ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Vol. 46, No. 1—January 2018

THE ANNALS OF PROBABILITY Vol. 46, No. 1, pp. 1–603 January 2018

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 46, Number 1, January 2018. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Probability
2018, Vol. 46, No. 1, 1–71
https://doi.org/10.1214/16-AOP1142

NONBACKTRACKING SPECTRUM OF RANDOM GRAPHS: COMMUNITY DETECTION AND NONREGULAR RAMANUJAN GRAPHS

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A nonbacktracking walk on a graph is a directed path such that no edge is the inverse of its preceding edge. The nonbacktracking matrix of a graph is indexed by its directed edges and can be used to count nonbacktracking walks of a given length. It has been used recently in the context of community detection and has appeared previously in connection with the Ihara zeta function and in some generalizations of Ramanujan graphs. In this work, we study the largest eigenvalues of the nonbacktracking matrix of the Erdős–Rényi random graph and of the stochastic block model in the regime where the number of edges is proportional to the number of vertices. Our results confirm the "spectral redemption conjecture" in the symmetric case and show that community detection can be made on the basis of the leading eigenvectors above the feasibility threshold.

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MSC2010 subject classifications. Primary 05C80, 60B20; secondary 62M15, 60J85.

Key words and phrases. Random graphs, nonbacktracking matrix, spectral gap, community detection.

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The Annals of Probability
2018, Vol. 46, No. 1, 72–125
https://doi.org/10.1214/17-AOP1180

SIZE BIASED COUPLINGS AND THE SPECTRAL GAP FOR RANDOM REGULAR GRAPHS

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Let λ be the second largest eigenvalue in absolute value of a uniform random d-regular graph on n vertices. It was famously conjectured by Alon and proved by Friedman that if d is fixed independent of n, then $\lambda = 2\sqrt{d-1} + o(1)$ with high probability. In the present work, we show that $\lambda = O(\sqrt{d})$ continues to hold with high probability as long as $d = O(n^{2/3})$, making progress toward a conjecture of Vu that the bound holds for all $1 \le d \le n/2$. Prior to this work the best result was obtained by Broder, Frieze, Suen and Upfal (1999) using the configuration model, which hits a barrier at $d = o(n^{1/2})$. We are able to go beyond this barrier by proving concentration of measure results directly for the uniform distribution on d-regular graphs. These come as consequences of advances we make in the theory of concentration by size biased couplings. Specifically, we obtain Bennett-type tail estimates for random variables admitting certain unbounded size biased couplings.

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MSC2010 subject classifications. Primary 05C80; secondary 60B20, 60E15.

Key words and phrases. Second eigenvalue, random regular graph, Alon's conjecture, size biased coupling, Stein's method, concentration.

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The Annals of Probability
2018, Vol. 46, No. 1, 126–174
https://doi.org/10.1214/17-AOP1181

PATH-DEPENDENT EQUATIONS AND VISCOSITY SOLUTIONS IN INFINITE DIMENSION[1]

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Path-dependent partial differential equations (PPDEs) are natural objects to study when one deals with non-Markovian models. Recently, after the introduction of the so-called pathwise (or functional or Dupire) calculus [see Dupire (2009)], in the case of finite-dimensional underlying space various papers have been devoted to studying the well-posedness of such kind of equations, both from the point of view of regular solutions [see, e.g., Dupire (2009) and Cont (2016) *Stochastic Integration by Parts and Functional Itô Calculus* 115–207, Birkhäuser] and viscosity solutions [see, e.g., Ekren et al. (2014) *Ann. Probab.* **42** 204–236]. In this paper, motivated by the study of models driven by path-dependent stochastic PDEs, we give a first well-posedness result for viscosity solutions of PPDEs when the underlying space is a separable Hilbert space. We also observe that, in contrast with the finite-dimensional case, our well-posedness result, even in the Markovian case, applies to equations which cannot be treated, up to now, with the known theory of viscosity solutions.

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MSC2010 subject classifications. 35D40, 35R15, 60H15, 60H30.
Key words and phrases. Viscosity solutions, path-dependent stochastic differential equations, path-dependent partial differential equations, partial differential equations in infinite dimension.

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The Annals of Probability
2018, Vol. 46, No. 1, 175–206
https://doi.org/10.1214/17-AOP1182

AN UPPER BOUND ON THE NUMBER OF SELF-AVOIDING POLYGONS VIA JOINING

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For $d \geq 2$ and $n \in \mathbb{N}$ even, let $p_n = p_n(d)$ denote the number of length n self-avoiding polygons in $\mathbb{Z}^d$ up to translation. The polygon cardinality grows exponentially, and the growth rate $\lim_{n \in 2\mathbb{N}} p_n^{1/n} \in (0, \infty)$ is called the connective constant and denoted by μ. Madras [*J. Stat. Phys.* **78** (1995) 681–699] has shown that $p_n \mu^{-n} \leq C n^{-1/2}$ in dimension $d = 2$. Here, we establish that $p_n \mu^{-n} \leq n^{-3/2+o(1)}$ for a set of even n of full density when $d = 2$. We also consider a certain variant of self-avoiding walk and argue that, when $d \geq 3$, an upper bound of $n^{-2+d^{-1}+o(1)}$ holds on a full density set for the counterpart in this variant model of this normalized polygon cardinality.

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The Annals of Probability
2018, Vol. 46, No. 1, 207–260
https://doi.org/10.1214/17-AOP1183

RANDOM PLANAR MAPS AND GROWTH-FRAGMENTATIONS

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We are interested in the cycles obtained by slicing at all heights random Boltzmann triangulations with a simple boundary. We establish a functional invariance principle for the lengths of these cycles, appropriately rescaled, as the size of the boundary grows. The limiting process is described using a self-similar growth-fragmentation process with explicit parameters. To this end, we introduce a branching peeling exploration of Boltzmann triangulations, which allows us to identify a crucial martingale involving the perimeters of cycles at given heights. We also use a recent result concerning self-similar scaling limits of Markov chains on the nonnegative integers. A motivation for this work is to give a new construction of the Brownian map from a growth-fragmentation process.

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MSC2010 subject classifications. Primary 60F17, 60D05; secondary 60G51, 60G18.
Key words and phrases. Planar maps, growth-fragmentations, self-similar Markov processes, scaling limits.

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The Annals of Probability
2018, Vol. 46, No. 1, 261–301
https://doi.org/10.1214/17-AOP1184

DIMENSIONAL IMPROVEMENTS OF THE LOGARITHMIC SOBOLEV, TALAGRAND AND BRASCAMP–LIEB INEQUALITIES[1]

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In this work, we consider dimensional improvements of the logarithmic Sobolev, Talagrand and Brascamp–Lieb inequalities. For this, we use optimal transport methods and the Borell–Brascamp–Lieb inequality. These refinements can be written as a deficit in the classical inequalities. They have the right scale with respect to the dimension. They lead to sharpened concentration properties as well as refined contraction bounds, convergence to equilibrium and short time behavior for the laws of solutions to stochastic differential equations.

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MSC2010 subject classifications. 60H10, 60J60, 60E15.

Key words and phrases. Logarithmic Sobolev inequality, Talagrand inequality, Brascamp–Lieb inequality, Fokker–Planck equations, optimal transport.

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The Annals of Probability
2018, Vol. 46, No. 1, 302–336
https://doi.org/10.1214/17-AOP1186

QUENCHED INVARIANCE PRINCIPLE FOR RANDOM WALKS WITH TIME-DEPENDENT ERGODIC DEGENERATE WEIGHTS

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We study a continuous-time random walk, X, on $\mathbb{Z}^d$ in an environment of dynamic random conductances taking values in $(0, \infty)$. We assume that the law of the conductances is ergodic with respect to space–time shifts. We prove a quenched invariance principle for the Markov process X under some moment conditions on the environment. The key result on the sublinearity of the corrector is obtained by Moser's iteration scheme.

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MSC2010 subject classifications. 60K37, 60F17, 82C41.
Key words and phrases. Time dependent dynamics, random walk, Moser iteration.

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The Annals of Probability
2018, Vol. 46, No. 1, 337–396
https://doi.org/10.1214/17-AOP1187

AN L^p THEORY OF SPARSE GRAPH CONVERGENCE II: LD CONVERGENCE, QUOTIENTS AND RIGHT CONVERGENCE

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We extend the L^p theory of sparse graph limits, which was introduced in a companion paper, by analyzing different notions of convergence. Under suitable restrictions on node weights, we prove the equivalence of metric convergence, quotient convergence, microcanonical ground state energy convergence, microcanonical free energy convergence and large deviation convergence. Our theorems extend the broad applicability of dense graph convergence to all sparse graphs with unbounded average degree, while the proofs require new techniques based on uniform upper regularity. Examples to which our theory applies include stochastic block models, power law graphs and sparse versions of W-random graphs.

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MSC2010 subject classifications. Primary 05C80; secondary 82B99.
Key words and phrases. Sparse graph limit, graphon, graph convergence, graph quotient.

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The Annals of Probability
2018, Vol. 46, No. 1, 397–455
https://doi.org/10.1214/17-AOP1188

LATTICE APPROXIMATION TO THE DYNAMICAL Φ_3^4 MODEL[1]

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We study the lattice approximations to the dynamical Φ_3^4 model by paracontrolled distributions proposed in [*Forum Math. Pi* **3** (2015) e6]. We prove that the solutions to the lattice systems converge to the solution to the dynamical Φ_3^4 model in probability, locally uniformly in time. Since the dynamical Φ_3^4 model is not well defined in the classical sense and renormalisation has to be performed in order to define the nonlinear term, a corresponding suitable drift term is added in the stochastic equations for the lattice systems.

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MSC2010 subject classifications. 60H15, 82C28.

Key words and phrases. Φ_3^4 model, regularity structure, paracontrolled distribution, space–time white noise, renormalisation.

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The Annals of Probability
2018, Vol. 46, No. 1, 456–490
https://doi.org/10.1214/17-AOP1189

RANDOM WALKS ON THE RANDOM GRAPH

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We study random walks on the giant component of the Erdős–Rényi random graph $\mathcal{G}(n, p)$ where $p = \lambda/n$ for $\lambda > 1$ fixed. The mixing time from a worst starting point was shown by Fountoulakis and Reed, and independently by Benjamini, Kozma and Wormald, to have order $\log^2 n$. We prove that starting from a uniform vertex (equivalently, from a fixed vertex conditioned to belong to the giant) both accelerates mixing to $O(\log n)$ and concentrates it (the cutoff phenomenon occurs): the typical mixing is at $(\nu\mathbf{d})^{-1} \log n \pm (\log n)^{1/2+o(1)}$, where ν and $\mathbf{d}$ are the speed of random walk and dimension of harmonic measure on a Poisson(λ)-Galton–Watson tree. Analogous results are given for graphs with prescribed degree sequences, where cutoff is shown both for the simple and for the nonbacktracking random walk.

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MSC2010 subject classifications. 60B10, 60J10, 60G50, 05C80.
Key words and phrases. Random walks, random graphs, cutoff phenomenon, Markov chain mixing times.

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The Annals of Probability
2018, Vol. 46, No. 1, 491–550
https://doi.org/10.1214/17-AOP1190

A CLASS OF GLOBALLY SOLVABLE MARKOVIAN QUADRATIC BSDE SYSTEMS AND APPLICATIONS

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We establish existence and uniqueness for a wide class of Markovian systems of backward stochastic differential equations (BSDE) with quadratic nonlinearities. This class is characterized by an abstract structural assumption on the generator, an a priori local-boundedness property, and a locally-Hölder-continuous terminal condition. We present easily verifiable sufficient conditions for these assumptions and treat several applications, including stochastic equilibria in incomplete financial markets, stochastic differential games and martingales on Riemannian manifolds.

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MSC2010 subject classifications. Primary 60G44, 60G99, 60H30; secondary 58J65, 91A15, 91B51.

Key words and phrases. BSDE, backward stochastic differential equations, systems of BSDE, quadratic nonlinearities, stochastic equilibrium, martingales on manifolds, nonzero-sum stochastic games.

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The Annals of Probability
2018, Vol. 46, No. 1, 551–603
https://doi.org/10.1214/17-AOP1191

STOCHASTIC CONTROL FOR A CLASS OF NONLINEAR KERNELS AND APPLICATIONS[1]

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We consider a stochastic control problem for a class of nonlinear kernels. More precisely, our problem of interest consists in the optimization, over a set of possibly nondominated probability measures, of solutions of backward stochastic differential equations (BSDEs). Since BSDEs are nonlinear generalizations of the traditional (linear) expectations, this problem can be understood as stochastic control of a family of nonlinear expectations, or equivalently of nonlinear kernels. Our first main contribution is to prove a dynamic programming principle for this control problem in an abstract setting, which we then use to provide a semimartingale characterization of the value function. We next explore several applications of our results. We first obtain a wellposedness result for second order BSDEs (as introduced in Soner, Touzi and Zhang [*Probab. Theory Related Fields* **153** (2012) 149–190]) which does not require any regularity assumption on the terminal condition and the generator. Then we prove a nonlinear optional decomposition in a robust setting, extending recent results of Nutz [*Stochastic Process. Appl.* **125** (2015) 4543–4555], which we then use to obtain a super-hedging duality in uncertain, incomplete and nonlinear financial markets. Finally, we relate, under additional regularity assumptions, the value function to a viscosity solution of an appropriate path–dependent partial differential equation (PPDE).

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MSC2010 subject classifications. 60H10, 60H30.

Key words and phrases. Stochastic control, measurable selection, nonlinear kernels, second-order BSDEs, path–dependent PDEs, robust super-hedging.

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The Annals of Probability

Vol. 46 **March 2018** **No. 2**

Articles