ISSN 0091-1798 (print)
ISSN 2168-894X (online)

# THE ANNALS *of* PROBABILITY

*An Official Journal of the*
Institute of Mathematical Statistics

**Articles**



Vol. 45, No. 6B—November 2017

THE ANNALS OF PROBABILITY Vol. 45, No. 6B, pp. 4167–4820 November 2017

***The Annals of Probability*** **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 45, Number 6B, November 2017. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

**POSTMASTER:** Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

*The Annals of Probability*
2017, Vol. 45, No. 6B, 4167–4221
https://doi.org/10.1214/16-AOP1162


# A CENTRAL LIMIT THEOREM FOR THE KPZ EQUATION

BY MARTIN HAIRER[1] AND HAO SHEN

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We consider the KPZ equation in one space dimension driven by a stationary centred space–time random field, which is sufficiently integrable and mixing, but not necessarily Gaussian. We show that, in the weakly asymmetric regime, the solution to this equation considered at a suitable large scale and in a suitable reference frame converges to the Hopf–Cole solution to the KPZ equation driven by space–time Gaussian white noise. While the limiting process depends only on the integrated variance of the driving field, the diverging constants appearing in the definition of the reference frame also depend on higher order moments.

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*MSC2010 subject classifications.* Primary 60H15; secondary 35K55, 60H30.
*Key words and phrases.* KPZ equation, central limit theorem, Wiener chaos, cumulants.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4222–4272
https://doi.org/10.1214/16-AOP1163


# THE SHARP THRESHOLD FOR THE DUARTE MODEL

BY BÉLA BOLLOBÁS[*,†,‡,1], HUGO DUMINIL-COPIN[§,2], ROBERT MORRIS[¶,3] AND PAUL SMITH[‖,4]

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The class of critical bootstrap percolation models in two dimensions was recently introduced by Bollobás, Smith and Uzzell, and the critical threshold for percolation was determined up to a constant factor for all such models by the authors of this paper. Here, we develop and refine the techniques introduced in that paper in order to determine a sharp threshold for the Duarte model. This resolves a question of Mountford from 1995, and is the first result of its type for a model with drift.

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*Key words and phrases.* Bootstrap percolation, monotone cellular automata, Duarte model, critical probability, sharp threshold.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4273–4306
https://doi.org/10.1214/16-AOP1164


# STOCHASTIC INTEGRATION WITH RESPECT TO CYLINDRICAL LÉVY PROCESSES

BY ADAM JAKUBOWSKI[1] AND MARKUS RIEDLE[2]

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A cylindrical Lévy process does not enjoy a cylindrical version of the semimartingale decomposition which results in the need to develop a completely novel approach to stochastic integration. In this work, we introduce a stochastic integral for random integrands with respect to cylindrical Lévy processes in Hilbert spaces. The space of admissible integrands consists of càglàd, adapted stochastic processes with values in the space of Hilbert–Schmidt operators. Neither the integrands nor the integrator is required to satisfy any moment or boundedness condition. The integral process is characterised as an adapted, Hilbert space valued semimartingale with càdlàg trajectories.

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*MSC2010 subject classifications.* Primary 60H05; secondary 60B11, 60G20, 28C20.
*Key words and phrases.* Cylindrical Lévy processes, stochastic integration, decoupled tangent sequence, cylindrical Brownian motion, random measures.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4307–4347
https://doi.org/10.1214/16-AOP1166


# CENTRAL LIMIT THEOREM FOR RANDOM WALKS IN DOUBLY STOCHASTIC RANDOM ENVIRONMENT: $\mathscr{H}_{-1}$ SUFFICES

BY GADY KOZMA[*,1,3] AND BÁLINT TÓTH[†,‡,2,3]

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We prove a central limit theorem under diffusive scaling for the displacement of a random walk on $\mathbb{Z}^d$ in stationary and ergodic doubly stochastic random environment, under the $\mathscr{H}_{-1}$-condition imposed on the drift field. The condition is equivalent to assuming that the stream tensor of the drift field be stationary and square integrable. This improves the best existing result [*Fluctuations in Markov Processes—Time Symmetry and Martingale Approximation* (2012) Springer], where it is assumed that the stream tensor is in $\mathscr{L}^{\max\{2+\delta,d\}}$, with $\delta > 0$. Our proof relies on an extension of the *relaxed sector condition* of [*Bull. Inst. Math. Acad. Sin. (N.S.)* **7** (2012) 463–476], and is technically rather simpler than existing earlier proofs of similar results by Oelschläger [*Ann. Probab.* **16** (1988) 1084–1126] and Komorowski, Landim and Olla [*Fluctuations in Markov Processes—Time Symmetry and Martingale Approximation* (2012) Springer].

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*MSC2010 subject classifications.* 60F05, 60G99, 60K37.

*Key words and phrases.* Random walk in random environment, central limit theorem, Kipnis–Varadhan theory, sector condition.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4348–4388
https://doi.org/10.1214/16-AOP1167


# A LEADER-ELECTION PROCEDURE USING RECORDS

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Motivated by the open problem of finding the asymptotic distributional behavior of the number of collisions in a Poisson–Dirichlet coalescent, the following version of a stochastic leader-election algorithm is studied. Consider an infinite family of persons, labeled by $1, 2, 3, \dots$, who generate i.i.d. random numbers from an arbitrary continuous distribution. Those persons who have generated a record value, that is, a value larger than the values of all previous persons, stay in the game, all others must leave. The remaining persons are relabeled by $1, 2, 3, \dots$ maintaining their order in the first round, and the election procedure is repeated independently from the past and indefinitely. We prove limit theorems for a number of relevant functionals for this procedure, notably the number of rounds $T(M)$ until all persons among $1, \dots, M$, except the first one, have left (as $M \to \infty$). For example, we show that the sequence $(T(M) - \log^* M)_{M\in\mathbb{N}}$, where $\log^*$ denotes the iterated logarithm, is tight, and study its weak subsequential limits. We further provide an appropriate and apparently new kind of normalization (based on tetrations) such that the original labels of persons who stay in the game until round $n$ converge (as $n \to \infty$) to some random non-Poissonian point process and study its properties. The results are applied to describe all subsequential distributional limits for the number of collisions in the Poisson–Dirichlet coalescent, thus providing a complete answer to the open problem mentioned above.

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*MSC2010 subject classifications.* Primary 60F05, 60G55; secondary 60J10.

*Key words and phrases.* Poisson–Dirichlet coalescent, leader-election procedure, absorption time, random recursion, tetration, iterated logarithm, records.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4389–4418
https://doi.org/10.1214/16-AOP1168


# LÉVY PROCESSES AND LÉVY WHITE NOISE AS TEMPERED DISTRIBUTIONS[1]

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We identify a necessary and sufficient condition for a Lévy white noise to be a tempered distribution. More precisely, we show that if the Lévy measure associated with this noise has a positive absolute moment, then the Lévy white noise almost surely takes values in the space of tempered distributions. If the Lévy measure does not have a positive absolute moment of any order, then the event on which the Lévy white noise is a tempered distribution has probability zero.

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*MSC2010 subject classifications.* Primary 60G51; secondary 60G60, 60G20, 60H40.
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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4419–4476
https://doi.org/10.1214/16-AOP1169


# LARGE DEVIATIONS FOR RANDOM PROJECTIONS OF $\ell^p$ BALLS

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Let $p \in [1, \infty]$. Consider the projection of a uniform random vector from a suitably normalized $\ell^p$ ball in $\mathbb{R}^n$ onto an independent random vector from the unit sphere. We show that sequences of such random projections, when suitably normalized, satisfy a large deviation principle (LDP) as the dimension $n$ goes to $\infty$, which can be viewed as an *annealed* LDP. We also establish a *quenched* LDP (conditioned on a fixed sequence of projection directions) and show that for $p \in (1, \infty]$ (but *not* for $p = 1$), the corresponding rate function is "universal," in the sense that it coincides for "almost every" sequence of projection directions. We also analyze some exceptional sequences of directions in the "measure zero" set, including the sequence of directions corresponding to the classical Cramér's theorem, and show that those sequences of directions yield LDPs with rate functions that are distinct from the universal rate function of the quenched LDP. Lastly, we identify a variational formula that relates the annealed and quenched LDPs, and analyze the minimizer of this variational formula. These large deviation results complement the central limit theorem for convex sets, specialized to the case of sequences of $\ell^p$ balls.

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*MSC2010 subject classifications.* Primary 60F10; secondary 52A20, 60K37, 60D05.
*Key words and phrases.* Large deviations, random projections, $\ell^p$-balls, annealed and quenched large deviations, self-normalized large deviations, central limit theorem for convex sets, variational formula.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4477–4528
https://doi.org/10.1214/16-AOP1170


# POWER VARIATION FOR A CLASS OF STATIONARY INCREMENTS LÉVY DRIVEN MOVING AVERAGES

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In this paper, we present some new limit theorems for power variation of $k$th order increments of stationary increments Lévy driven moving averages. In the infill asymptotic setting, where the sampling frequency converges to zero while the time span remains fixed, the asymptotic theory gives novel results, which (partially) have no counterpart in the theory of discrete moving averages. More specifically, we show that the first-order limit theory and the mode of convergence strongly depend on the interplay between the given order of the increments $k \geq 1$, the considered power $p > 0$, the Blumenthal–Getoor index $\beta \in [0, 2)$ of the driving pure jump Lévy process $L$ and the behaviour of the kernel function $g$ at 0 determined by the power $\alpha$. First-order asymptotic theory essentially comprises three cases: stable convergence towards a certain infinitely divisible distribution, an ergodic type limit theorem and convergence in probability towards an integrated random process. We also prove a second-order limit theorem connected to the ergodic type result. When the driving Lévy process $L$ is a symmetric $\beta$-stable process, we obtain two different limits: a central limit theorem and convergence in distribution towards a $(k - \alpha)\beta$-stable totally right skewed random variable.

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*MSC2010 subject classifications.* Primary 60F05, 60F15, 60G22; secondary 60G48, 60H05.

*Key words and phrases.* Power variation, limit theorems, moving averages, fractional processes, stable convergence, high frequency data.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4529–4560
https://doi.org/10.1214/16-AOP1171


# EQUILIBRIUM FLUCTUATION OF THE ATLAS MODEL

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We study the fluctuation of the Atlas model, where a unit drift is assigned to the lowest ranked particle among a semi-infinite ($\mathbb{Z}_+$-indexed) system of otherwise independent Brownian particles, initiated according to a Poisson point process on $\mathbb{R}_+$. In this context, we show that the joint law of ranked particles, after being centered and scaled by $t^{-\frac{1}{4}}$, converges as $t \to \infty$ to the Gaussian field corresponding to the solution of the Additive Stochastic Heat Equation (ASHE) on $\mathbb{R}_+$ with the Neumann boundary condition at zero. This allows us to express the asymptotic fluctuation of the lowest ranked particle in terms of a fractional Brownian Motion (fBM). In particular, we prove a conjecture of Pal and Pitman [*Ann. Appl. Probab.* **18** (2008) 2179–2207] about the asymptotic Gaussian fluctuation of the ranked particles.

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*MSC2010 subject classifications.* Primary 60K35; secondary 60H15, 82C22.

*Key words and phrases.* Equilibrium fluctuation, fractional Brownian motion, interacting particles, reflected Brownian motion, stochastic heat equation.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4561–4616
https://doi.org/10.1214/16-AOP1172


# STOCHASTIC HEAT EQUATION WITH ROUGH DEPENDENCE IN SPACE

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This paper studies the nonlinear one-dimensional stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in the space variable. The existence and uniqueness of the solution $u$ are proved assuming the nonlinear coefficient $\sigma(u)$ is differentiable with a Lipschitz derivative and $\sigma(0) = 0$.

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*MSC2010 subject classifications.* 60G15, 60H07, 60H10, 65C30.
*Key words and phrases.* Stochastic heat equation, fractional Brownian motion, Feynman–Kac formula, Wiener chaos expansion, intermittency.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4617–4631
https://doi.org/10.1214/16-AOP1173


# PARISI FORMULA FOR THE GROUND STATE ENERGY IN THE MIXED $p$-SPIN MODEL

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We show that the thermodynamic limit of the ground state energy in the mixed $p$-spin model can be identified as a variational problem. This gives a natural generalization of the Parisi formula at zero temperature.

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*MSC2010 subject classifications.* 60K35.
*Key words and phrases.* Spin glasses, ground state energy, Parisi formula.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4632–4674
https://doi.org/10.1214/17-AOP1174


# INVARIANCE TIMES[1]

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On a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we consider two filtrations $\mathbb{F} \subset \mathbb{G}$ and a $\mathbb{G}$ stopping time $\theta$ such that the $\mathbb{G}$ predictable processes coincide with $\mathbb{F}$ predictable processes on $(0, \theta]$. In this setup, it is well known that, for any $\mathbb{F}$ semimartingale $X$, the process $X^{\theta-}$ ($X$ stopped "right before $\theta$") is a $\mathbb{G}$ semimartingale. Given a positive constant $T$, we call $\theta$ an invariance time if there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale $X$, $X^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale. We characterize invariance times in terms of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale of $\theta$ and we derive a mild and tractable invariance time sufficiency condition. We discuss invariance times in mathematical finance and BSDE applications.

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*MSC2010 subject classifications.* Primary 60G07; secondary 60G44.

*Key words and phrases.* Random time, enlargement of filtration, measure change, mathematical finance.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4675–4699
https://doi.org/10.1214/17-AOP1175


# ASYMPTOTIC EXPANSION OF THE INVARIANT MEASURE FOR BALLISTIC RANDOM WALK IN THE LOW DISORDER REGIME[1]

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We consider a random walk in random environment in the low disorder regime on $\mathbb{Z}^d$, that is, the probability that the random walk jumps from a site $x$ to a nearest neighboring site $x + e$ is given by $p(e) + \varepsilon\xi(x, e)$, where $p(e)$ is deterministic, $\{\{\xi(x, e) : |e|_1 = 1\} : x \in \mathbb{Z}^d\}$ are i.i.d. and $\varepsilon > 0$ is a parameter, which is eventually chosen small enough. We establish an asymptotic expansion in $\varepsilon$ for the invariant measure of the environmental process whenever a ballisticity condition is satisfied. As an application of our expansion, we derive a numerical expression up to first order in $\varepsilon$ for the invariant measure of random perturbations of the simple symmetric random walk in dimensions $d = 2$.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4700–4751
https://doi.org/10.1214/17-AOP1176


# POLARITY OF POINTS FOR GAUSSIAN RANDOM FIELDS

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We show that for a wide class of Gaussian random fields, points are polar in the critical dimension. Examples of such random fields include solutions of systems of linear stochastic partial differential equations with deterministic coefficients, such as the stochastic heat equation or wave equation with space–time white noise, or colored noise in spatial dimensions $k \geq 1$. Our approach builds on a delicate covering argument developed by M. Talagrand [*Ann. Probab.* **23** (1995) 767–775; *Probab. Theory Related Fields* **112** (1998) 545–563] for the study of fractional Brownian motion, and uses a harmonizable representation of the solutions of these stochastic PDEs.

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*MSC2010 subject classifications.* 60G15, 60J45, 60G60.
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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4752–4785
https://doi.org/10.1214/17-AOP1177


# THE VACANT SET OF TWO-DIMENSIONAL CRITICAL RANDOM INTERLACEMENT IS INFINITE[1]

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For the model of two-dimensional random interlacements in the critical regime (i.e., $\alpha = 1$), we prove that the vacant set is a.s. infinite, thus solving an open problem from [*Commun. Math. Phys.* **343** (2016) 129–164]. Also, we prove that the entrance measure of simple random walk on annular domains has certain regularity properties; this result is useful when dealing with soft local times for excursion processes.

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*MSC2010 subject classifications.* Primary 60K35; secondary 60G50, 82C41.

*Key words and phrases.* Random interlacements, vacant set, critical regime, simple random walk, Doob's $h$-transform, annular domain.

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*The Annals of Probability*
2017, Vol. 45, No. 6B, 4786–4820
https://doi.org/10.1214/17-AOP1179


# EXISTENCE CONDITIONS OF PERMANENTAL AND MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS

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Existence conditions of permanental distributions are deeply connected to existence conditions of multivariate negative binomial distributions. The aim of this paper is twofold. It answers several questions generated by recent works on this subject, but it also goes back to the roots of this field and fixes existing gaps in older papers concerning conditions of infinite divisibility for these distributions.

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*MSC2010 subject classifications.* 60G15, 60E05, 60E07, 60E10, 15A15, 15B48.
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# THE ANNALS *of* PROBABILITY

*AN OFFICIAL JOURNAL OF THE*
INSTITUTE OF MATHEMATICAL STATISTICS

VOLUME 45

2017

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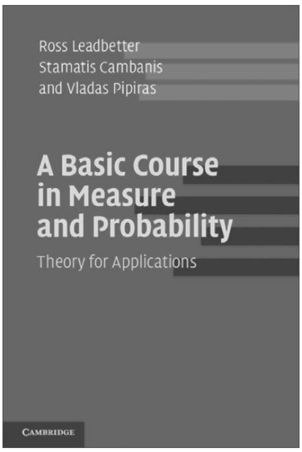
**Vol. 46** **January 2018** **No. 1**

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