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A STRATONOVICH–SKOROHOD INTEGRAL FORMULA FOR GAUSSIAN ROUGH PATHS

BY THOMAS CASS[1] AND NENGLI LIM[2]

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Given a Gaussian process X, its canonical geometric rough path lift $\mathbf{X}$, and a solution Y to the rough differential equation (RDE) $\mathrm{d}Y_t = V(Y_t) \circ \mathrm{d}\mathbf{X}_t$, we present a closed-form correction formula for $\int Y \circ \mathrm{d}\mathbf{X} - \int Y \,\mathrm{d}X$, that is, the difference between the rough and Skorohod integrals of Y with respect to X. When X is standard Brownian motion, we recover the classical Stratonovich-to-Itô conversion formula, which we generalize to Gaussian rough paths with finite p-variation, $p < 3$, and satisfying an additional natural condition. This encompasses many familiar examples, including fractional Brownian motion with $H > \frac{1}{3}$. To prove the formula, we first show that the Riemann-sum approximants of the Skorohod integral converge in $L^2(\Omega)$ by using a novel characterization of the Cameron–Martin norm in terms of higher-dimensional Young–Stieltjes integrals. Next, we append the approximants of the Skorohod integral with a suitable compensation term without altering the limit, and the formula is finally obtained after a rebalancing of terms.

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Key words and phrases. Rough paths theory, Malliavin calculus, generalized Itô–Stratonovich correction formulas.

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BERRY–ESSEEN BOUNDS OF NORMAL AND NONNORMAL APPROXIMATION FOR UNBOUNDED EXCHANGEABLE PAIRS[1]

BY QI-MAN SHAO AND ZHUO-SONG ZHANG

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An exchangeable pair approach is commonly taken in the normal and nonnormal approximation using Stein's method. It has been successfully used to identify the limiting distribution and provide an error of approximation. However, when the difference of the exchangeable pair is not bounded by a small deterministic constant, the error bound is often not optimal. In this paper, using the exchangeable pair approach of Stein's method, a new Berry–Esseen bound for an arbitrary random variable is established without a bound on the difference of the exchangeable pair. An optimal convergence rate for normal and nonnormal approximation is achieved when the result is applied to various examples including the quadratic forms, general Curie–Weiss model, mean field Heisenberg model and colored graph model.

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Key words and phrases. Stein's method, exchangeable pairs, Berry–Esseen bound, quadratic forms, simple random sampling, general Curie–Weiss model, mean field Heisenberg model, monochromatic edges.

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STRUCTURE OF OPTIMAL MARTINGALE TRANSPORT PLANS IN GENERAL DIMENSIONS

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Given two probability measures μ and ν in "convex order" on $\mathbb{R}^d$, we study the profile of one-step martingale plans π on $\mathbb{R}^d \times \mathbb{R}^d$ that optimize the expected value of the modulus of their increment among all martingales having μ and ν as marginals. While there is a great deal of results for the real line (i.e., when $d = 1$), much less is known in the richer and more delicate higher-dimensional case that we tackle in this paper. We show that many structural results can be obtained, provided the initial measure μ is absolutely continuous with respect to the Lebesgue measure. One such a property is that μ-almost every x in $\mathbb{R}^d$ is transported by the optimal martingale plan into a probability measure π_x concentrated on the extreme points of the closed convex hull of its support. This will be established for the distance cost $c(x, y) = |x - y|$ in the two-dimensional case, and also for any $d \geq 3$ as long as the marginals are in "subharmonic order." In some cases, π_x is supported on the vertices of a $k(x)$-dimensional polytope, such as when the target measure is discrete. Duality plays a crucial role in our approach, even though, in contrast to standard optimal transports, the dual extremal problem may not be attained in general. We show however that "martingale supporting" Borel subsets of $\mathbb{R}^d \times \mathbb{R}^d$ can be decomposed into a collection of mutually disjoint components by means of a "convex paving" of the source space, in such a way that when the martingale is optimal for a general cost function, each of the components then supports a restricted optimal martingale transport whose dual problem is attained. This decomposition is used to obtain structural results in cases where global duality is not attained. On the other hand, it shows that certain "optimal martingale supporting" Borel sets can be viewed as higher-dimensional versions of Nikodym-type sets. The paper focuses on the distance cost, but much of the results hold for general Lipschitz cost functions.

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Key words and phrases. Optimal transport, martingale, Choquet boundary, duality, convex paving.

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REGULARIZATION BY NOISE AND FLOWS OF SOLUTIONS FOR A STOCHASTIC HEAT EQUATION

BY OLEG BUTKOVSKY[1] AND LEONID MYTNIK[2]

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Motivated by the regularization by noise phenomenon for SDEs, we prove existence and uniqueness of the flow of solutions for the non-Lipschitz stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + b\big(u(t,z)\big) + \dot{W}(t,z),$$

where $\dot{W}$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ and b is a bounded measurable function on $\mathbb{R}$. As a byproduct of our proof, we also establish the so-called path-by-path uniqueness for any initial condition in a certain class on the same set of probability one. To obtain these results, we develop a new approach that extends Davie's method (2007) to the context of stochastic partial differential equations.

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BROWNIAN MOTION ON SOME SPACES WITH VARYING DIMENSION[1]

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In this paper, we introduce and study Brownian motion on a class of state spaces with varying dimension. Starting with a concrete case of such state spaces that models a big square with a flag pole, we construct a Brownian motion on it and study how heat propagates on such a space. We derive sharp two-sided global estimates on its transition density function (also called heat kernel). These two-sided estimates are of Gaussian type, but the measure on the underlying state space does not satisfy volume doubling property. Parabolic Harnack inequality fails for such a process. Nevertheless, we show Hölder regularity holds for its parabolic functions. We also derive the Green function estimates for this process on bounded smooth domains. Brownian motion on some other state spaces with varying dimension are also constructed and studied in this paper.

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RÉNYI DIVERGENCE AND THE CENTRAL LIMIT THEOREM

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We explore properties of the χ^2 and Rényi distances to the normal law and in particular propose necessary and sufficient conditions under which these distances tend to zero in the central limit theorem (with exact rates with respect to the increasing number of summands).

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TOWARDS A UNIVERSALITY PICTURE FOR THE RELAXATION TO EQUILIBRIUM OF KINETICALLY CONSTRAINED MODELS[1]

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Recent years have seen a great deal of progress in our understanding of bootstrap percolation models, a particular class of *monotone cellular automata*. In the two-dimensional lattice $\mathbb{Z}^2$, there is now a quite satisfactory understanding of their evolution starting from a random initial condition, with a strikingly beautiful universality picture for their critical behavior. Much less is known for their nonmonotone stochastic counterpart, namely *kinetically constrained models* (KCM). In KCM, each vertex is resampled (independently) at rate one by tossing a p-coin iff it can be infected in the next step by the bootstrap model. In particular, an infection can also heal, hence the nonmonotonicity. Besides the connection with bootstrap percolation, KCM have an interest in their own as they feature some of the most striking features of the liquid/glass transition, a major and still largely open problem in condensed matter physics. In this paper, we pave the way towards proving universality results for the characteristic time scales of KCM. Our novel and general approach gives the right tools to establish a close connection between the critical scaling of characteristic time scales for KCM and the scaling of the critical length in critical bootstrap models. When applied to the Fredrickson–Andersen k-facilitated models in dimension $d \geq 2$, among the most studied KCM, and to the Gravner–Griffeath model, our results are close to optimal.

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THE SPECTRAL GAP OF DENSE RANDOM REGULAR GRAPHS

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For any $\alpha \in (0, 1)$ and any $n^\alpha \leq d \leq n/2$, we show that $\lambda(\mathbf{G}) \leq C_\alpha \sqrt{d}$ with probability at least $1 - \frac{1}{n}$, where $\mathbf{G}$ is the uniform random undirected d-regular graph on n vertices, $\lambda(\mathbf{G})$ denotes its second largest eigenvalue (in absolute value) and C_α is a constant depending only on α. Combined with earlier results in this direction covering the case of sparse random graphs, this completely settles the problem of estimating the magnitude of $\lambda(\mathbf{G})$, up to a multiplicative constant, for all values of n and d, confirming a conjecture of Vu. The result is obtained as a consequence of an estimate for the second largest singular value of adjacency matrices of random *directed* graphs with predefined degree sequences. As the main technical tool, we prove a concentration inequality for arbitrary linear forms on the space of matrices, where the probability measure is induced by the adjacency matrix of a random directed graph with prescribed degree sequences. The proof is a nontrivial application of the Freedman inequality for martingales, combined with self-bounding and tensorization arguments. Our method bears considerable differences compared to the approach used by Broder et al. [*SIAM J. Comput.* **28** (1999) 541–573] who established the upper bound for $\lambda(\mathbf{G})$ for $d = o(\sqrt{n})$, and to the argument of Cook, Goldstein and Johnson [*Ann. Probab.* **46** (2018) 72–125] who derived a concentration inequality for linear forms and estimated $\lambda(\mathbf{G})$ in the range $d = O(n^{2/3})$ using size-biased couplings.

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CANONICAL RDES AND GENERAL SEMIMARTINGALES AS ROUGH PATHS

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In the spirit of Marcus canonical stochastic differential equations, we study a similar notion of rough differential equations (RDEs), notably dropping the assumption of continuity prevalent in the rough path literature. A new metric is exhibited in which the solution map is a continuous function of the driving rough path *and* a so-called path function, which directly models the effect of the jump on the system. In a second part, we show that general multidimensional semimartingales admit canonically defined rough path lifts. An extension of Lépingle's BDG inequality to this setting is given, and in turn leads to a number of novel limit theorems for semimartingale driven differential equations, both in law and in probability, conveniently phrased a uniformly-controlled-variations (UCV) condition (Kurtz–Protter, Jakubowski–Mémin–Pagès). A number of examples illustrate the scope of our results.

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RATE OF CONVERGENCE TO EQUILIBRIUM OF FRACTIONAL DRIVEN STOCHASTIC DIFFERENTIAL EQUATIONS WITH ROUGH MULTIPLICATIVE NOISE

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We investigate the problem of the rate of convergence to equilibrium for ergodic stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H \in (1/3, 1)$ and multiplicative noise component σ. When σ is constant and for every $H \in (0, 1)$, it was proved in [*Ann. Probab.* **33** (2005) 703–758] that, under some mean-reverting assumptions, such a process converges to its equilibrium at a rate of order $t^{-\alpha}$ where $\alpha \in (0, 1)$ (depending on H). In [*Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 503–538], this result has been extended to the multiplicative case when $H > 1/2$. In this paper, we obtain these types of results in the rough setting $H \in (1/3, 1/2)$. Once again, we retrieve the rate orders of the additive setting. Our methods also extend the multiplicative results of [*Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 503–538] by deleting the gradient assumption on the noise coefficient σ. The main theorems include some existence and uniqueness results for the invariant distribution.

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GLOBAL SOLUTIONS TO STOCHASTIC REACTION–DIFFUSION EQUATIONS WITH SUPER-LINEAR DRIFT AND MULTIPLICATIVE NOISE

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Let $\xi(t,x)$ denote space–time white noise and consider a reaction–diffusion equation of the form

$$\dot{u}(t,x) = \frac{1}{2}u''(t,x) + b\big(u(t,x)\big) + \sigma\big(u(t,x)\big)\xi(t,x),$$

on $\mathbb{R}_+ \times [0,1]$, with homogeneous Dirichlet boundary conditions and suitable initial data, in the case that there exists $\varepsilon > 0$ such that $|b(z)| \geq |z|(\log|z|)^{1+\varepsilon}$ for all sufficiently-large values of $|z|$. When $\sigma \equiv 0$, it is well known that such PDEs frequently have nontrivial stationary solutions. By contrast, Bonder and Groisman [*Phys. D* **238** (2009) 209–215] have recently shown that there is finite-time blowup when σ is a nonzero constant. In this paper, we prove that the Bonder–Groisman condition is unimprovable by showing that the reaction–diffusion equation with noise is "typically" well posed when $|b(z)| = O(|z|\log_+|z|)$ as $|z| \to \infty$. We interpret the word "typically" in two essentially-different ways without altering the conclusions of our assertions.

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SHARP INTERFACE LIMIT FOR STOCHASTICALLY PERTURBED MASS CONSERVING ALLEN–CAHN EQUATION

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This paper studies the sharp interface limit for a mass conserving Allen–Cahn equation, added an external noise and derives a stochastically perturbed mass conserving mean curvature flow in the limit. The stochastic term destroys the precise conservation law, instead the total mass changes like a Brownian motion in time. For our equation, the comparison argument does not work, so that to study the limit we adopt the asymptotic expansion method, which extends that for deterministic equations used originally in de Mottoni and Schatzman [*Interfaces Free Bound.* **12** (2010) 527–549] for the nonconservative case and then in Chen et al. [*Trans. Amer. Math. Soc.* **347** (1995) 1533–1589] for the conservative case. Differently from the deterministic case, each term except the leading term appearing in the expansion of the solution in a small parameter ε diverges as ε tends to 0, since our equation contains the noise which converges to a white noise and the products or the powers of the white noise diverge. To derive the error estimate for our asymptotic expansion, we need to establish the Schauder estimate for a diffusion operator with coefficients determined from higher order derivatives of the noise and their powers. We show that one can choose the noise sufficiently mild in such a manner that it converges to the white noise and at the same time its diverging speed is slow enough for establishing a necessary error estimate.

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