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HIGH-DIMENSIONAL LIPSCHITZ FUNCTIONS ARE TYPICALLY FLAT[1]

BY RON PELED

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A *homomorphism height function* on the d-dimensional torus $\mathbb{Z}_n^d$ is a function on the vertices of the torus taking integer values and constrained to have adjacent vertices take adjacent integer values. A *Lipschitz height function* is defined similarly but may also take equal values on adjacent vertices. For each of these models, we consider the uniform distribution over all such functions with predetermined values at some fixed vertices (boundary conditions). Our main result is that in high dimensions and with zero boundary values, the random function obtained is typically very flat, having bounded variance at any fixed vertex and taking at most $C(\log n)^{1/d}$ values with high probability. This result matches, up to constants, a lower bound of Benjamini, Yadin and Yehudayoff. Our results extend to any dimension $d \geq 2$; if one replaces the torus $\mathbb{Z}_n^d$ by an enhanced version of it, the torus $\mathbb{Z}_n^d \times \mathbb{Z}_2^{d_0}$ for some fixed d_0. Consequently, we establish one side of a conjectured roughening transition in two dimensions. The full transition is established for a class of tori with nonequal side lengths, including, for example, the $n \times \lfloor \frac{1}{10} \log n \rfloor$ torus. In another case of interest, we find that when the dimension d is taken to infinity while n remains fixed, the random function takes at most r values with high probability, where $r = 5$ for the homomorphism model and $r = 4$ for the Lipschitz model. Suitable generalizations are obtained when n grows with d. Our results have consequences also for the related model of uniform 3-coloring and establish that for certain boundary conditions, a uniformly sampled proper 3-coloring of $\mathbb{Z}_n^d$ will be nearly constant on either the even or odd sublattice.

Our proofs are based on the construction of a combinatorial transformation suitable to the homomorphism model and on a careful analysis of the properties of a class of cutsets which we term *odd cutsets*. For the Lipschitz model, our results rely also on a bijection of Yadin. This work generalizes results of Galvin and Kahn, refutes a conjecture of Benjamini, Yadin and Yehudayoff and answers a question of Benjamini, Häggström and Mossel.

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CHARACTERIZATION OF CUTOFF FOR REVERSIBLE MARKOV CHAINS[1]

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A sequence of Markov chains is said to exhibit (total variation) cutoff if the convergence to stationarity in total variation distance is abrupt. We consider reversible lazy chains. We prove a necessary and sufficient condition for the occurrence of the cutoff phenomena in terms of concentration of hitting time of "worst" (in some sense) sets of stationary measure at least α, for some $\alpha \in (0, 1)$.

We also give general bounds on the total variation distance of a reversible chain at time t in terms of the probability that some "worst" set of stationary measure at least α was not hit by time t. As an application of our techniques, we show that a sequence of lazy Markov chains on finite trees exhibits a cutoff iff the product of their spectral gaps and their (lazy) mixing-times tends to ∞.

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MSC2010 subject classifications. 60J10.
Key words and phrases. Cutoff, mixing-time, finite reversible Markov chains, hitting times, trees, maximal inequality.

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REGULARITY OF WIENER FUNCTIONALS UNDER A HÖRMANDER TYPE CONDITION OF ORDER ONE

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We study the local existence and regularity of the density of the law of a functional on the Wiener space which satisfies a criterion that generalizes the Hörmander condition of order one (i.e., involving the first-order Lie brackets) for diffusion processes.

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Key words and phrases. Malliavin calculus, local integration by parts formulas, total variation distance, variance of the Brownian path.

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A LIMIT THEOREM FOR MOMENTS IN SPACE OF THE INCREMENTS OF BROWNIAN LOCAL TIME

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We prove a limit theorem for moments in space of the increments of Brownian local time. As special cases for the second and third moments, previous results by Chen et al. [*Ann. Prob.* **38** (2010) 396–438] and Rosen [*Stoch. Dyn.* **11** (2011) 5–48], which were later reproven by Hu and Nualart [*Electron. Commun. Probab.* **15** (2010) 396–410] and Rosen [In *Séminaire de Probabilités XLIII* (2011) 95–104 Springer] are included. Furthermore, a conjecture of Rosen for the fourth moment is settled. In comparison to the previous methods of proof, we follow a fundamentally different approach by exclusively working in the space variable of the Brownian local time, which allows to give a unified argument for arbitrary orders. The main ingredients are Perkins' semimartingale decomposition, the Kailath–Segall identity and an asymptotic Ray–Knight theorem by Pitman and Yor.

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MSC2010 subject classifications. 60F05, 60G44, 60H05.
Key words and phrases. Kailath–Segall identity, Brownian local time, central limit theorem, asymptotic Ray–Knight theorem.

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LOCAL ALGORITHMS FOR INDEPENDENT SETS ARE HALF-OPTIMAL

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We show that the largest density of factor of i.i.d. independent sets in the d-regular tree is asymptotically at most $(\log d)/d$ as $d \to \infty$. This matches the lower bound given by previous constructions. It follows that the largest independent sets given by local algorithms on random d-regular graphs have the same asymptotic density. In contrast, the density of the largest independent sets in these graphs is asymptotically $2(\log d)/d$. We prove analogous results for Poisson–Galton–Watson trees, which yield bounds for local algorithms on sparse Erdős–Rényi graphs.

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Key words and phrases. Independent set, local algorithm, factor of i.i.d., random graphs.

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INVARIANCE PRINCIPLES UNDER THE MAXWELL–WOODROOFE CONDITION IN BANACH SPACES

BY CHRISTOPHE CUNY

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We prove that, for (adapted) stationary processes, the so-called Maxwell–Woodroofe condition is sufficient for the law of the iterated logarithm and that it is optimal in some sense. That result actually holds in the context of Banach valued stationary processes, including the case of L^p-valued random variables, with $1 \leq p < \infty$. In this setting, we also prove the weak invariance principle, hence generalizing a result of Peligrad and Utev [*Ann. Probab.* **33** (2005) 798–815]. The proofs make use of a new maximal inequality and of approximation by martingales, for which some of our results are also new.

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A STRUCTURE THEOREM FOR POORLY ANTICONCENTRATED POLYNOMIALS OF GAUSSIANS AND APPLICATIONS TO THE STUDY OF POLYNOMIAL THRESHOLD FUNCTIONS[1]

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We prove a structural result for degree-d polynomials. In particular, we show that any degree-d polynomial, p can be approximated by another polynomial, p_0, which can be decomposed as some function of polynomials $q_1, \dots, q_m$ with q_i normalized and $m = O_d(1)$, so that if X is a Gaussian random variable, the probability distribution on $(q_1(X), \dots, q_m(X))$ does not have too much mass in any small box.

Using this result, we prove improved versions of a number of results about polynomial threshold functions, including producing better pseudorandom generators, obtaining a better invariance principle, and proving improved bounds on noise sensitivity.

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ROBUSTNESS OF SCALE-FREE SPATIAL NETWORKS

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A growing family of random graphs is called robust if it retains a giant component after percolation with arbitrary positive retention probability. We study robustness for graphs, in which new vertices are given a spatial position on the d-dimensional torus and are connected to existing vertices with a probability favouring short spatial distances and high degrees. In this model of a scale-free network with clustering, we can independently tune the power law exponent τ of the degree distribution and the rate $-\delta d$ at which the connection probability decreases with the distance of two vertices. We show that the network is robust if $\tau < 2 + \frac{1}{\delta}$, but fails to be robust if $\tau > 3$. In the case of one-dimensional space, we also show that the network is not robust if $\tau > 2 + \frac{1}{\delta - 1}$. This implies that robustness of a scale-free network depends not only on its power-law exponent but also on its clustering features. Other than the classical models of scale-free networks, our model is not locally tree-like, and hence we need to develop novel methods for its study, including, for example, a surprising application of the BK-inequality.

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POINT-MAP-PROBABILITIES OF A POINT PROCESS AND MECKE'S INVARIANT MEASURE EQUATION[1]

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A compatible point-shift F maps, in a translation invariant way, each point of a stationary point process Φ to some point of Φ. It is fully determined by its associated point-map, f, which gives the image of the origin by F. It was proved by J. Mecke that if F is bijective, then the Palm probability of Φ is left invariant by the translation of $-f$. The initial question motivating this paper is the following generalization of this invariance result: in the nonbijective case, what probability measures on the set of counting measures are left invariant by the translation of $-f$? The point-map-probabilities of Φ are defined from the action of the semigroup of point-map translations on the space of Palm probabilities, and more precisely from the compactification of the orbits of this semigroup action. If the point-map-probability exists, is uniquely defined and if it satisfies certain continuity properties, it then provides a solution to this invariant measure problem. Point-map-probabilities are objects of independent interest. They are shown to be a strict generalization of Palm probabilities: when F is bijective, the point-map-probability of Φ boils down to the Palm probability of Φ. When it is not bijective, there exist cases where the point-map-probability of Φ is singular with respect to its Palm probability. A tightness based criterion for the existence of the point-map-probabilities of a stationary point process is given. An interpretation of the point-map-probability as the conditional law of the point process given that the origin has F-pre-images of all orders is also provided. The results are illustrated by a few examples.

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CUTOFF FOR NONBACKTRACKING RANDOM WALKS ON SPARSE RANDOM GRAPHS

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A finite ergodic Markov chain exhibits *cutoff* if its distance to stationarity remains close to 1 over a certain number of iterations and then abruptly drops to near 0 on a much shorter time scale. Discovered in the context of card shuffling (Aldous–Diaconis, 1986), this phenomenon is now believed to be rather typical among fast mixing Markov chains. Yet, establishing it rigorously often requires a challengingly detailed understanding of the underlying chain. Here, we consider nonbacktracking random walks on random graphs with a given degree sequence. Under a general sparsity condition, we establish the cutoff phenomenon, determine its precise window and prove that the cutoff profile approaches a remarkably simple, universal shape.

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KPZ EQUATION LIMIT OF HIGHER-SPIN EXCLUSION PROCESSES

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We prove that under a particular *weak scaling*, the 4-parameter interacting particle system introduced by Corwin and Petrov [*Comm. Math. Phys.* **343** (2016) 651–700] converges to the Kardar–Parisi–Zhang (KPZ) equation. This expands the relatively small number of systems for which *weak universality* of the KPZ equation has been demonstrated.

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Key words and phrases. Exclusion processes, Hopf–Cole transform, higher-spin, Kardar–Parisi–Zhang equation, stochastic heat equation.

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FUNDAMENTAL SOLUTIONS OF NONLOCAL HÖRMANDER'S OPERATORS II[1]

BY XICHENG ZHANG

Dedicated to the memory of Professor Paul Malliavin

Wuhan University

Consider the following nonlocal integro-differential operator: for $\alpha \in (0,2)$:

$$\mathcal{L}^{(\alpha)}_{\sigma,b} f(x) := \text{p.v.} \int_{|z|<\delta} \frac{f(x+\sigma(x)z) - f(x)}{|z|^{d+\alpha}} \mathrm{d}z + b(x) \cdot \nabla f(x) + \mathscr{L} f(x),$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions and have bounded partial derivatives of all orders greater than 1, δ is a small positive number, p.v. stands for the Cauchy principal value and $\mathscr{L}$ is a bounded linear operator in Sobolev spaces. Let $B_1(x) := \sigma(x)$ and $B_{j+1}(x) := b(x) \cdot \nabla B_j(x) - \nabla b(x) \cdot B_j(x)$ for $j \in \mathbb{N}$. Suppose $B_j \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ for each $j \in \mathbb{N}$. Under the following uniform Hörmander's type condition: for some $j_0 \in \mathbb{N}$,

$$\inf_{x \in \mathbb{R}^d} \inf_{|u|=1} \sum_{j=1}^{j_0} |u B_j(x)|^2 > 0,$$

by using Bismut's approach to the Malliavin calculus with jumps, we prove the existence of fundamental solutions to operator $\mathcal{L}^{(\alpha)}_{\sigma,b}$. In particular, we answer a question proposed by Nualart [*Sankhyā A* **73** (2011) 46–49] and Varadhan [*Sankhyā A* **73** (2011) 50–51].

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RANDOM WALKS ON INFINITE PERCOLATION CLUSTERS IN MODELS WITH LONG-RANGE CORRELATIONS

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For a general class of percolation models with long-range correlations on $\mathbb{Z}^d$, $d \geq 2$, introduced in [*J. Math. Phys.* **55** (2014) 083307], we establish regularity conditions of Barlow [*Ann. Probab.* **32** (2004) 3024–3084] that mesoscopic subballs of all large enough balls in the unique infinite percolation cluster have regular volume growth and satisfy a weak Poincaré inequality. As immediate corollaries, we deduce quenched heat kernel bounds, parabolic Harnack inequality, and finiteness of the dimension of harmonic functions with at most polynomial growth. Heat kernel bounds and the quenched invariance principle of [*Probab. Theory Related Fields* **166** (2016) 619–657] allow to extend various other known results about Bernoulli percolation by mimicking their proofs, for instance, the local central limit theorem of [*Electron. J. Probab.* **14** (209) 1–27] or the result of [*Ann. Probab.* **43** (2015) 2332–2373] that the dimension of at most linear harmonic functions on the infinite cluster is $d + 1$.

In terms of specific models, all these results are new for random interlacements at every level in any dimension $d \geq 3$, as well as for the vacant set of random interlacements [*Ann. of Math. (2)* **171** (2010) 2039–2087; *Comm. Pure Appl. Math.* **62** (2009) 831–858] and the level sets of the Gaussian free field [*Comm. Math. Phys.* **320** (2013) 571–601] in the regime of the so-called local uniqueness (which is believed to coincide with the whole supercritical regime for these models).

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PERCOLATION ON THE STATIONARY DISTRIBUTIONS OF THE VOTER MODEL

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The voter model on $\mathbb{Z}^d$ is a particle system that serves as a rough model for changes of opinions among social agents or, alternatively, competition between biological species occupying space. When $d \geq 3$, the set of (extremal) stationary distributions is a family of measures μ_α, for α between 0 and 1. A configuration sampled from μ_α is a strongly correlated field of 0's and 1's on $\mathbb{Z}^d$ in which the density of 1's is α. We consider such a configuration as a site percolation model on $\mathbb{Z}^d$. We prove that if $d \geq 5$, the probability of existence of an infinite percolation cluster of 1's exhibits a phase transition in α. If the voter model is allowed to have sufficiently spread-out interactions, we prove the same result for $d \geq 3$.

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REAL SELF-SIMILAR PROCESSES STARTED FROM THE ORIGIN

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Since the seminal work of Lamperti, there is a lot of interest in the understanding of the general structure of self-similar Markov processes. Lamperti gave a representation of positive self-similar Markov processes with initial condition strictly larger than 0 which subsequently was extended to zero initial condition.

For real self-similar Markov processes (rssMps), there is a generalization of Lamperti's representation giving a one-to-one correspondence between Markov additive processes and rssMps with initial condition different from the origin.

We develop fluctuation theory for Markov additive processes and use Kuznetsov measures to construct the law of transient real self-similar Markov processes issued from the origin. The construction gives a pathwise representation through two-sided Markov additive processes extending the Lamperti–Kiu representation to the origin.

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LIMIT THEOREMS FOR POINT PROCESSES UNDER GEOMETRIC CONSTRAINTS (AND TOPOLOGICAL CRACKLE)

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We study the asymptotic nature of geometric structures formed from a point cloud of observations of (generally heavy tailed) distributions in a Euclidean space of dimension greater than one. A typical example is given by the Betti numbers of Čech complexes built over the cloud. The structure of dependence and sparcity (away from the origin) generated by these distributions leads to limit laws expressible via nonhomogeneous, random, Poisson measures. The parametrisation of the limits depends on both the tail decay rate of the observations and the particular geometric constraint being considered.

The main theorems of the paper generate a new class of results in the well established theory of extreme values, while their applications are of significance for the fledgling area of rigorous results in topological data analysis. In particular, they provide a broad theory for the empirically well-known phenomenon of homological "crackle;" the continued presence of spurious homology in samples of topological structures, despite increased sample size.

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ERRATUM TO "SCALING FOR A ONE-DIMENSIONAL DIRECTED POLYMER WITH BOUNDARY CONDITIONS"

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