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INVARIANCE PRINCIPLE FOR VARIABLE SPEED RANDOM WALKS ON TREES

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We consider stochastic processes on complete, locally compact tree-like metric spaces (T,r) on their "natural scale" with boundedly finite speed measure ν. Given a triple (T,r,ν) such a speed-ν motion on (T,r) can be characterized as the unique strong Markov process which if restricted to compact subtrees satisfies for all $x, y \in T$ and all positive, bounded measurable f,

$$\mathbb{E}^x\left[\int_0^{\tau_y} \mathrm{d}s f(X_s)\right] = 2\int_T \nu(\mathrm{d}z) r\big(y, c(x,y,z)\big) f(z) < \infty, \tag{0.1}$$

where $c(x,y,z)$ denotes the branch point generated by x, y, z. If (T,r) is a discrete tree, X is a continuous time nearest neighbor random walk which jumps from v to $v' \sim v$ at rate $\frac{1}{2} \cdot (\nu(\{v\}) \cdot r(v,v'))^{-1}$. If (T,r) is path-connected, X has continuous paths and equals the ν-Brownian motion which was recently constructed in [*Trans. Amer. Math. Soc.* **365** (2013) 3115–3150]. In this paper, we show that speed-ν_n motions on (T_n, r_n) converge weakly in path space to the speed-ν motion on (T,r) provided that the underlying triples of metric measure spaces converge in the Gromov–Hausdorff-vague topology introduced in [*Stochastic Process. Appl.* **126** (2016) 2527–2553].

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SHARP DIMENSION FREE QUANTITATIVE ESTIMATES FOR THE GAUSSIAN ISOPERIMETRIC INEQUALITY[1]

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We provide a full quantitative version of the Gaussian isoperimetric inequality: the difference between the Gaussian perimeter of a given set and a half-space with the same mass controls the gap between the norms of the corresponding barycenters. In particular, it controls the Gaussian measure of the symmetric difference between the set and the half-space oriented so to have the barycenter in the same direction of the set. Our estimate is independent of the dimension, sharp on the decay rate with respect to the gap and with optimal dependence on the mass.

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Key words and phrases. Quantitative estimates, Gaussian isoperimetric inequality.

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RANDOM CURVES, SCALING LIMITS AND LOEWNER EVOLUTIONS

BY ANTTI KEMPPAINEN[*,1,2] AND STANISLAV SMIRNOV[†,‡,2]

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In this paper, we provide a framework of estimates for describing 2D scaling limits by Schramm's SLE curves. In particular, we show that a weak estimate on the probability of an annulus crossing implies that a random curve arising from a statistical mechanics model will have scaling limits and those will be well described by Loewner evolutions with random driving forces. Interestingly, our proofs indicate that existence of a nondegenerate observable with a conformally-invariant scaling limit seems sufficient to deduce the required condition.

Our paper serves as an important step in establishing the convergence of Ising and FK Ising interfaces to SLE curves; moreover, the setup is adapted to branching interface trees, conjecturally describing the full interface picture by a collection of branching SLEs.

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Key words and phrases. Random curve, lattice model, weak convergence of probability measures, Schramm–Loewner evolution, Brownian motion, conformal mapping.

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FERROMAGNETIC ISING MEASURES ON LARGE LOCALLY TREE-LIKE GRAPHS

BY ANIRBAN BASAK AND AMIR DEMBO[1]

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We consider the ferromagnetic Ising model on a sequence of graphs G_n converging locally weakly to a rooted random tree. Generalizing [*Probab. Theory Related Fields* **152** (2012) 31–51], under an appropriate "continuity" property, we show that the Ising measures on these graphs converge locally weakly to a measure, which is obtained by first picking a random tree, and then the symmetric mixture of Ising measures with $+$ and $-$ boundary conditions on that tree. Under the extra assumptions that G_n are edge-expanders, we show that the local weak limit of the Ising measures conditioned on positive magnetization is the Ising measure with $+$ boundary condition on the limiting tree. The "continuity" property holds except possibly for countable many choices of β, which for limiting trees of minimum degree at least three, are all within certain explicitly specified compact interval. We further show the edge-expander property for (most of) the configuration model graphs corresponding to limiting (multi-type) Galton–Watson trees.

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MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATIONS AND ASSOCIATED PDES

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In this paper, we consider a mean-field stochastic differential equation, also called the McKean–Vlasov equation, with initial data $(t,x)\in[0,T]\times\mathbb{R}^d$, whose coefficients depend on both the solution $X_s^{t,x}$ and its law. By considering square integrable random variables ξ as initial condition for this equation, we can easily show the flow property of the solution $X_s^{t,\xi}$ of this new equation. Associating it with a process X_s^{t,x,P_ξ} which coincides with $X_s^{t,\xi}$, when one substitutes ξ for x, but which has the advantage to depend on ξ only through its law P_ξ, we characterize the function $V(t,x,P_\xi)=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]$ under appropriate regularity conditions on the coefficients of the stochastic differential equation as the unique classical solution of a nonlocal partial differential equation of mean-field type, involving the first- and the second-order derivatives of V with respect to its space variable and the probability law. The proof bases heavily on a preliminary study of the first- and second-order derivatives of the solution of the mean-field stochastic differential equation with respect to the probability law and a corresponding Itô formula. In our approach, we use the notion of derivative with respect to a probability measure with finite second moment, introduced by Lions in [Cours au Collège de France: Théorie des jeu à champs moyens (2013)], and we extend it in a direct way to the second-order derivatives.

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A LOWER BOUND FOR DISCONNECTION BY SIMPLE RANDOM WALK

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We consider simple random walk on $\mathbb{Z}^d$, $d \geq 3$. Motivated by the work of A.-S. Sznitman and the author in [*Probab. Theory Related Fields* **161** (2015) 309–350] and [*Electron. J. Probab.* **19** (2014) 1–26], we investigate the asymptotic behavior of the probability that a large body gets disconnected from infinity by the set of points visited by a simple random walk. We derive asymptotic lower bounds that bring into play random interlacements. Although open at the moment, some of the lower bounds we obtain possibly match the asymptotic upper bounds recently obtained in [Disconnection, random walks, and random interlacements (2014)]. This potentially yields special significance to the tilted walks that we use in this work, and to the strategy that we employ to implement disconnection.

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RANDOM CURVES ON SURFACES INDUCED FROM THE LAPLACIAN DETERMINANT

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We define natural probability measures on finite multicurves (finite collections of pairwise disjoint simple closed curves) on curved surfaces. These measures arise as universal scaling limits of probability measures on cycle-rooted spanning forests (CRSFs) on graphs embedded on a surface with a Riemannian metric, in the limit as the mesh size tends to zero. These in turn are defined from the Laplacian determinant and depend on the choice of a unitary connection on the surface.

Wilson's algorithm for generating spanning trees on a graph generalizes to a cycle-popping algorithm for generating CRSFs for a general family of weights on the cycles. We use this to sample the above measures. The sampling algorithm, which relates these measures to the loop-erased random walk, is also used to prove tightness of the sequence of measures, a key step in the proof of their convergence.

We set the framework for the study of these probability measures and their scaling limits and state some of their properties.

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WHEN DOES A DISCRETE-TIME RANDOM WALK IN $\mathbb{R}^n$ ABSORB THE ORIGIN INTO ITS CONVEX HULL?

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We connect this question to a problem of estimating the probability that the image of certain random matrices does not intersect with a subset of the unit sphere $\mathbb{S}^{n-1}$. In this way, the case of a discretized Brownian motion is related to Gordon's escape theorem dealing with standard Gaussian matrices. We show that for the random walk $\mathrm{BM}_n(i)$, $i \in \mathbb{N}$, the convex hull of the first C^n steps (for a sufficiently large universal constant C) contains the origin with probability close to one. Moreover, the approach allows us to prove that with high probability the $\pi/2$-covering time of certain random walks on $\mathbb{S}^{n-1}$ is of order n. For certain spherical simplices on $\mathbb{S}^{n-1}$, we prove an extension of Gordon's theorem dealing with a broad class of random matrices; as an application, we show that C^n steps are sufficient for the standard walk on $\mathbb{Z}^n$ to absorb the origin into its convex hull with a high probability. Finally, we prove that the aforementioned bound is sharp in the following sense: for some universal constant $c > 1$, the convex hull of the n-dimensional Brownian motion $\mathrm{conv}\{\mathrm{BM}_n(t) : t \in [1, c^n]\}$ does not contain the origin with probability close to one.

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MIXING TIMES FOR A CONSTRAINED ISING PROCESS ON THE TORUS AT LOW DENSITY

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We study a kinetically constrained Ising process (KCIP) associated with a graph G and density parameter p; this process is an interacting particle system with state space $\{0,1\}^G$, the location of the particles. The number of particles at stationarity follows the Binomial$(|G|, p)$ distribution, conditioned on having at least one particle. The "constraint" in the name of the process refers to the rule that a vertex cannot change its state unless it has at least one neighbour in state "1". The KCIP has been proposed by statistical physicists as a model for the glass transition, and more recently as a simple algorithm for data storage in computer networks. In this note, we study the mixing time of this process on the torus $G = \mathbb{Z}_L^d$, $d \geq 3$, in the low-density regime $p = \frac{c}{|G|}$ for arbitrary $0 < c < \infty$; this regime is the subject of a conjecture of Aldous and is natural in the context of computer networks. Our results provide a counterexample to Aldous' conjecture, suggest a natural modification of the conjecture, and show that this modification is correct up to logarithmic factors. The methods developed in this paper also provide a strategy for tackling Aldous' conjecture for other graphs.

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DISCRETE MALLIAVIN–STEIN METHOD: BERRY–ESSEEN BOUNDS FOR RANDOM GRAPHS AND PERCOLATION[1]

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A new Berry–Esseen bound for nonlinear functionals of nonsymmetric and nonhomogeneous infinite Rademacher sequences is established. It is based on a discrete version of the Malliavin–Stein method and an analysis of the discrete Ornstein–Uhlenbeck semigroup. The result is applied to subgraph counts and to the number of vertices having a prescribed degree in the Erdős–Rényi random graph. A further application deals with a percolation problem on trees.

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CONVERGENCE AND REGULARITY OF PROBABILITY LAWS BY USING AN INTERPOLATION METHOD

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Fournier and Printems [*Bernoulli* **16** (2010) 343–360] have recently established a methodology which allows to prove the absolute continuity of the law of the solution of some stochastic equations with Hölder continuous coefficients. This is of course out of reach by using already classical probabilistic methods based on Malliavin calculus. By employing some Besov space techniques, Debussche and Romito [*Probab. Theory Related Fields* **158** (2014) 575–596] have substantially improved the result of Fournier and Printems. In our paper, we show that this kind of problem naturally fits in the framework of interpolation spaces: we prove an interpolation inequality (see Proposition 2.5) which allows to state (and even to slightly improve) the above absolute continuity result. Moreover, it turns out that the above interpolation inequality has applications in a completely different framework: we use it in order to estimate the error in total variance distance in some convergence theorems.

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CLIMBING DOWN GAUSSIAN PEAKS

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How likely is the high level of a continuous Gaussian random field on an Euclidean space to have a "hole" of a certain dimension and depth? Questions of this type are difficult, but in this paper we make progress on questions shedding new light in existence of holes. How likely is the field to be above a high level on one compact set (e.g., a sphere) and to be below a fraction of that level on some other compact set, for example, at the center of the corresponding ball? How likely is the field to be below that fraction of the level *anywhere* inside the ball? We work on the level of large deviations.

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EXTREMAL CUTS OF SPARSE RANDOM GRAPHS

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For Erdős–Rényi random graphs with average degree γ, and uniformly random γ-regular graph on n vertices, we prove that with high probability the size of both the Max-Cut and maximum bisection are $n(\frac{\gamma}{4} + \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o(\sqrt{\gamma})) + o(n)$ while the size of the minimum bisection is $n(\frac{\gamma}{4} - \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o(\sqrt{\gamma})) + o(n)$. Our derivation relates the free energy of the anti-ferromagnetic Ising model on such graphs to that of the Sherrington–Kirkpatrick model, with $\mathsf{P}_* \approx 0.7632$ standing for the ground state energy of the latter, expressed analytically via Parisi's formula.

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POLY-ADIC FILTRATIONS, STANDARDNESS, COMPLEMENTABILITY AND MAXIMALITY

BY CHRISTOPHE LEURIDAN

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Given some essentially separable filtration $(\mathcal{Z}_n)_{n\leq 0}$ indexed by the nonpositive integers, we define the notion of complementability for the filtrations contained in $(\mathcal{Z}_n)_{n\leq 0}$. We also define and characterize the notion of maximality for the poly-adic sub-filtrations of $(\mathcal{Z}_n)_{n\leq 0}$. We show that any poly-adic sub-filtration of $(\mathcal{Z}_n)_{n\leq 0}$ which can be complemented by a Kolmogorovian filtration is maximal in $(\mathcal{Z}_n)_{n\leq 0}$. We show that the converse is false, and we prove a partial converse, which generalizes Vershik's lacunary isomorphism theorem for poly-adic filtrations.

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Key words and phrases. Filtrations indexed by the nonpositive integers, product-type filtrations, standard filtrations, complementability, maximality, exchange property.

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LOCALITY OF PERCOLATION FOR ABELIAN CAYLEY GRAPHS[1]

BY SÉBASTIEN MARTINEAU AND VINCENT TASSION

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We prove that the value of the critical probability for percolation on an Abelian Cayley graph is determined by its local structure. This is a partial positive answer to a conjecture of Schramm: the function p_c defined on the set of Cayley graphs of Abelian groups of rank at least 2 is continuous for the Benjamini–Schramm topology. The proof involves group-theoretic tools and a new block argument.

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BEHAVIOR OF THE GENERALIZED ROSENBLATT PROCESS AT EXTREME CRITICAL EXPONENT VALUES[1]

BY SHUYANG BAI AND MURAD S. TAQQU

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The generalized Rosenblatt process is obtained by replacing the single critical exponent characterizing the Rosenblatt process by two different exponents living in the interior of a triangular region. What happens to that generalized Rosenblatt process as these critical exponents approach the boundaries of the triangle? We show by two different methods that on each of the two symmetric boundaries, the limit is non-Gaussian. On the third boundary, the limit is Brownian motion. The rates of convergence to these boundaries are also given. The situation is particularly delicate as one approaches the corners of the triangle, because the limit process will depend on how these corners are approached. All limits are in the sense of weak convergence in $C[0, 1]$. These limits cannot be strengthened to convergence in $L^2(\Omega)$.

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SYNCHRONIZATION BY NOISE FOR ORDER-PRESERVING RANDOM DYNAMICAL SYSTEMS

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We provide sufficient conditions for weak synchronization/stabilization by noise for order-preserving random dynamical systems on Polish spaces. That is, under these conditions we prove the existence of a weak point attractor consisting of a single random point. This generalizes previous results in two directions: First, we do not restrict to Banach spaces, and second, we do not require the partial order to be admissible nor normal. As a second main result and application, we prove weak synchronization by noise for stochastic porous media equations with additive noise.

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