ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

continued

Vol. 45, No. 5—September 2017

ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Vol. 45, No. 5—September 2017

The Annals of Probability
2017, Vol. 45, No. 5, 2767–2825
DOI: 10.1214/16-AOP1124

THE SCALING LIMIT OF RANDOM SIMPLE TRIANGULATIONS AND RANDOM SIMPLE QUADRANGULATIONS

BY LOUIGI ADDARIO-BERRY[1] AND MARIE ALBENQUE[2]

McGill University and CNRS

Let M_n be a simple triangulation of the sphere $\mathbb{S}^2$, drawn uniformly at random from all such triangulations with n vertices. Endow M_n with the uniform probability measure on its vertices. After rescaling graph distance by $(3/(4n))^{1/4}$, the resulting random measured metric space converges in distribution, in the Gromov–Hausdorff–Prokhorov sense, to the Brownian map. In proving the preceding fact, we introduce a labelling function for the vertices of M_n. Under this labelling, distances to a distinguished point are essentially given by vertex labels, with an error given by the winding number of an associated closed loop in the map. We establish similar results for simple quadrangulations.

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MSC2010 subject classifications. 60F17, 05C12, 82B41.

Key words and phrases. Random maps, Brownian map, Brownian snake, spatial branching process.

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The Annals of Probability
2017, Vol. 45, No. 5, 2826–2854
DOI: 10.1214/16-AOP1125

RECURRENCE AND TRANSIENCE FOR THE FROG MODEL ON TREES

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The frog model is a growing system of random walks where a particle is added whenever a new site is visited. A longstanding open question is how often the root is visited on the infinite d-ary tree. We prove the model undergoes a phase transition, finding it recurrent for $d = 2$ and transient for $d \geq 5$. Simulations suggest strong recurrence for $d = 2$, weak recurrence for $d = 3$, and transience for $d \geq 4$. Additionally, we prove a 0–1 law for all d-ary trees, and we exhibit a graph on which a 0–1 law does not hold.

To prove recurrence when $d = 2$, we construct a recursive distributional equation for the number of visits to the root in a smaller process and show the unique solution must be infinity a.s. The proof of transience when $d = 5$ relies on computer calculations for the transition probabilities of a large Markov chain. We also include the proof for $d \geq 6$, which uses similar techniques but does not require computer assistance.

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MSC2010 subject classifications. Primary 60K35; secondary 60J80, 60J10.
Key words and phrases. Frog model, transience, recurrence, phase transition, zero–one law.

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The Annals of Probability
2017, Vol. 45, No. 5, 2855–2866
DOI: 10.1214/16-AOP1126

STOCHASTIC DE GIORGI ITERATION AND REGULARITY OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

BY ELTON P. HSU, YU WANG AND ZHENAN WANG

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Under general conditions, we devise a stochastic version of De Giorgi iteration scheme for semilinear stochastic parabolic partial differential equation of the form

$$\partial_t u = \operatorname{div}(A\nabla u) + f(t,x,u) + g_i(t,x,u)\dot{w}_t^i$$

with progressively measurable diffusion coefficients. We use the scheme to show that the solution of the equation is almost surely Hölder continuous in both space and time variables.

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MSC2010 subject classifications. 60H15.
Key words and phrases. Stochastic De Giorgi iteration, measurable coefficients.

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The Annals of Probability
2017, Vol. 45, No. 5, 2867–2898
DOI: 10.1214/16-AOP1127

FINITARY COLORING

BY ALEXANDER E. HOLROYD, ODED SCHRAMM AND DAVID B. WILSON

Microsoft Research

Suppose that the vertices of $\mathbb{Z}^d$ are assigned random colors via a finitary factor of independent identically distributed (i.i.d.) vertex-labels. That is, the color of vertex v is determined by a rule that examines the labels within a finite (but random and perhaps unbounded) distance R of v, and the same rule applies at all vertices. We investigate the tail behavior of R if the coloring is required to be proper (i.e., if adjacent vertices must receive different colors). When $d \geq 2$, the optimal tail is given by a power law for 3 colors, and a tower (iterated exponential) function for 4 or more colors (and also for 3 or more colors when $d = 1$). If proper coloring is replaced with any shift of finite type in dimension 1, then, apart from trivial cases, tower function behavior also applies.

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MSC2010 subject classifications. 60G10, 05C15, 37A50.
Key words and phrases. Coloring, finitary factor, tower function, shift of finite type.

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The Annals of Probability
2017, Vol. 45, No. 5, 2899–2940
DOI: 10.1214/16-AOP1128

A $(2+1)$-DIMENSIONAL GROWTH PROCESS WITH EXPLICIT STATIONARY MEASURES[1]

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We introduce a class of $(2+1)$-dimensional stochastic growth processes, that can be seen as irreversible random dynamics of discrete interfaces. "Irreversible" means that the interface has an average nonzero drift. Interface configurations correspond to height functions of dimer coverings of the infinite hexagonal or square lattice. The model can also be viewed as an interacting driven particle system and in the totally asymmetric case the dynamics corresponds to an infinite collection of mutually interacting Hammersley processes.

When the dynamical asymmetry parameter $(p-q)$ equals zero, the infinite-volume Gibbs measures π_ρ (with given slope ρ) are stationary and reversible. When $p \neq q$, π_ρ are not reversible any more but, remarkably, they are still stationary. In such stationary states, we find that the average height function at any given point x grows linearly with time t with a nonzero speed: $\mathbb{E}Q_x(t) := \mathbb{E}(h_x(t) - h_x(0)) = V(\rho)t$ while the typical fluctuations of $Q_x(t)$ are smaller than any power of t as $t \to \infty$.

In the totally asymmetric case of $p = 0, q = 1$ and on the hexagonal lattice, the dynamics coincides with the "anisotropic KPZ growth model" introduced by A. Borodin and P. L. Ferrari in [*J. Stat. Mech. Theory Exp.* **2009** (2009) P02009, *Comm. Math. Phys.* **325** 603–684]. For a suitably chosen, "integrable", initial condition (that is very far from the stationary state), they were able to determine the hydrodynamic limit and a CLT for interface fluctuations on scale $\sqrt{\log t}$, exploiting the fact that in that case certain space-time height correlations can be computed exactly. In the same setting, they proved that, asymptotically for $t \to \infty$, the local statistics of height fluctuations tends to that of a Gibbs state (which led to the prediction that Gibbs states should be stationary).

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MSC2010 subject classifications. 82C20, 60J10, 60K35, 82C24.

Key words and phrases. Interface growth, interacting particle system, Lozenge and domino tilings, Hammersley process, anisotropic KPZ equation.

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The Annals of Probability
2017, Vol. 45, No. 5, 2941–2970
DOI: 10.1214/16-AOP1129

ASYMPTOTICS FOR 2*D* CRITICAL FIRST PASSAGE PERCOLATION

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We consider first passage percolation on $\mathbb{Z}^2$ with i.i.d. weights, whose distribution function satisfies $F(0) = p_c = 1/2$. This is sometimes known as the "critical case" because large clusters of zero-weight edges force passage times to grow at most logarithmically, giving zero time constant. Denote $T(\mathbf{0}, \partial B(n))$ as the passage time from the origin to the boundary of the box $[-n, n] \times [-n, n]$. We characterize the limit behavior of $T(\mathbf{0}, \partial B(n))$ by conditions on the distribution function F. We also give exact conditions under which $T(\mathbf{0}, \partial B(n))$ will have uniformly bounded mean or variance. These results answer several questions of Kesten and Zhang from the 1990s and, in particular, disprove a conjecture of Zhang from 1999. In the case when both the mean and the variance go to infinity as $n \to \infty$, we prove a CLT under a minimal moment assumption. The main tool involves a new relation between first passage percolation and invasion percolation: up to a constant factor, the passage time in critical first passage percolation has the same first-order behavior as the passage time of an optimal path constrained to lie in an embedded invasion cluster.

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MSC2010 subject classifications. Primary 60K35; secondary 60F05, 82B43.
Key words and phrases. First passage percolation, critical percolation, correlation length, invasion percolation, central limit theorem.

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The Annals of Probability
2017, Vol. 45, No. 5, 2971–3037
DOI: 10.1214/16-AOP1130

OBLIQUELY REFLECTED BROWNIAN MOTION IN NONSMOOTH PLANAR DOMAINS

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We construct obliquely reflected Brownian motions in all bounded simply connected planar domains, including nonsmooth domains, with general reflection vector fields on the boundary. Conformal mappings and excursion theory are our main technical tools. A key intermediate step, which may be of independent interest, is an alternative characterization of reflected Brownian motions in smooth bounded planar domains with a given field of angles of oblique reflection on the boundary in terms of a pair of quantities, namely an integrable positive harmonic function, which represents the stationary distribution of the process, and a real number that represents, in a suitable sense, the asymptotic rate of rotation of the process around a reference point in the domain. Furthermore, we also show that any obliquely reflected Brownian motion in a simply connected Jordan domain can be obtained as a suitable limit of obliquely reflected Brownian motions in smooth domains.

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MSC2010 subject classifications. Primary 60J65, 58J65, 60H20; secondary 30C20, 30J99.

Key words and phrases. Reflected Brownian motion, oblique reflection, simply connected domains, conformal mapping, stationary distribution, excursion reflected Brownian motion, Brownian motion with darning, Excursion reflected Brownian motion, rate of rotation of obliquely reflected Brownian motion.

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The Annals of Probability
2017, Vol. 45, No. 5, 3038–3074
DOI: 10.1214/16-AOP1131

COMPLETE DUALITY FOR MARTINGALE OPTIMAL TRANSPORT ON THE LINE

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We study the optimal transport between two probability measures on the real line, where the transport plans are laws of one-step martingales. A quasi-sure formulation of the dual problem is introduced and shown to yield a complete duality theory for general marginals and measurable reward (cost) functions: absence of a duality gap and existence of dual optimizers. Both properties are shown to fail in the classical formulation. As a consequence of the duality result, we obtain a general principle of cyclical monotonicity describing the geometry of optimal transports.

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MSC2010 subject classifications. 60G42, 49N05.
Key words and phrases. Martingale optimal transport, Kantorovich duality.

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The Annals of Probability
2017, Vol. 45, No. 5, 3075–3144
DOI: 10.1214/16-AOP1132

THE SCALING LIMIT OF THE MINIMUM SPANNING TREE OF THE COMPLETE GRAPH

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Consider the minimum spanning tree (MST) of the complete graph with n vertices, when edges are assigned independent random weights. Endow this tree with the graph distance renormalized by $n^{1/3}$ and with the uniform measure on its vertices. We show that the resulting space converges in distribution as $n \to \infty$ to a random compact measured metric space in the Gromov–Hausdorff–Prokhorov topology. We additionally show that the limit is a random binary $\mathbb{R}$-tree and has Minkowski dimension 3 almost surely. In particular, its law is mutually singular with that of the Brownian continuum random tree or any rescaled version thereof. Our approach relies on a coupling between the MST problem and the Erdős–Rényi random graph. We exploit the explicit description of the scaling limit of the Erdős–Rényi random graph in the so-called critical window, established in [*Probab. Theory Related Fields* **152** (2012) 367–406], and provide a similar description of the scaling limit for a "critical minimum spanning forest" contained within the MST. In order to accomplish this, we introduce the notion of $\mathbb{R}$-graphs, which generalise $\mathbb{R}$-trees, and are of independent interest.

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MSC2010 subject classifications. Primary 60C05; secondary 60F05.
Key words and phrases. Minimum spanning tree, scaling limit, R-tree, R-graph, Gromov–Hausdorff–Prokhorov distance, critical random graphs.

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The Annals of Probability
2017, Vol. 45, No. 5, 3145–3201
DOI: 10.1214/16-AOP1133

INVARIANT MEASURE FOR THE STOCHASTIC NAVIER–STOKES EQUATIONS IN UNBOUNDED 2D DOMAINS

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Building upon a recent work by two of the authors and J. Seidler on bw-Feller property for stochastic nonlinear beam and wave equations, we prove the existence of an invariant measure to stochastic 2-D Navier–Stokes (with multiplicative noise) equations in unbounded domains. This answers an open question left after the first author and Y. Li proved a corresponding result in the case of an additive noise.

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MSC2010 subject classifications. Primary 60H15, 35Q30, 37L40; secondary 76M35, 60J25.
Key words and phrases. Invariant measure, bw-Feller semigroup, stochastic Navier–Stokes equations.

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The Annals of Probability
2017, Vol. 45, No. 5, 3202–3222
DOI: 10.1214/16-AOP1134

ON THE BEHAVIOR OF DIFFUSION PROCESSES WITH TRAPS

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We consider processes that coincide with a given diffusion process outside a finite collection of domains. In each of the domains, there is, additionally, a large drift directed towards the interior of the domain. We describe the limiting behavior of the processes as the magnitude of the drift tends to infinity, and thus the domains become trapping, with the time to exit the domains being exponentially large. In particular, in exponential time scales, metastable distributions between the trapping regions are considered.

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MSC2010 subject classifications. 60F10, 35J25, 47D07, 60J60.

Key words and phrases. Metastable distributions, large deviations, exit problem, nonstandard boundary problem.

The Annals of Probability
2017, Vol. 45, No. 5, 3223–3265
DOI: 10.1214/16-AOP1135

INTEGRABILITY CONDITIONS FOR SDES AND SEMILINEAR SPDES[1]

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By using the local dimension-free Harnack inequality established on incomplete Riemannian manifolds, integrability conditions on the coefficients are presented for SDEs to imply the nonexplosion of solutions as well as the existence, uniqueness and regularity estimates of invariant probability measures. These conditions include a class of drifts unbounded on compact domains such that the usual Lyapunov conditions cannot be verified. The main results are extended to second-order differential operators on Hilbert spaces and semilinear SPDEs.

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MSC2010 subject classifications. Primary 60H15; secondary 60J45.
Key words and phrases. Nonexplosion, invariant probability measure, local Harnack inequality, SDE, SPDE.

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The Annals of Probability
2017, Vol. 45, No. 5, 3266–3292
DOI: 10.1214/16-AOP1136

A CLARK–OCONE FORMULA FOR TEMPORAL POINT PROCESSES AND APPLICATIONS

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We provide a Clark–Ocone formula for square-integrable functionals of a general temporal point process satisfying only a mild moment condition, generalizing known results on the Poisson space. Some classical applications are given, namely a deviation bound and the construction of a hedging portfolio in a pure-jump market model. As a more modern application, we provide a bound on the total variation distance between two temporal point processes, improving in some sense a recent result in this direction.

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MSC2010 subject classifications. 60G55, 60H07.
Key words and phrases. Clark–Ocone formula, point processes, Malliavin calculus, conditional intensity, deviation inequalities, option hedging.

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The Annals of Probability
2017, Vol. 45, No. 5, 3293–3335
DOI: 10.1214/16-AOP1137

A SYSTEM OF COALESCING HEAVY DIFFUSION PARTICLES ON THE REAL LINE

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We construct a modified Arratia flow with mass and energy conservation. We suppose that particles have a mass obeying the conservation law, and their diffusion is inversely proportional to the mass. Our main result asserts that such a system exists under the assumption of the uniform mass distribution on an interval at the starting moment. We introduce a stochastic integral with respect to such a flow and obtain the total local time as the density of the occupation measure for all particles.

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MSC2010 subject classifications. Primary 60K35, 82B21; secondary 60H05, 60J55.

Key words and phrases. Interacting particle system, Arratia flow, coalescing, stochastic integral with respect to flow, Itô formula, local time.

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The Annals of Probability
2017, Vol. 45, No. 5, 3336–3384
DOI: 10.1214/16-AOP1138

THE FEYNMAN–KAC FORMULA AND HARNACK INEQUALITY FOR DEGENERATE DIFFUSIONS

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We study various probabilistic and analytical properties of a class of degenerate diffusion operators arising in population genetics, the so-called generalized Kimura diffusion operators Epstein and Mazzeo [*SIAM J. Math. Anal.* **42** (2010) 568–608; *Degenerate Diffusion Operators Arising in Population Biology* (2013) Princeton University Press; *Applied Mathematics Research Express* (2016)]. Our main results are a stochastic representation of weak solutions to a degenerate parabolic equation with singular lower-order coefficients and the proof of the scale-invariant Harnack inequality for nonnegative solutions to the Kimura parabolic equation. The stochastic representation of solutions that we establish is a considerable generalization of the classical results on Feynman–Kac formulas concerning the assumptions on the degeneracy of the diffusion matrix, the boundedness of the drift coefficients and the a priori regularity of the weak solutions.

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MSC2010 subject classifications. Primary 35J90; secondary 60J60.
Key words and phrases. Degenerate elliptic equations, degenerate diffusions, generalized Kimura diffusions, Markov processes, Feynman–Kac formulas, Girsanov formula, weighted Sobolev spaces, anisotropic Hölder spaces.

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The Annals of Probability
2017, Vol. 45, No. 5, 3385–3450
DOI: 10.1214/16-AOP1139

THE COMPLEXITY OF SPHERICAL p-SPIN MODELS—A SECOND MOMENT APPROACH[1]

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Recently, Auffinger, Ben Arous and Černý initiated the study of critical points of the Hamiltonian in the spherical pure p-spin spin glass model, and established connections between those and several notions from the physics literature. Denoting the number of critical values less than Nu by $\mathrm{Crt}_N(u)$, they computed the asymptotics of $\frac{1}{N}\log(\mathbb{E}\mathrm{Crt}_N(u))$, as N, the dimension of the sphere, goes to ∞. We compute the asymptotics of the corresponding second moment and show that, for $p \geq 3$ and sufficiently negative u, it matches the first moment:

$$\mathbb{E}\{(\mathrm{Crt}_N(u))^2\}/(\mathbb{E}\{\mathrm{Crt}_N(u)\})^2 \to 1.$$

As an immediate consequence we obtain that $\mathrm{Crt}_N(u)/\mathbb{E}\{\mathrm{Crt}_N(u)\} \to 1$, in L^2, and thus in probability. For any u for which $\mathbb{E}\,\mathrm{Crt}_N(u)$ does not tend to 0 we prove that the moments match on an exponential scale.

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The Annals of Probability
2017, Vol. 45, No. 5, 3451–3479
DOI: 10.1214/16-AOP1140

STABILITY OF GEODESICS IN THE BROWNIAN MAP

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The Brownian map is a random geodesic metric space arising as the scaling limit of random planar maps. We strengthen the so-called *confluence of geodesics* phenomenon observed at the root of the map, and with this, reveal several properties of its rich geodesic structure.

Our main result is the continuity of the cut locus at typical points. A small shift from such a point results in a small, local modification to the cut locus. Moreover, the cut locus is uniformly stable, in the sense that any two cut loci coincide outside a closed, nowhere dense set of zero measure.

We obtain similar stability results for the set of points inside geodesics to a fixed point. Furthermore, we show that the set of points inside geodesics of the map is of first Baire category. Hence, most points in the Brownian map are endpoints.

Finally, we classify the types of geodesic networks which are dense. For each $k \in \{1, 2, 3, 4, 6, 9\}$, there is a dense set of pairs of points which are joined by networks of exactly k geodesics and of a specific topological form. We find the Hausdorff dimension of the set of pairs joined by each type of network. All other geodesic networks are nowhere dense.

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MSC2010 subject classifications. Primary 60D05; secondary 05C80, 54E52, 54G99.

Key words and phrases. Brownian map, planar map, geodesic, cut locus, real tree, scaling limit, Gromov–Hausdorff convergence, random geometry, quantum gravity.

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