ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 46, No. 2—March 2018

THE ANNALS OF PROBABILITY Vol. 46, No. 2, pp. 605–1200 March 2018

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 46, Number 2, March 2018. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Probability
2018, Vol. 46, No. 2, 605–686
https://doi.org/10.1214/16-AOP1159

SCALING LIMITS FOR SUB-BALLISTIC BIASED RANDOM WALKS IN RANDOM CONDUCTANCES

BY ALEXANDER FRIBERGH AND DANIEL KIOUS

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We consider biased random walks in positive random conductances on the d-dimensional lattice in the zero-speed regime and study their scaling limits. We obtain a functional law of large numbers for the position of the walker, properly rescaled. Moreover, we state a functional central limit theorem where an atypical process, related to the fractional kinetics, appears in the limit.

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MSC2010 subject classifications. Primary 60K37; secondary 82D30.

Key words and phrases. Random walks in random environments, random conductances, scaling limit, trap model, zero-speed.

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The Annals of Probability
2018, Vol. 46, No. 2, 687–774
https://doi.org/10.1214/16-AOP1165

GROWTH EXPONENT FOR LOOP-ERASED RANDOM WALK IN THREE DIMENSIONS

BY DAISUKE SHIRAISHI

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Let M_n be the number of steps of the loop-erasure of a simple random walk on $\mathbb{Z}^3$ run until its first exit from a ball of radius n. In the paper, we will show the existence of the growth exponent, that is, we show that there exists $\beta > 0$ such that

$$\lim_{n\to\infty} \frac{\log E(M_n)}{\log n} = \beta.$$

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MSC2010 subject classifications. 60G17, 60K35, 60D05.
Key words and phrases. Loop-erased random walk, simple random walk, ergodic theory.

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The Annals of Probability
2018, Vol. 46, No. 2, 775–828
https://doi.org/10.1214/17-AOP1192

LARGE DEVIATIONS OF THE TRAJECTORY OF EMPIRICAL DISTRIBUTIONS OF FELLER PROCESSES ON LOCALLY COMPACT SPACES

BY RICHARD C. KRAAIJ[1]

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We study the large deviation behaviour of the trajectories of empirical distributions of independent copies of time-homogeneous Feller processes on locally compact metric spaces. Under the condition that we can find a suitable core for the generator of the Feller process, we are able to define a notion of absolutely continuous trajectories of measures in terms of some topology on this core. Also, we define a Hamiltonian in terms of the linear generator and a Lagrangian as its Legendre transform.

We prove the large deviation principle and show that the rate function can be decomposed as a rate function for the initial time and an integral over the Lagrangian, finite only for absolutely continuous trajectories of measures.

We apply this result for diffusion and Lévy processes on $\mathbb{R}^d$, for pure jump processes with bounded jump kernel on arbitrary locally compact spaces and for discrete interacting particle systems. For diffusion processes, the theorem partly extends the Dawson and Gärtner theorem for noninteracting copies in the sense that it only holds for time-homogeneous processes, but on the other hand it holds for processes with degenerate diffusion matrix.

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MSC2010 subject classifications. 60F10, 60J25.

Key words and phrases. Markov processes, large deviation principle, Lagrangian, Doob transform, Nisio semigroup.

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The Annals of Probability
2018, Vol. 46, No. 2, 829–864
https://doi.org/10.1214/17-AOP1193

FREE ENERGY IN THE POTTS SPIN GLASS[1]

BY DMITRY PANCHENKO

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We study the Potts spin glass model, which generalizes the Sherrington–Kirkpatrick model to the case when spins take more than two values but their interactions are counted only if the spins are equal. We obtain the analogue of the Parisi variational formula for the free energy, with the order parameter now given by a monotone path in the set of positive-semidefinite matrices. The main idea of the paper is a novel synchronization mechanism for blocks of overlaps. This mechanism can be used to solve a more general version of the Sherrington–Kirkpatrick model with vector spins interacting through their scalar product, which includes the Potts spin glass as a special case. As another example of application, one can show that Talagrand's bound for multiple copies of the mixed p-spin model with constrained overlaps is asymptotically sharp. We will consider these problems in the subsequent paper and illustrate the main new idea on the technically more transparent case of the Potts spin glass.

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MSC2010 subject classifications. 60K35, 60G15, 60F10, 82B44.
Key words and phrases. Spin glasses, Sherrington–Kirkpatrick model, Potts spin glass, free energy.

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The Annals of Probability
2018, Vol. 46, No. 2, 865–896
https://doi.org/10.1214/17-AOP1194

FREE ENERGY IN THE MIXED p-SPIN MODELS WITH VECTOR SPINS[1]

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Using the synchronization mechanism developed in the previous work on the Potts spin glass model, we obtain the analogue of the Parisi formula for the free energy in the mixed even p-spin models with vector spins, which include the Sherrington–Kirkpatrick model with vector spins interacting through their scalar product. As a special case, this also establishes the sharpness of Talagrand's upper bound for the free energy of multiple mixed p-spin systems coupled by constraining their overlaps.

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The Annals of Probability
2018, Vol. 46, No. 2, 897–955
https://doi.org/10.1214/17-AOP1196

A FRACTIONAL KINETIC PROCESS DESCRIBING THE INTERMEDIATE TIME BEHAVIOUR OF CELLULAR FLOWS[1]

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Dedicated to the memory of Joe Keller

This paper studies the *intermediate time* behaviour of a small random perturbation of a periodic cellular flow. Our main result shows that on time scales shorter than the diffusive time scale, the limiting behaviour of trajectories that start close enough to cell boundaries is a fractional kinetic process: a Brownian motion time changed by the local time of an independent Brownian motion. Our proof uses the Freidlin–Wentzell framework, and the key step is to establish an analogous averaging principle on shorter time scales.

As a consequence of our main theorem, we obtain a homogenization result for the associated advection diffusion equation. We show that on intermediate time scales the effective equation is a *fractional time* PDE that arises in modelling anomalous diffusion.

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MSC2010 subject classifications. 60H10, 60H30, 60F17, 26A33, 35R11, 76R50.
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The Annals of Probability
2018, Vol. 46, No. 2, 956–1003
https://doi.org/10.1214/17-AOP1198

QUASI-SYMMETRIES OF DETERMINANTAL POINT PROCESSES[1]

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The main result of this paper is that determinantal point processes on $\mathbb{R}$ corresponding to projection operators with integrable kernels are quasi-invariant, in the continuous case, under the group of diffeomorphisms with compact support (Theorem 1.4); in the discrete case, under the group of all finite permutations of the phase space (Theorem 1.6). The Radon–Nikodym derivative is computed explicitly and is given by a regularized multiplicative functional. Theorem 1.4 applies, in particular, to the sine-process, as well as to determinantal point processes with the Bessel and the Airy kernels; Theorem 1.6 to the discrete sine-process and the Gamma kernel process. The paper answers a question of Grigori Olshanski.

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MSC2010 subject classifications. Primary 60G55; secondary 22E66, 60B10, 60B15, 33C10.
Key words and phrases. Determinantal point processes, Gibbs property, Palm measures, integrable kernels, multiplicative functionals.

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The Annals of Probability
2018, Vol. 46, No. 2, 1004–1041
https://doi.org/10.1214/17-AOP1199

FIRST-PASSAGE PERCOLATION ON CARTESIAN POWER GRAPHS[1]

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We consider first-passage percolation on the class of "high-dimensional" graphs that can be written as an iterated Cartesian product $G\square G\square \dots \square G$ of some base graph G as the number of factors tends to infinity. We propose a natural asymptotic lower bound on the first-passage time between $(v, v, \dots, v)$ and $(w, w, \dots, w)$ as n, the number of factors, tends to infinity, which we call the critical time $t^*_G(v, w)$. Our main result characterizes when this lower bound is sharp as $n \to \infty$. As a corollary, we are able to determine the limit of the so-called diagonal time-constant in $\mathbb{Z}^n$ as $n \to \infty$ for a large class of distributions of passage times.

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MSC2010 subject classifications. Primary 60C05; secondary 60K35, 82B43.

Key words and phrases. First-passage percolation, power graph, high dimension, time constant, hypercube.

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The Annals of Probability
2018, Vol. 46, No. 2, 1042–1069
https://doi.org/10.1214/17-AOP1200

SPDE LIMIT OF THE GLOBAL FLUCTUATIONS IN RANK-BASED MODELS[1]

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We consider systems of diffusion processes ("particles") interacting through their ranks (also referred to as "rank-based models" in the mathematical finance literature). We show that, as the number of particles becomes large, the process of fluctuations of the empirical cumulative distribution functions converges to the solution of a linear parabolic SPDE with additive noise. The coefficients in the limiting SPDE are determined by the hydrodynamic limit of the particle system which, in turn, can be described by the porous medium PDE. The result opens the door to a thorough investigation of large equity markets and investment therein. In the course of the proof, we also derive quantitative propagation of chaos estimates for the particle system.

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MSC2010 subject classifications. 60H15, 82C22, 91G80.

Key words and phrases. Central limit theorems, Gaussian random fields, fluctuations in interacting particle systems, large equity markets, mean field interaction, porous medium equation, quantitative propagation of chaos estimates, rank-based models, stochastic partial differential equations, stochastic portfolio theory, Wasserstein distances.

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The Annals of Probability
2018, Vol. 46, No. 2, 1070–1113
https://doi.org/10.1214/17-AOP1201

EXPONENTIALLY CONCAVE FUNCTIONS AND A NEW INFORMATION GEOMETRY[1]

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A function is exponentially concave if its exponential is concave. We consider exponentially concave functions on the unit simplex. In a previous paper, we showed that gradient maps of exponentially concave functions provide solutions to a Monge–Kantorovich optimal transport problem and give a better gradient approximation than those of ordinary concave functions. The approximation error, called L-divergence, is different from the usual Bregman divergence. Using tools of information geometry and optimal transport, we show that L-divergence induces a new information geometry on the simplex consisting of a Riemannian metric and a pair of dually coupled affine connections which defines two kinds of geodesics. We show that the induced geometry is dually projectively flat but not flat. Nevertheless, we prove an analogue of the celebrated generalized Pythagorean theorem from classical information geometry. On the other hand, we consider displacement interpolation under a Lagrangian integral action that is consistent with the optimal transport problem and show that the action minimizing curves are dual geodesics. The Pythagorean theorem is also shown to have an interesting application of determining the optimal trading frequency in stochastic portfolio theory.

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MSC2010 subject classifications. Primary 60E05; secondary 52A41.

Key words and phrases. Information geometry, optimal transport, exponential concavity, L-divergence, generalized Pythagorean theorem, functionally generated portfolio, stochastic portfolio theory.

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The Annals of Probability
2018, Vol. 46, No. 2, 1114–1169
https://doi.org/10.1214/17-AOP1202

ON THE CYCLE STRUCTURE OF MALLOWS PERMUTATIONS[1]

BY ALEXEY GLADKICH AND RON PELED

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We study the length of cycles of random permutations drawn from the Mallows distribution. Under this distribution, the probability of a permutation $\pi \in \mathbb{S}_n$ is proportional to $q^{\mathrm{inv}(\pi)}$ where $q > 0$ and $\mathrm{inv}(\pi)$ is the number of inversions in π.

We focus on the case that $q < 1$ and show that the expected length of the cycle containing a given point is of order $\min\{(1-q)^{-2}, n\}$. This marks the existence of two asymptotic regimes: with high probability, when n tends to infinity with $(1-q)^{-2} \ll n$ then all cycles have size $o(n)$ whereas when n tends to infinity with $(1-q)^{-2} \gg n$ then macroscopic cycles, of size proportional to n, emerge. In the second regime, we prove that the distribution of normalized cycle lengths follows the Poisson–Dirichlet law, as in a uniformly random permutation. The results bear formal similarity with a conjectured localization transition for random band matrices.

Further results are presented for the variance of the cycle lengths, the expected diameter of cycles and the expected number of cycles. The proofs rely on the exact sampling algorithm for the Mallows distribution and make use of a special diagonal exposure process for the graph of the permutation.

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The Annals of Probability
2018, Vol. 46, No. 2, 1170–1200
https://doi.org/10.1214/17-AOP1203

INTERLACEMENTS AND THE WIRED UNIFORM SPANNING FOREST

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We extend the Aldous–Broder algorithm to generate the wired uniform spanning forests (WUSFs) of infinite, transient graphs. We do this by replacing the simple random walk in the classical algorithm with Sznitman's random interlacement process. We then apply this algorithm to study the WUSF, showing that every component of the WUSF is one-ended almost surely in any graph satisfying a certain weak anchored isoperimetric condition, that the number of 'excessive ends' in the WUSF is nonrandom in any graph, and also that every component of the WUSF is one-ended almost surely in any transient unimodular random rooted graph. The first two of these results answer positively two questions of Lyons, Morris and Schramm [*Electron. J. Probab.* **13** (2008) 1702–1725], while the third extends a recent result of the author.

Finally, we construct a counterexample showing that almost sure one-endedness of WUSF components is not preserved by rough isometries of the underlying graph, answering negatively a further question of Lyons, Morris and Schramm.

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The Annals of Probability

Vol. 46 **May 2018** **No. 3**

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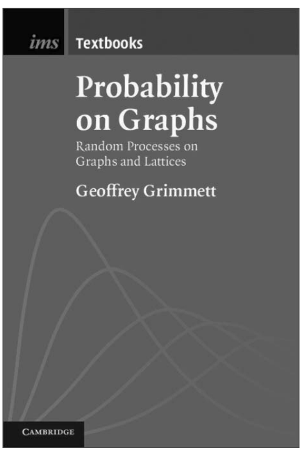

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Geoffrey Grimmett

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