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CONDITIONS FOR PERMANENTAL PROCESSES TO BE UNBOUNDED[1]

BY MICHAEL B. MARCUS AND JAY ROSEN

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An α-permanental process $\{X_t, t \in T\}$ is a stochastic process determined by a kernel $K = \{K(s,t), s,t \in T\}$, with the property that for all $t_1, \dots, t_n \in T$, $|I + K(t_1, \dots, t_n)S|^{-\alpha}$ is the Laplace transform of $(X_{t_1}, \dots, X_{t_n})$, where $K(t_1, \dots, t_n)$ denotes the matrix $\{K(t_i, t_j)\}_{i,j=1}^n$ and S is the diagonal matrix with entries $s_1, \dots, s_n$. $(X_{t_1}, \dots, X_{t_n})$ is called a permanental vector.

Under the condition that K is the potential density of a transient Markov process, $(X_{t_1}, \dots, X_{t_n})$ is represented as a random mixture of n-dimensional random variables with components that are independent gamma random variables. This representation leads to a Sudakov-type inequality for the sup-norm of $(X_{t_1}, \dots, X_{t_n})$ that is used to obtain sufficient conditions for a large class of permanental processes to be unbounded almost surely. These results are used to obtain conditions for permanental processes associated with certain Lévy processes to be unbounded.

Because K is the potential density of a transient Markov process, for all $t_1, \dots, t_n \in T$, $A(t_1, \dots, t_n) := (K(t_1, \dots, t_n))^{-1}$ are M-matrices. The results in this paper are obtained by working with these M-matrices.

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MSC2010 subject classifications. 60K99, 60J55, 60G15, 60G17.
Key words and phrases. Permanental processes, M-matrices.

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FUNCTIONAL CENTRAL LIMIT THEOREM FOR A CLASS OF NEGATIVELY DEPENDENT HEAVY-TAILED STATIONARY INFINITELY DIVISIBLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

BY PAUL JUNG[1], TAKASHI OWADA[2] AND GENNADY SAMORODNITSKY[3]

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We prove a functional central limit theorem for partial sums of symmetric stationary long-range dependent heavy tailed infinitely divisible processes. The limiting stable process is particularly interesting due to its long memory which is quantified by a Mittag–Leffler process induced by an associated Harris chain, at the discrete-time level. Previous results in Owada and Samorodnitsky [*Ann. Probab.* **43** (2015) 240–285] dealt with positive dependence in the increment process, whereas this paper derives the functional limit theorems under negative dependence. The negative dependence is due to cancellations arising from Gaussian-type fluctuations of functionals of the associated Harris chain. The new types of limiting processes involve stable random measures, due to heavy tails, Mittag–Leffler processes, due to long memory, and Brownian motions, due to the Gaussian second order cancellations. Along the way, we prove a function central limit theorem for fluctuations of functionals of Harris chains which is of independent interest as it extends a result of Chen [*Probab. Theory Related Fields* **116** (2000) 89–123].

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MSC2010 subject classifications. Primary 60F17, 60G18; secondary 37A40, 60G52.
Key words and phrases. Infinitely divisible process, conservative flow, Harris recurrent Markov chain, functional central limit theorem, self-similar process, pointwise dual ergodicity, Darling–Kac theorem, fractional stable motion.

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MOMENT BOUNDS FOR A CLASS OF FRACTIONAL STOCHASTIC HEAT EQUATIONS[1]

BY MOHAMMUD FOONDUN, WEI LIU AND MCSYLVESTER OMABA

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We consider fractional stochastic heat equations of the form $\frac{\partial u_t(x)}{\partial t} = -(-\Delta)^{\alpha/2} u_t(x) + \lambda\sigma(u_t(x))\dot{F}(t,x)$. Here, $\dot{F}$ denotes the noise term. Under suitable assumptions, we show that the second moment of the solution grows exponentially with time. Since we do not assume that the initial condition is bounded below, this solves an open problem stated in [*Probab. Theory Related Fields* **152** (2012) 681–701]. Along the way, we prove a number of other interesting results about continuity properties and noise excitation indices. These extend and complement results in [*Stochastic Process. Appl.* **124** (2014) 3429–3440], [Khoshnevisan and Kim (2013)] and [Khoshnevisan and Kim (2014)].

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Key words and phrases. Stochastic partial differential equations, intermittence.

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UNIVERSALITY FOR THE PINNING MODEL IN THE WEAK COUPLING REGIME

BY FRANCESCO CARAVENNA[*,1], FABIO LUCIO TONINELLI[†,‡,2] AND NICCOLÒ TORRI[†,3]

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We consider disordered pinning models, when the return time distribution of the underlying renewal process has a polynomial tail with exponent $\alpha \in (\frac{1}{2}, 1)$. This corresponds to a regime where disorder is known to be *relevant*, that is, to change the critical exponent of the localization transition and to induce a nontrivial shift of the critical point. We show that the free energy and critical curve have an explicit universal asymptotic behavior in the weak coupling regime, depending only on the tail of the return time distribution and not on finer details of the models. This is obtained comparing the partition functions with corresponding continuum quantities, through coarse-graining techniques.

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MSC2010 subject classifications. Primary 82B44; secondary 82D60, 60K35.

Key words and phrases. Scaling limit, disorder relevance, weak disorder, pinning model, random polymer, universality, free energy, critical curve, coarse-graining.

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AN ITERATED AZÉMA–YOR TYPE EMBEDDING FOR FINITELY MANY MARGINALS

BY JAN OBŁÓJ[1] AND PETER SPOIDA[2]

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We solve the n-marginal Skorokhod embedding problem for a continuous local martingale and a sequence of probability measures $\mu_1, \dots, \mu_n$ which are in convex order and satisfy an additional technical assumption. Our construction is explicit and is a multiple marginal generalization of the Azéma and Yor [In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)* (1979) 90–115 Springer] solution. In particular, we recover the stopping boundaries obtained by Brown, Hobson and Rogers [*Probab. Theory Related Fields* **119** (2001) 558–578] and Madan and Yor [*Bernoulli* **8** (2002) 509–536]. Our technical assumption is necessary for the explicit embedding, as demonstrated with a counterexample. We discuss extensions to the general case giving details when $n = 3$.

In our analysis we compute the law of the maximum at each of the n stopping times. This is used in Henry-Labordère et al. [*Ann. Appl. Probab.* **26** (2016) 1–44] to show that the construction maximizes the distribution of the maximum among all solutions to the n-marginal Skorokhod embedding problem. The result has direct implications for robust pricing and hedging of Lookback options.

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CUT-OFF PHENOMENON IN THE UNIFORM PLANE KAC WALK

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We consider an analogue of the Kac random walk on the special orthogonal group SO(N), in which at each step a random rotation is performed in a randomly chosen 2-plane of $\mathbb{R}^N$. We obtain sharp asymptotics for the rate of convergence in total variance distance, establishing a cut-off phenomenon in the large N limit. In the special case where the angle of rotation is deterministic, this confirms a conjecture of Rosenthal [*Ann. Probab.* **22** (1994) 398–423]. Under mild conditions, we also establish a cut-off for convergence of the walk to stationarity under the L^2 norm. Depending on the distribution of the randomly chosen angle of rotation, several surprising features emerge. For instance, it is sometimes the case that the mixing times differ in the total variation and L^2 norms. Our estimates use an integral representation of the characters of the special orthogonal group together with saddle point analysis.

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CENTRAL LIMIT THEOREMS AND BOOTSTRAP IN HIGH DIMENSIONS

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This paper derives central limit and bootstrap theorems for probabilities that sums of centered high-dimensional random vectors hit hyperrectangles and sparsely convex sets. Specifically, we derive Gaussian and bootstrap approximations for probabilities $\mathrm{P}(n^{-1/2}\sum_{i=1}^{n} X_i \in A)$ where $X_1, \dots, X_n$ are independent random vectors in $\mathbb{R}^p$ and A is a hyperrectangle, or more generally, a sparsely convex set, and show that the approximation error converges to zero even if $p = p_n \to \infty$ as $n \to \infty$ and $p \gg n$; in particular, p can be as large as $O(e^{Cn^c})$ for some constants $c, C > 0$. The result holds uniformly over all hyperrectangles, or more generally, sparsely convex sets, and does not require any restriction on the correlation structure among coordinates of X_i. Sparsely convex sets are sets that can be represented as intersections of many convex sets whose indicator functions depend only on a small subset of their arguments, with hyperrectangles being a special case.

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LIMITS OF LOCAL ALGORITHMS OVER SPARSE RANDOM GRAPHS[1]

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Local algorithms on graphs are algorithms that run in parallel on the nodes of a graph to compute some global structural feature of the graph. Such algorithms use only local information available at nodes to determine local aspects of the global structure, while also potentially using some randomness. Recent research has shown that such algorithms show significant promise in computing structures like large independent sets in graphs locally. Indeed the promise led to a conjecture by Hatami, Lovász and Szegedy [*Geom. Funct. Anal.* **24** (2014) 269–296] that local algorithms defined specifically as so-called i.i.d. factors may be able to find approximately largest independent sets in random d-regular graphs. In this paper, we refute this conjecture and show that every independent set produced by local algorithms is multiplicative factor $1/2+1/(2\sqrt{2})$ smaller than the largest, asymptotically as $d \to \infty$.

Our result is based on an important clustering phenomena predicted first in the literature on spin glasses, and recently proved rigorously for a variety of constraint satisfaction problems on random graphs. Such properties suggest that the geometry of the solution space can be quite intricate. The specific clustering property that we prove and apply in this paper shows that typically every two large independent sets in a random graph either have a significant intersection, or have a very small intersection. As a result, large independent sets are clustered according to the proximity to each other. While the clustering property was postulated earlier as an obstruction for the success of local algorithms, our result is the first one where the clustering property is used to formally prove limits on local algorithms.

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Key words and phrases. Random graphs, algorithms.

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QUADRATIC BSDE WITH $\mathbb{L}^2$-TERMINAL DATA: KRYLOV'S ESTIMATE, ITÔ–KRYLOV'S FORMULA AND EXISTENCE RESULTS[1]

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We establish a Krylov-type estimate and an Itô–Krylov change of variable formula for the solutions of one-dimensional quadratic backward stochastic differential equations (QBSDEs) with a measurable generator and an arbitrary terminal datum. This allows us to prove various existence and uniqueness results for some classes of QBSDEs with a square integrable terminal condition and sometimes a merely measurable generator. It turns out that neither the existence of exponential moments of the terminal datum nor the continuity of the generator are necessary to the existence and/or uniqueness of solutions. We also establish a comparison theorem for solutions of a particular class of QBSDEs with measurable generator. As a byproduct, we obtain the existence of viscosity solutions for a particular class of quadratic partial differential equations (QPDEs) with a square integrable terminal datum.

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GLOBAL WELL-POSEDNESS OF THE DYNAMIC Φ^4 MODEL IN THE PLANE

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We show global well-posedness of the dynamic Φ^4 model in the plane. The model is a nonlinear stochastic PDE that can only be interpreted in a "renormalised" sense. Solutions take values in suitable weighted Besov spaces of negative regularity.

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TAIL ESTIMATES FOR MARKOVIAN ROUGH PATHS

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The accumulated local p-variation functional [*Ann. Probab.* **41** (213) 3026–3050] arises naturally in the theory of rough paths in estimates both for solutions to rough differential equations (RDEs), and for the higher-order terms of the signature (or Lyons lift). In stochastic examples, it has been observed that the tails of the accumulated local p-variation functional typically decay much faster than the tails of classical p-variation. This observation has been decisive, for example, for problems involving Malliavin calculus for Gaussian rough paths [*Ann. Probab.* **43** (2015) 188–239].

All of the examples treated so far have been in this Gaussian setting that contains a great deal of additional structure. In this paper, we work in the context of Markov processes on a locally compact Polish space E, which are associated to a class of Dirichlet forms. In this general framework, we first prove a better-than-exponential tail estimate for the accumulated local p-variation functional derived from the intrinsic metric of this Dirichlet form. By then specialising to a class of Dirichlet forms on the step $\lfloor p \rfloor$ free nilpotent group, which are sub-elliptic in the sense of Fefferman–Phong, we derive a better than exponential tail estimate for a class of Markovian rough paths. This class includes the examples studied in [*Probab. Theory Related Fields* **142** (2008) 475–523]. We comment on the significance of these estimates to recent papers, including the results of Ni Hao [Personal communication (2014)] and Chevyrev and Lyons [*Ann. Probab.* To appear].

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RELATIVE COMPLEXITY OF RANDOM WALKS IN RANDOM SCENERY IN THE ABSENCE OF A WEAK INVARIANCE PRINCIPLE FOR THE LOCAL TIMES

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We answer a question of Aaronson about the relative complexity of Random Walks in Random Sceneries driven by either aperiodic two-dimensional random walks, two-dimensional Simple Random walk, or by aperiodic random walks in the domain of attraction of the Cauchy distribution. A key step is proving that the range of the random walk satisfies the Fölner property almost surely.

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EINSTEIN RELATION AND STEADY STATES FOR THE RANDOM CONDUCTANCE MODEL[1]

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We consider random walk among i.i.d., uniformly elliptic conductances on $\mathbb{Z}^d$, and prove the Einstein relation (see Theorem 1). It says that the derivative of the velocity of a biased walk as a function of the bias equals the diffusivity in equilibrium. For fixed bias, we show that there is an invariant measure for the environment seen from the particle. These invariant measures are often called steady states. The Einstein relation follows at least for $d \geq 3$, from an expansion of the steady states as a function of the bias (see Theorem 2), which can be considered our main result. This expansion is proved for $d \geq 3$. In contrast to Guo [*Ann. Probab.* **44** (2016) 324–359], we need not only convergence of the steady states, but an estimate on the rate of convergence (see Theorem 4).

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UNIVERSALITY FOR FIRST PASSAGE PERCOLATION ON SPARSE RANDOM GRAPHS

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We consider first passage percolation on the configuration model with n vertices, and general independent and identically distributed edge weights assumed to have a density. Assuming that the degree distribution satisfies a uniform $X^2 \log X$-condition, we analyze the asymptotic distribution for the minimal weight path between a pair of typical vertices, as well the number of edges on this path namely the hopcount.

Writing L_n for the weight of the optimal path, we show that $L_n - (\log n)/\alpha_n$ converges to a limiting random variable, for some sequence α_n. Furthermore, the hopcount satisfies a central limit theorem (CLT) with asymptotic mean and variance of order $\log n$. The sequence α_n and the norming constants for the CLT are expressible in terms of the parameters of an associated continuous-time branching process that describes the growth of neighborhoods around a uniformly chosen vertex in the random graph. The limit of $L_n - (\log n)/\alpha_n$ equals the sum of the logarithm of the product of two independent martingale limits, and a Gumbel random variable. So far, for sparse random graph models, such results have only been shown for the special case where the edge weights have an exponential distribution, wherein the Markov property of this distribution plays a crucial role in the technical analysis of the problem.

The proofs in the paper rely on a refined coupling between shortest path trees and continuous-time branching processes, and on a Poisson point process limit for the potential closing edges of shortest-weight paths between the source and destination.

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ON STRUCTURE OF REGULAR DIRICHLET SUBSPACES FOR ONE-DIMENSIONAL BROWNIAN MOTION

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The main purpose of this paper is to explore the structure of regular Dirichlet subspaces of one-dimensional Brownian motion. As stated in [*Osaka J. Math.* **42** (2005) 27–41], every such regular Dirichlet subspace can be characterized by a measure-dense set G. When G is open, $F = G^c$ is the boundary of G and, before leaving G, the diffusion associated with the regular Dirichlet subspace is nothing but Brownian motion. Their traces on F still inherit the inclusion relation, in other words, the trace Dirichlet form of regular Dirichlet subspace on F is still a regular Dirichlet subspace of trace Dirichlet form of one-dimensional Brownian motion on F. Moreover, we shall prove that the trace of Brownian motion on F may be decomposed into two parts; one is the trace of the regular Dirichlet subspace on F, which has only the nonlocal part and the other comes from the orthogonal complement of the regular Dirichlet subspace, which has only the local part. Actually the orthogonal complement of regular Dirichlet subspace corresponds to a time-changed absorbing Brownian motion after a darning transform.

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Key words and phrases. Regular Dirichlet subspaces, trace Dirichlet forms, time-changed Brownian motions.

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EDGE- AND VERTEX-REINFORCED RANDOM WALKS WITH SUPER-LINEAR REINFORCEMENT ON INFINITE GRAPHS

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In this paper, we introduce a new simple but powerful general technique for the study of edge- and vertex-reinforced processes with super-linear reinforcement, based on the use of order statistics for the number of edge, respectively of vertex, traversals. The technique relies on upper bound estimates for the number of edge traversals, proved in a different context by Cotar and Limic [*Ann. Appl. Probab.* **19** (2009) 1972–2007] for finite graphs with edge reinforcement. We apply our new method both to edge- and to vertex-reinforced random walks with super-linear reinforcement on arbitrary infinite connected graphs of bounded degree. We stress that, unlike all previous results for processes with super-linear reinforcement, we make no other assumption on the graphs.

For edge-reinforced random walks, we complete the results of Limic and Tarrès [*Ann. Probab.* **35** (2007) 1783–1806] and we settle a conjecture of Sellke (1994) by showing that for any reciprocally summable reinforcement weight function w, the walk traverses a random attracting edge at all large times.

For vertex-reinforced random walks, we extend results previously obtained on $\mathbb{Z}$ by Volkov [*Ann. Probab.* **29** (2001) 66–91] and by Basdevant, Schapira and Singh [*Ann. Probab.* **42** (2014) 527–558], and on complete graphs by Benaim, Raimond and Schapira [*ALEA Lat. Am. J. Probab. Math. Stat.* **10** (2013) 767–782]. We show that on any infinite connected graph of bounded degree, with reinforcement weight function w taken from a general class of reciprocally summable reinforcement weight functions, the walk traverses two random neighbouring attracting vertices at all large times.

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GENERAL ROUGH INTEGRATION, LÉVY ROUGH PATHS AND A LÉVY–KINTCHINE-TYPE FORMULA

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We consider rough paths with jumps. In particular, the analogue of Lyons' extension theorem and rough integration are established in a jump setting, offering a pathwise view on stochastic integration against càdlàg processes. A class of Lévy rough paths is introduced and characterized by a sub-ellipticity condition on the left-invariant diffusion vector fields and a certain integrability property of the Carnot–Caratheodory norm with respect to the Lévy measure on the group, using Hunt's framework of Lie group valued Lévy processes. Examples of Lévy rough paths include a standard multidimensional Lévy process enhanced with a stochastic area as constructed by D. Williams, the pure area Poisson process and Brownian motion in a magnetic field. An explicit formula for the expected signature is given.

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