vers des distributions de valeurs propres de matrices aléatoires, dans un régime de basse température. Nous clarifions également le lien entre différentes représentations de ces quantités en termes de déterminants de Fredholm, et expliquons comment nos résultats peuvent être utiles pour l'analyse asymptotique et les estimations des grandes déviations.

MSC2020 subject classifications: Primary 82D60; 60B20; secondary 60G55

Keywords: Log Gamma polymer; O'Connel-Yor polymer; Random matrices; Biorthogonal ensembles

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Gaussian beta ensembles: The perfect freezing transition and its characterization in terms of Beurling–Landau densities

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Abstract. The Gaussian β-ensemble is a real n-point configuration $\{x_j\}_1^n$ picked randomly with respect to the Boltzmann factor $e^{-\frac{\beta}{2}H_n}$, where $H_n = \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + n \sum_{i=1}^n \frac{1}{2} x_i^2$. It is well known that the point process $\{x_j\}_1^n$ tends to follow the semicircle law $\sigma(x) = \frac{1}{2\pi} \sqrt{(4 - x^2)_+}$ in certain average senses.

A Fekete configuration (minimizer of H_n) is spread out in a much more uniform way in the interval $[-2, 2]$ with respect to the regularization $\sigma_n(x) = \max\{\sigma(x), n^{-\frac{1}{3}}\}$ of the semicircle law. In particular, Fekete configurations are "equidistributed" with respect to $\sigma_n(x)$, in a certain technical sense of Beurling–Landau densities.

We consider the problem of characterizing sequences β_n of inverse temperatures, which guarantee almost sure equidistribution as $n \to \infty$. We find that a necessary and sufficient condition is that β_n grows at least logarithmically in n:

$$\beta_n \gtrsim \log n.$$

We call this growth rate the *perfect freezing regime*. Along the way, we give several further results on the distribution of particles when $\beta_n \gtrsim \log n$, for example on minimal spacing and discrepancies, and that with high probability a random sample solves certain sampling and interpolation problems for weighted polynomials. (In this context, Fekete sets correspond to $\beta \equiv \infty$.)

The condition $\beta_n \gtrsim \log n$ was introduced in earlier works due to some of the authors in the context of two-dimensional Coulomb gas ensembles, where it is shown to be sufficient for equidistribution. Interestingly, although the technical implementation requires some considerable modifications, the strategy from dimension two adapts well to prove sufficiency also for one-dimensional Gaussian ensembles. On a technical level, we use estimates for weighted polynomials due to Levin, Lubinsky, Gustavsson and others. The other direction (necessity) involves estimates due to Ledoux and Rider on the distribution of particles which fall near or outside the boundary.

Résumé. Le β-ensemble gaussien est une configuration de n points réels $\{x_j\}_1^n$ choisis aléatoirement par rapport au facteur de Boltzmann $e^{-\frac{\beta}{2}H_n}$, où $H_n = \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + n \sum_{i=1}^n \frac{1}{2} x_i^2$. On sait que le processus ponctuel $\{x_j\}_1^n$ tend à suivre la loi du demi-cercle $\sigma(x) = \frac{1}{2\pi} \sqrt{(4 - x^2)_+}$ dans un certain sens des moyennes.

Une configuration de Fekete (un minimiseur de H_n) est répartie de manière beaucoup plus uniforme dans l'intervalle $[-2, 2]$ par rapport à la régularisation $\sigma_n(x) = \max\{\sigma(x), n^{-\frac{1}{3}}\}$ de la loi du demi-cercle. En particulier, les configurations de Fekete sont "équidistribuées" par rapport à $\sigma_n(x)$, dans un certain sens technique des densités de Beurling–Landau.

Nous considérons le problème consistant à caractériser les suites β_n de températures inverses qui garantissent une équidistribution presque sûre lorsque $n \to \infty$. Nous trouvons qu'une condition nécessaire et suffisante est que β_n croisse au moins logarithmiquement en n :

$$\beta_n \gtrsim \log n.$$

Nous appelons ce taux de croissance le *régime de congélation parfaite*. En cours de route, nous donnons plusieurs résultats supplémentaires sur la distribution des particules lorsque $\beta_n \gtrsim \log n$, par exemple sur l'espacement minimal et les écarts, et montrons que, avec une haute probabilité, un échantillon aléatoire résout certains problèmes d'échantillonnage et d'interpolation pour des polynômes pondérés. (Dans ce contexte, les ensembles de Fekete correspondent à $\beta \equiv \infty$.) La condition $\beta_n \gtrsim \log n$ a été introduite dans des travaux antérieurs par certains des auteurs dans le contexte des ensembles de gaz de Coulomb bidimensionnels, où il est montré qu'elle est suffisante pour l'équidistribution. Il est intéressant de noter que, bien que la mise en œuvre technique nécessite des modifications

considérables, la stratégie de la dimension deux s'adapte bien pour prouver la suffisance également pour les ensembles gaussiens unidimensionnels. Sur le plan technique, nous utilisons des estimations pour les polynômes pondérés dues entre autres à Levin, Lubinsky, Gustavsson. L'autre direction (la nécessité) implique des estimations dues à Ledoux et Rider sur la distribution des particules qui tombent près ou en dehors de la frontière.

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About the asymptotic behaviour of the martingale associated with the Vertex Reinforced Jump Process on trees and $\mathbb{Z}^d$

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Abstract. We study the asymptotic behaviour of the martingale $(\psi_n(o))_{n\in\mathbb{N}}$ associated with the Vertex Reinforced Jump Process (VRJP). We show that it is bounded in L^p for every $p > 1$ on trees and uniformly integrable on $\mathbb{Z}^d$ in all the transient phase of the VRJP. Moreover, when the VRJP is recurrent on trees, we have good estimates on the moments of $\psi_n(o)$ and we can compute the exact decreasing rate τ such that $n^{-1}\ln(\psi_n(o)) \sim -\tau$ almost surely where τ is related to standard quantities for branching random walks. Besides, on trees, at the critical point, we show that $n^{-1/3}\ln(\psi_n(o)) \sim -\rho_c$ almost surely where ρ_c can be computed explicitly. Furthermore, at the critical point on trees again, we prove that the discrete process associated with the VRJP is a mixture of positive recurrent Markov chains. Our proofs use properties of the β-potential associated with the VRJP and techniques coming from the domain of branching random walks.

Résumé. Nous étudions le comportement asymptotique de la martingale $(\psi_n(o))_{n\in\mathbb{N}}$ associée au Processus de saut renforcé par sommets (VRJP). On montre que cette martingale est bornée dans L^p dans le cas des arbres et uniformément intégrable sur $\mathbb{Z}^d$ dans toute la phase transiente. De plus, lorsque le VRJP est récurrent sur les arbres, on a de bonnes estimées sur les moments de $\psi_n(o)$. et on peut calculer de manière exacte le taux de décroissance τ tel que $n^{-1}\ln(\psi_n(o)) \sim -\tau$ presque sûrement. τ est lié à certaines quantités standard dans l'étude des marches aléatoires branchantes. Par ailleurs, sur les arbres, au point critique, on montre que $n^{-1/3}\ln(\psi_n(o)) \sim -\rho_c$ presque sûrement avec ρ_c qui peut être calculé explicitement. En outre, toujours au point critique sur les arbres, on prouve que le processus discret associé au VRJP est un mélange de chaines de Markov récurrentes positives. Nos preuves utilisent des propriétés du potentiel β associé au VRJP et des techniques provenant des marches aléatoires branchantes.

MSC2020 subject classifications: Primary 60K37; secondary 60J80

Keywords: Reinforced random processes; Martingales; Branching random walks

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Convergences of looptrees coded by excursions

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Abstract. In order to study convergences of looptrees, we construct continuum trees and looptrees from real-valued càdlàg functions without negative jumps called excursions. We then provide a toolbox to manipulate the two resulting codings of metric spaces by excursions and we formalize the principle that jumps correspond to loops and that continuous growths correspond to branches. Combining these codings creates new metric spaces from excursions that we call vernation trees. They consist of a collection of loops and trees glued along a tree structure so that they unify trees and looptrees. We also propose a topological definition for vernation trees, which yields what we argue to be the right space to study convergences of looptrees. However, those first codings lack some functional continuity, so we adjust them. We thus obtain several limit theorems. Finally, we present some probabilistic applications, such as proving an invariance principle for random discrete looptrees.

Résumé. Afin d'étudier les convergences d'arbres à boucles, on construit des arbres et des arbres à boucles continus à partir d'excursions, c'est-à-dire des fonctions à valeurs réelles, càdlàg, et dont les sauts sont positifs. On fournit plusieurs outils pour manipuler au mieux les deux codages résultants d'espaces métriques par des excursions. Ceux-ci sont régis par le principe que les boucles correspondent aux sauts de l'excursion et que les branches correspondent aux portions de croissance continue. En combinant les deux codages, on définit de nouveaux espaces métriques, généralisant à la fois arbres et arbres à boucles car formés par une famille de boucles et de branches greffées entre elles selon une structure arborescente, que l'on nomme arbres à vernation. On propose aussi une caractérisation topologique des limites des arbres à vernation. Cependant, nos codages se comportent mal avec la convergence fonctionnelle des excursions codantes, donc on les ajuste pour pouvoir répondre à notre objectif initial. Enfin, on donne quelques applications probabilistes, comme la preuve d'un principe d'invariance pour des arbres à boucles discrets aléatoires.

MSC2020 subject classifications: Primary 60F17; 54C30; 54E70; secondary 05C05; 54F50

Keywords: Looptree; Tree; Coding by real-valued functions; Limit theorem; Scaling limit; Random metric space; Geodesic space

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Height and contour processes of Crump–Mode–Jagers forests (III): The binary, homogeneous universality class

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Abstract. This paper belongs to a series of papers which investigate scaling limits of Crump–Mode–Jagers (CMJ) forests. In the previous two papers we identified general conditions under which CMJ forests belong to the universality class of Galton–Watson and Bellman–Harris forests. In this paper we identify general conditions for CMJ forests to belong to the universality class of binary, homogeneous CMJ forests. These conditions state that the long-term growth of the offspring process should be linear, and also involve a technical condition on its tail behavior. We show in particular that any renewal process with finite mean satisfies these conditions.

Résumé. Cet article fait partie d'une série d'articles qui étudient les limites d'échelle des forêts de Crump–Mode–Jagers (CMJ). Dans les deux articles précédents, nous avons identifié des conditions générales sous lesquelles les forêts CMJ appartiennent à la classe d'universalité des forêts de Galton–Watson et de Bellman–Harris. Dans cet article, nous identifions des conditions générales pour que les forêts CMJ appartiennent à la classe d'universalité des forêts CMJ binaires et homogènes. Ces conditions stipulent que le processus de descendance doit croître linéairement en temps long, et impliquent aussi une condition technique sur sa queue de distribution. Nous montrons en particulier que tout processus de renouvellement d'espérance finie satisfait à ces conditions.

MSC2020 subject classifications: Primary 60J80; 60F17; 60F05; secondary 60G50

Keywords: Crump–Mode–Jagers branching process; Chronological tree; Scaling limit; Universality class

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Quenched invariance principle for biased random walks in random conductances in the sub-ballistic regime

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Abstract. We consider a biased random walk in positive random conductances on $\mathbb{Z}^d$ for $d \geq 5$. In the sub-ballistic regime, we prove the quenched convergence of the properly rescaled random walk towards a Fractional Kinetics.

Résumé. On étudie la marche aléatoire biaisée en conductances aléatoires sur $\mathbb{Z}^d$ en dimension $d \geq 5$. Dans le régime sous-balistique, sous la loi quenched, on prouve que la marche aléatoire renormalisée converge en loi vers un Fractional Kinetics.

MSC2020 subject classifications: Primary 60K37; 60F17; secondary 60K50; 60G22

Keywords: Scaling limit; Random walk in random environment; Fractional kinetics

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Non-atomicity of the extremal decomposition of the free state for finite-spin models on Cayley trees

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Abstract. We prove the non-atomicity of the extremal decomposition measure of the free state of low temperature Potts models, and more generally of ferromagnetic finite-spin models, on a regular tree, including general clock models. The decomposition is supported on uncountably many inhomogeneous extremal states, that we call *glassy states*. The method of proof provides explicit concentration bounds on *branch overlaps*, which play the role of an order parameter for typical extremal states. The result extends to the counterpart of the free state (called *central state*) in a wide range of models which have no symmetry, allowing also the presence of sufficiently small field terms. Our work shows in particular that the decomposition of central states into uncountably many glassy states in finite-spin models on trees at low temperature is a generic phenomenon, and does not rely on symmetries of the Hamiltonian.

Résumé. Nous prouvons la non-atomicité de la mesure de décomposition extrémale de l'état libre pour les modèles de Potts à basse température, et plus généralement les modèles ferromagnétiques à spin fini, modèles d'horloge généraux compris, sur un arbre régulier. La décomposition est portée par d'indénombrables états extrémaux inhomogènes, que nous appelons *états vitreux*. La méthode de preuve fournit des bornes de concentration explicites sur les *recouvrements de branches*, qui jouent le rôle de paramètres d'ordre pour les états extrémaux typiques. Le résultat s'étend à l'analogue de l'état libre (appelé *état central*) dans une grande famille de modèles sans symétrie, permettant notamment la présence de champs suffisamment petits. Nos travaux montrent en particulier que, dans des modèles à spin fini sur des arbres réguliers, à basse température, le phénomène de décomposition extrémale des états centraux portée par d'indénombrables états vitreux est un phénomène générique, et ne repose pas sur les symétries du hamiltonien.

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Localized bounds on log-derivatives of the heat kernel on incomplete Riemannian manifolds

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Abstract. Bounds on the logarithmic derivatives of the heat kernel on a compact Riemannian manifolds have been long known, and were recently extended, for the log-gradient and log-Hessian, to general complete Riemannian manifolds, by Chen–Li–Wu. Here, building on earlier heat kernel asymptotics of Léandre and Bailleul–Norris, we further extend these bounds to incomplete Riemannan manifolds under a mild condition on the distance to infinity, for derivatives of all orders. Moreover, we consider not only the usual heat kernel associated to the Laplace–Beltrami operator, but we also allow the addition of a conservative vector field. We show that these bounds are sharp in general, even for compact manifolds, and we discuss the difficulties that arise when the operator incorporates non-conservative vector fields or when the Riemannian structure is weakened to a sub-Riemannian structure.

Résumé. Des bornes sur les dérivées logarithmiques du noyau de la chaleur sur une variété riemannienne compacte sont connues depuis longtemps et ont été récemment étendues, dans le cas du log-gradient et de la log-hessienne, aux variétés riemanniennes complètes générales, par Chen–Li–Wu. Ici, en s'appuyant sur des asymptotiques du noyau de la chaleur antérieures de Léandre et Bailleul–Norris, nous étendons ces bornes aux variétés riemanniennes incomplètes sous une faible hypothèse de distance à l'infini, pour des dérivées de tout ordre. De plus, nous ne considérons pas uniquement le noyau de la chaleur usuel associé à l'opérateur de Laplace–Beltrami, mais permettons également l'ajout d'un champ de vecteurs conservatif. Nous montrons que ces bornes sont optimales en général, même pour des variétés compactes, et nous discutons les difficultés qui émergent lorsque l'opérateur comprend des champs de vecteurs non conservatifs ou que la structure riemannienne est affaiblie en une structure sous-riemannienne.

MSC2020 subject classifications: Primary 58J65; secondary 58J35; 58K55; 35K08

Keywords: Heat kernel bounds; Logarithmic derivatives; Small-time asymptotics; Incomplete manifold

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Local times of deterministic paths and self-similar processes with stationary increments as normalized numbers of interval crossings

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Abstract. We prove a general result on a relationship between a limit of normalized numbers of interval crossings by a càdlàg path and an occupation measure associated with this path.

Using this result, we define local times of fractional Brownian motions (classically defined as densities of relevant occupation measure) as weak limits of properly normalized numbers of interval crossings.

We also discuss a similar result for càdlàg semimartingales, in particular for alpha-stable processes, and Rosenblatt processes, and provide natural examples of deterministic paths which possess quadratic or higher-order variation but no local times.

Résumé. Nous montrons un résultat général sur la relation entre une limite du nombre renormalisé de traversées d'intervalles par un chemin càdlàg et une mesure d'occupation associée à ce chemin.

En utilisant ce résultat, nous définissons les temps locaux des mouvements brownien fractionnaires (traditionnellement définis comme des densités de la mesure d'occupation correspondante) comme des limites faibles des nombres proprement renormalisés de traversées d'intervalles.

Nous discutons également d'un résultat similaire pour les semi-martingales càdlàg, en particulier pour les processus alpha-stables et les processus de Rosenblatt, et fournissons des exemples naturels de chemins déterministes qui possèdent une variation quadratique ou d'ordre supérieur mais pas de temps locaux.

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On the robustness of spatial quantiles

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Abstract. Spatial quantiles are among the most successful concepts of multivariate quantiles. In particular, they are essentially the only quantiles that can be computed in high dimensions. There has been an intense research activity to study spatial quantiles in the last two decades, yet surprisingly little is known about their robustness properties. In the present work, we carefully study the breakdown point of spatial quantiles. We offer three approaches, that show diverse distinctive advantages. The first approach is a constructive one: it is conceptually simple and allows us to derive the finite-sample breakdown point of spatial quantiles. While the second approach is not constructive and does not identify the global breakdown point of spatial quantiles, it provides an upper bound on the breakdown point under contamination in any fixed direction. It also allows us to determine the breakdown point of spatial L_p-quantiles for any $p > 1$. Last but not least, the third approach characterizes precisely when breakdown occurs under any given contamination scheme, hence provides the breakdown points associated with very diverse contamination scenarios. Quite nicely, this last approach further covers cases where the contamination and/or the ground probability measures are continuous distributions. An intriguing corollary of our results states that, in high dimensions, the "practical" breakdown point exceeds the theoretical one. Throughout, our theoretical results are illustrated through numerical exercises. Part of our results cover infinite-dimensional Hilbert spaces as well.

Résumé. Les quantiles spatiaux comptent parmi les concepts de quantiles multivariés qui ont rencontré le plus grand succès. En particulier, ces quantiles sont essentiellement les seuls qui peuvent être calculés en grande dimension. Ils ont fait l'objet d'une activité de recherche intense au cours des deux dernières décennies, mais, de façon surprenante, on sait très peu au sujet de leur robustesse. Dans ce travail, nous étudions en détail le point de rupture des quantiles spatiaux. Nous adoptons trois approches, qui ont chacune des avantages propres. La première approche est constructive : elle est conceptuellement simple et nous permet d'obtenir le point de rupture exact des quantiles spatiaux. Si la seconde approche n'est pas constructive et ne permet pas d'identifier le point de rupture global des quantiles spatiaux, elle fournit une borne supérieure sur le point de rupture associé à des contaminations dans une direction fixée. Elle nous permet par ailleurs de déterminer le point de rupture des quantiles spatiaux L_p pour tout $p > 1$. Enfin et surtout, la troisième approche caractérise précisément quand la rupture survient sous un quelconque schéma de contamination fixé, ce qui fournit les points de rupture associés à des scenarios de contamination très divers. De façon intéressante, cette dernière approche couvre de plus les cas où la contamination et/ou la mesure de probabilité de référence sont des distributions continues. Un corollaire surprenant de nos résultats montre que le point de rupture "pratique" en grande dimension excède le point de rupture théorique. Tout au long de l'article, nos résultats théoriques sont illustrés par des exercices numériques. Une partie de nos résultats couvrent également les espaces de Hilbert de dimension infinie.

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Wasserstein posterior contraction rates in non-dominated Bayesian nonparametric models

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Abstract. Posterior contractions rates (PCRs) strengthen the notion of Bayesian consistency, quantifying the speed at which the posterior distribution concentrates on arbitrarily small neighborhoods of the "true model", with probability tending to 1 or almost surely, as the sample size goes to infinity. In this paper, we study PCRs in non-dominated Bayesian nonparametric models for the observations, thus assuming that posterior distributions are available through a more general disintegration argument than Bayes formula. By means of a suitable sieve construction on the space of observations, in combination with an assumption of Lipschitz-continuity for the posterior distribution, our main result provides Wasserstein PCRs in non-dominated Bayesian nonparametric models. Besides the Lipschitz-continuity, our result relies on minimal modeling assumptions, and it is stated in a general form that allows for any nonparametric prior distributions. Refinements of our result are presented under additional assumptions on the prior distribution and the "true model", which lead to PCRs that, up to a constant, are of the form $n^{-1/(d+2)}$, with n being the sample size and d being the dimension of the observations. Such a PCR is slightly slower than the optimal minimax rate for the estimation of smooth densities in Wasserstein distance, which, however, is obtained for dominated models for the observations, and under smoothness conditions for the (density of the) model. To date, we are not aware of any classical (frequentist) study on Wasserstein rates of consistency in the context of non-dominated nonparametric models, which would provide a fair term of comparison for our PCRs. Applications of our results are given with respect to the Dirichlet process prior, which is a conjugate prior, and the normalized extended Gamma process prior, which is a non-conjugate prior.

Résumé. Les vitesses de contraction a posteriori (VCPs) renforcent la notion de consistance bayésienne, en quantifiant la vitesse à laquelle la distribution a posteriori se concentre autour de voisinages arbitrairement petits du «vrai modèle», avec une probabilité tendant vers 1 ou presque sûrement, lorsque la taille de l'échantillon tend vers l'infini. Dans cet article, nous étudions les VCPs dans des modèles bayésiens non paramétriques non dominés, en supposant uniquement que les distributions a posteriori sont disponibles sous une forme plus générale de désintégration que la formule de Bayes. Au moyen d'une construction de tamis sur l'espace des observations, et d'une hypothèse de continuité lipschitzienne pour la distribution a posteriori, notre résultat principal fournit des VCPs relativement à la distance de Wasserstein dans des modèles bayésiens non paramétriques non dominés. Outre la continuité lipschitzi-enne, notre résultat repose sur des hypothèses de modélisation minimales, et il est énoncé sous une forme générale applicable à toutes les distributions a priori non paramétriques. Des raffinements de notre résultat sont présentés sous des hypothèses supplémentaires sur la distribution a priori et le «vrai modèle», ce qui conduit à des VCPs qui, à une constante près, sont de la forme $n^{-1/(d+2)}$, où n est la taille de l'échantillon et d est la dimension des observations. Une telle VCP est légèrement plus lente que le taux minimax optimal pour l'estimation de densités lisses en distance de Wasserstein, qui cependant, est obtenu pour des modèles dominés et sous des conditions de régularité sur la densité. À ce jour, nous ne sommes pas au courant de l'existence de résultats classiques (fréquentistes) sur les vitesses d'estimation en distance de Wasserstein dans le contexte de modèles non paramétriques non dominés, ce qui fournirait un élément de comparaison équitable pour nos VCPs. Des applications de nos résultats sont données par rapport à l'a priori processus de Dirichlet, qui est conjugué, et à l'a priori processus Gamma étendu normalisé, qui est non conjugué.

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Strong existence and uniqueness of solutions of SDEs with time dependent Kato class coefficients

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Abstract. Consider stochastic differential equation (SDE) in $\mathbb{R}^d$ $(d \geq 2)$: $dX_t = dW_t + b(t, X_t)\, dt$, where W is a Brownian motion and $b(\cdot, \cdot)$ is a measurable vector field. It is known that if $|b|^2(\cdot, \cdot) = |b|^2(\cdot)$ belongs to the Kato class $\mathbf{K}_{d,2}$, then there is a unique weak solution to the above SDE. In this article we show that if $|b|^2$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ for some $\alpha \in (0, 2)$ (α can be arbitrarily close to 2), then there exists a unique strong solution to the SDE, extending the results in the existing literature as demonstrated by examples. Furthermore, we allow the drift to be time-dependent. The new regularity estimates we established for the solutions of parabolic equations with Kato class coefficients play a crucial role.

Résumé. Considérons une équation différentielle stochastique (EDS) dans $\mathbb{R}^d$ $(d \geq 2)$: $dX_t = dW_t + b(t, X_t)\, dt$, où W est un mouvement brownien et $b(\cdot, \cdot)$ est un champ vectoriel mesurable. On sait que si $|b|^2(\cdot, \cdot) = |b|^2(\cdot)$ appartient à la classe de Kato $\mathbf{K}_{d,2}$, alors il existe une unique solution faible à l'EDS ci-dessus. Dans cet article, nous montrons que si $|b|^2$ appartient à la classe de Kato $\mathbf{K}_{d,\alpha}$ pour un certain $\alpha \in (0, 2)$ (α peut être arbitrairement proche de 2), alors il existe une unique solution forte à l'EDS, étendant les résultats de la littérature existante comme le démontrent des exemples. De plus, nous permettons à la dérive d'être dépendante du temps. Les nouvelles estimations de régularité que nous avons établies pour les solutions d'équations paraboliques à coefficients de classe Kato jouent un rôle crucial.

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Independent projections of diffusions: Gradient flows for variational inference and optimal mean field approximations

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Abstract. What is the optimal way to approximate a high-dimensional diffusion process by one in which the coordinates are *independent*? This paper presents a construction, called the *independent projection*, which is optimal for two natural criteria. First, when the original diffusion is reversible with invariant measure ρ_*, the independent projection serves as the Wasserstein gradient flow for the relative entropy $H(\cdot \mid \rho_*)$ constrained to the space of product measures. This is related to recent Langevin-based sampling schemes proposed in the statistical literature on mean field variational inference. In addition, we provide both qualitative and quantitative results on the long-time convergence of the independent projection, with quantitative results in the log-concave case derived via a new variant of the logarithmic Sobolev inequality. Second, among all processes with independent coordinates, the independent projection is shown to exhibit the slowest growth rate of path-space entropy relative to the original diffusion. This sheds new light on the classical McKean–Vlasov equation and recent variants proposed for non-exchangeable systems, which can be viewed as special cases of the independent projection.

Résumé. Quelle est la manière optimale d'approximer un processus de diffusion de grande dimension par un processus pour lequel les coordonnées sont *indépendantes* ? Cet article présente une construction, appelée *projection indépendante*, qui est optimale pour deux critères naturels. Premièrement, lorsque la diffusion originale est réversible avec une mesure invariante ρ_*, la projection indépendante sert de flot gradient pour la métrique de Wasserstein pour l'entropie relative $H(\cdot \mid \rho_*)$ contrainte à l'espace des mesures du produit. Ceci est lié aux récents schémas d'échantillonnage basés sur les processus de Langevin proposés dans la littérature statistique sur l'inférence variationnelle de champ moyen. De plus, nous fournissons des résultats qualitatifs et quantitatifs sur la convergence en temps longs de la projection indépendante, avec des résultats quantitatifs, dans le cas log-concave, dérivés via une nouvelle variante de l'inégalité logarithmique de Sobolev. Deuxièmement, parmi tous les processus avec des coordonnées indépendantes, nous montrons que la projection indépendante a le taux de croissance le plus lent de l'entropie dans l'espace de chemin par rapport à la diffusion originale. Cela donne un nouvel éclairage sur l'équation classique de McKean–Vlasov et sur les variantes récentes proposées pour les systèmes non échangeables, qui peuvent être considérés comme des cas particuliers de la projection indépendante.

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Poisson equations with locally-Lipschitz coefficients and uniform in time averaging for stochastic differential equations via strong exponential stability

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Abstract. We study averaging for Stochastic Differential Equations (SDEs) and Poisson equations. We succeed in obtaining a *uniform in time* (UiT) averaging result, with a rate, for fully coupled SDE models with super-linearly growing coefficients. This is the main result of this paper and it is, to the best of our knowledge, the first UiT multiscale result with a rate. More precisely, the main feature of our averaging theorem is that it holds uniformly in time; the technique of proof we use gives, as a biproduct, a rate of convergence as well. Very few UiT averaging results exist in the literature, and they almost exclusively apply to multiscale systems of Ordinary Differential Equations. Among these few, none of those we are aware of comes with a rate of convergence. The UiT nature of this result and the fact that the main theorem comes with a rate of convergence as well, make it important as theoretical underpinning for a range of applications, such as applications to statistical methodology, molecular dynamics etc. Key to obtaining both our UiT averaging result and to enable dealing with the super-linear growth of the coefficients (of the slow-fast system and of the associated Poisson equation) is conquering exponential decay in time of the space-derivatives of appropriate Markov semigroups. We refer to semigroups which enjoy this property as being *Strongly Exponentially Stable*.

There are various approaches in the literature to proving averaging results. The analytic approach we take here requires studying a family of Poisson problems associated with the generator of the (fast component of the) SDE dynamics. The study of Poisson equations in non-compact state space is notoriously difficult, with current literature mostly covering the case when the coefficients of the Partial Differential Equation (PDE) are either bounded or satisfy linear growth assumptions (with the latter case having been achieved only recently). In this paper we treat Poisson equations on non-compact state spaces for coefficients that can grow super-linearly. In particular, we demonstrate how Strong Exponential Stability can be employed not only to prove the UiT result for the slow-fast system but also to overcome some of the technical hurdles in the analysis of Poisson problems. Poisson equations are essential tools in both probability theory and PDE theory. Their vast range of applications includes the study of the asymptotic behaviour of solutions of parabolic PDEs, the treatment of multi-scale and homogenization problems as well as the theoretical analysis of approximations of solutions of Stochastic Differential Equations (SDEs). So our result on Poisson equations is of independent interest as well.

Résumé. Nous étudions la moyenne pour les équations différentielles stochastiques (EDS) et les équations de Poisson. Nous réussissons à obtenir un résultat de moyenne *uniforme en temps* (UiT), avec un taux de convergence, pour des modèles d'EDS entièrement couplés avec des coefficients à croissance surlinéaire. Il s'agit du résultat principal de cet article et, à notre connaissance, du premier résultat multi-échelle UiT avec un taux. Plus précisément, la caractéristique principale de notre théorème de moyenne est qu'il est valable uniformément en temps ; la technique de preuve que nous utilisons fournit également, comme sous-produit, un taux de convergence. Très peu de résultats de moyenne UiT existent dans la littérature, et ils s'appliquent presque exclusivement à des systèmes multi-échelles d'équations différentielles ordinaires. Parmi ce petit nombre, aucun de ceux que nous connaissons n'est accompagné d'un taux de convergence. La nature UiT de ce résultat et le fait que le théorème principal soit également assorti d'un taux de convergence en font un fondement théorique important pour un éventail d'applications, telles que les applications à la méthodologie statistique, à la dynamique moléculaire, etc. La clé pour obtenir à la fois notre résultat de moyenne UiT et permettre de traiter la croissance surlinéaire des coefficients (du système lent-rapide et de l'équation de Poisson associée) est de surmonter la décroissance exponentielle en temps des dérivées spatiales des semi-groupes de Markov appropriés. Nous faisons référence aux semi-groupes qui possèdent cette propriété comme étant *fortement exponentiellement stables*. Il existe diverses approches dans la littérature pour prouver des résultats de moyenne. L'approche analytique que nous adoptons ici nécessite l'étude d'une famille de problèmes de Poisson associés au générateur

de la (composante rapide de la) dynamique EDS. L'étude des équations de Poisson dans un espace d'état non compact est notoirement difficile, la littérature actuelle couvrant principalement le cas où les coefficients de l'équation aux dérivées partielles (EDP) sont soit bornés, soit satisfont des hypothèses de croissance linéaire (ce dernier cas n'ayant été traité que récemment). Dans cet article, nous traitons des équations de Poisson sur des espaces d'état non compacts pour des coefficients pouvant croître de manière surlinéaire. En particulier, nous démontrons comment la forte stabilité exponentielle peut être utilisée non seulement pour prouver le résultat UiT pour le système lent-rapide, mais aussi pour surmonter certains des obstacles techniques dans l'analyse des problèmes de Poisson. Les équations de Poisson sont des outils essentiels en théorie des probabilités et en théorie des EDP. Leur vaste champ d'application inclut l'étude du comportement asymptotique des solutions d'EDP paraboliques, le traitement des problèmes multi-échelles et d'homogénéisation, ainsi que l'analyse théorique des approximations des solutions des EDS. Ainsi, notre résultat sur les équations de Poisson possède également un intérêt indépendant.

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Differentiability of the shape function for directed polymers in continuous space

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Abstract. For directed polymers, the shape function computes the limiting average energy accrued by paths with a given average slope. We prove that, for a large family of directed polymer models in discrete time and continuous space in dimension $1 + 1$, for positive and zero temperature, the shape function is differentiable with respect to the slope on the entire real line.

Résumé. Pour les polymères dirigés, la fonction de forme calcule l'énergie moyenne limite accumulée par les chemins ayant une pente moyenne donnée. Nous démontrons que, pour une grande famille de modèles de polymères dirigés en temps discret et en espace continu de dimension $1 + 1$, pour des températures positives et nulles, la fonction de forme est différentiable par rapport à la pente sur toute la droite réelle.

MSC2020 subject classifications: 60K35; 0K37; 82B44; 82D60

Keywords: Directed polymers; Geodesics; Random environment; Shape function; Shear invariance

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Finitely dependent random colorings of bounded degree graphs

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Abstract. We prove that every (possibly infinite) graph of degree at most d has a 4-dependent random proper $4^{d(d+1)/2}$-coloring, and one can construct it as a finitary factor of iid. For unimodular transitive (or unimodular random) graphs we construct an automorphism-invariant (respectively, unimodular) 2-dependent coloring by $3^{d(d+1)/2}$ colors. In particular, there exist random proper colorings for $\mathbb{Z}^d$ and for the regular tree that are 2-dependent and automorphism-invariant, or 4-dependent and finitary factor of iid.

Résumé. Nous démontrons que tout graphe (éventuellement infini) de degré au plus d admet une coloration aléatoire propre 4-dépendante à $4^{d(d+1)/2}$ couleurs, et que celle-ci peut être construite comme un facteur finitaire d'iid. Pour les graphes unimodulaires transitifs (ou les graphes aléatoires unimodulaires), nous construisons une coloration 2-dépendante, invariante par automorphisme (respectivement unimodulaire), à $3^{d(d+1)/2}$ couleurs. En particulier, il existe des colorations aléatoires propres de $\mathbb{Z}^d$ et de l'arbre régulier qui sont 2-dépendantes et invariantes par automorphisme, ou bien 4-dépendantes et obtenues comme facteurs finitaires d'iid.

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