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GENEALOGICAL CONSTRUCTIONS OF POPULATION MODELS

BY ALISON M. ETHERIDGE[1] AND THOMAS G. KURTZ[2]

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Representations of population models in terms of countable systems of particles are constructed, in which each particle has a "type," typically recording both spatial position and genetic type, and a level. For finite intensity models, the levels are distributed on $[0, \lambda]$, whereas in the infinite intensity limit $\lambda \to \infty$, at each time t, the joint distribution of types and levels is conditionally Poisson, with mean measure $\Xi(t) \times \ell$ where ℓ denotes Lebesgue measure and $\Xi(t)$ is a measure-valued population process. The time-evolution of the levels captures the genealogies of the particles in the population.

Key forces of ecology and genetics can be captured within this common framework. Models covered incorporate both individual and event based births and deaths, one-for-one replacement, immigration, independent "thinning" and independent or exchangeable spatial motion and mutation of individuals. Since birth and death probabilities can depend on type, they also include natural selection. The primary goal of the paper is to present particle-with-level or lookdown constructions for each of these elements of a population model. Then the elements can be combined to specify the desired model. In particular, a nontrivial extension of the spatial Λ-Fleming–Viot process is constructed.

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INTERMITTENCY FOR THE STOCHASTIC HEAT EQUATION WITH LÉVY NOISE

BY CARSTEN CHONG[*,1] AND PÉTER KEVEI[*,†,2]

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We investigate the moment asymptotics of the solution to the stochastic heat equation driven by a $(d+1)$-dimensional Lévy space-time white noise. Unlike the case of Gaussian noise, the solution typically has no finite moments of order $1+2/d$ or higher. Intermittency of order p, that is, the exponential growth of the pth moment as time tends to infinity, is established in dimension $d=1$ for all values $p \in (1,3)$, and in higher dimensions for some $p \in (1, 1+2/d)$. The proof relies on a new moment lower bound for stochastic integrals against compensated Poisson measures. The behavior of the intermittency exponents when $p \to 1+2/d$ further indicates that intermittency in the presence of jumps is much stronger than in equations with Gaussian noise. The effect of other parameters like the diffusion constant or the noise intensity on intermittency will also be analyzed in detail.

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MSC2010 subject classifications. Primary 60H15, 37H15; secondary 60G51, 35B40.

Key words and phrases. Comparison principle, intermittency, intermittency fronts, Lévy noise, moment Lyapunov exponents, stochastic heat equation, stochastic PDE.

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UNIQUENESS OF GIBBS MEASURES FOR CONTINUOUS HARDCORE MODELS

BY DAVID GAMARNIK[1] AND KAVITA RAMANAN[2]

Massachusetts Institute of Technology and Brown University

We formulate a continuous version of the well-known discrete hardcore (or independent set) model on a locally finite graph, parameterized by the so-called activity parameter $\lambda > 0$. In this version the state or "spin value" x_u of any node u of the graph lies in the interval $[0, 1]$, the hardcore constraint $x_u + x_v \leq 1$ is satisfied for every edge (u, v) of the graph, and the space of feasible configurations is given by a convex polytope. When the graph is a regular tree, we show that there is a unique Gibbs measure associated to each activity parameter $\lambda > 0$. Our result shows that, in contrast to the standard discrete hardcore model, the continuous hardcore model does not exhibit a phase transition on the infinite regular tree. We also consider a family of continuous models that interpolate between the discrete and continuous hardcore models on a regular tree when $\lambda = 1$ and show that each member of the family has a unique Gibbs measure, even when the discrete model does not. In each case the proof entails the analysis of an associated Hamiltonian dynamical system that describes a certain limit of the marginal distribution at a node. Furthermore, given any sequence of regular graphs with fixed degree and girth diverging to infinity, we apply our results to compute the asymptotic limit of suitably normalized volumes of the corresponding sequence of convex polytopes of feasible configurations. In particular this yields an approximation for the partition function of the continuous hard core model on a regular graph with large girth in the case $\lambda = 1$.

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MSC2010 subject classifications. Primary 60K35, 82B820; secondary 82B27, 68W25.

Key words and phrases. Hardcore model, independent set, Gibbs measures, phase transition, partition function, linear programming polytope, volume computation, convex polytope, computational hardness, regular graphs.

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COUPLINGS AND QUANTITATIVE CONTRACTION RATES FOR LANGEVIN DYNAMICS[1]

BY ANDREAS EBERLE*, ARNAUD GUILLIN† AND RAPHAEL ZIMMER*

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We introduce a new probabilistic approach to quantify convergence to equilibrium for (kinetic) Langevin processes. In contrast to previous analytic approaches that focus on the associated kinetic Fokker–Planck equation, our approach is based on a specific combination of reflection and synchronous coupling of two solutions of the Langevin equation. It yields contractions in a particular Wasserstein distance, and it provides rather precise bounds for convergence to equilibrium at the borderline between the overdamped and the underdamped regime. In particular, we are able to recover kinetic behaviour in terms of explicit lower bounds for the contraction rate. For example, for a rescaled double-well potential with local minima at distance a, we obtain a lower bound for the contraction rate of order $\Omega(a^{-1})$ provided the friction coefficient is of order $\Theta(a^{-1})$.

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Key words and phrases. Langevin diffusion, kinetic Fokker–Planck equation, stochastic Hamiltonian dynamics, reflection coupling, convergence to equilibrium, hypocoercivity, quantitative bounds, Wasserstein distance, Lyapunov functions.

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POLY-LOGARITHMIC LOCALIZATION FOR RANDOM WALKS AMONG RANDOM OBSTACLES[1]

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Place an obstacle with probability $1 - \mathsf{p}$ independently at each vertex of $\mathbb{Z}^d$, and run a simple random walk until hitting one of the obstacles. For $d \geq 2$ and p strictly above the critical threshold for site percolation, we condition on the environment where the origin is contained in an infinite connected component free of obstacles, and we show that the following *path localization* holds for environments with probability tending to 1 as $n \to \infty$: conditioned on survival up to time n we have that ever since $o(n)$ steps the simple random walk is localized in a region of volume poly-logarithmic in n with probability tending to 1. The previous best result of this type went back to Sznitman (1996) on Brownian motion among Poisson obstacles, where a localization (only for the end point) in a region of volume $t^{o(1)}$ was derived conditioned on the survival of Brownian motion up to time t.

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THE SCALING LIMIT OF CRITICAL ISING INTERFACES IS CLE_3

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In this paper, we consider the set of interfaces between + and − spins arising for the critical planar Ising model on a domain with + boundary conditions, and show that it converges to nested CLE_3.

Our proof relies on the study of the coupling between the Ising model and its random cluster (FK) representation, and of the interactions between FK and Ising interfaces. The main idea is to construct an exploration process starting from the boundary of the domain, to discover the Ising loops and to establish its convergence to a conformally invariant limit. The challenge is that Ising loops do not touch the boundary; we use the fact that FK loops touch the boundary (and hence can be explored from the boundary) and that Ising loops in turn touch FK loops, to construct a recursive exploration process that visits all the macroscopic loops.

A key ingredient in the proof is the convergence of Ising free arcs to the Free Arc Ensemble (FAE), established in [*Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2016) 1784–1798]. Qualitative estimates about the Ising interfaces then allow one to identify the scaling limit of Ising loops as a conformally invariant collection of simple, disjoint SLE_3-like loops, and thus by the Markovian characterization of Sheffield and Werner [*Ann. of Math. (2)* **176** (2012) 1827–1917] as a CLE_3.

A technical point of independent interest contained in this paper is an investigation of double points of interfaces in the scaling limit of critical FK-Ising. It relies on the technology of Kemppainen and Smirnov [*Ann. Probab.* **45** (2017) 698–779].

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A SOBOLEV SPACE THEORY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH TIME-FRACTIONAL DERIVATIVES

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In this article, we present an L_p-theory ($p \geq 2$) for the semi-linear stochastic partial differential equations (SPDEs) of type

$$\partial_t^\alpha u = L(\omega, t, x)u + f(u) + \partial_t^\beta \sum_{k=1}^{\infty} \int_0^t \left(\Lambda^k(\omega, t, x)u + g^k(u)\right) dw_t^k,$$

where $\alpha \in (0, 2)$, $\beta < \alpha + \frac{1}{2}$ and ∂_t^α and ∂_t^β denote the Caputo derivatives of order α and β, respectively. The processes w_t^k, $k \in \mathbb{N} = \{1, 2, \ldots\}$, are independent one-dimensional Wiener processes, L is either divergence or nondivergence-type second-order operator, and Λ^k are linear operators of order up to two. This class of SPDEs can be used to describe random effects on transport of particles in medium with thermal memory or particles subject to sticking and trapping.

We prove uniqueness and existence results of strong solutions in appropriate Sobolev spaces, and obtain maximal L_p-regularity of the solutions. By converting SPDEs driven by d-dimensional space–time white noise into the equations of above type, we also obtain an L_p-theory for SPDEs driven by space–time white noise if the space dimension $d < 4 - 2(2\beta - 1)\alpha^{-1}$. In particular, if $\beta < 1/2 + \alpha/4$ then we can handle space–time white noise driven SPDEs with space dimension $d = 1, 2, 3$.

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A GENERAL METHOD FOR LOWER BOUNDS ON FLUCTUATIONS OF RANDOM VARIABLES[1]

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There are many ways of establishing upper bounds on fluctuations of random variables, but there is no systematic approach for lower bounds. As a result, lower bounds are unknown in many important problems. This paper introduces a general method for lower bounds on fluctuations. The method is used to obtain new results for the stochastic traveling salesman problem, the stochastic minimal matching problem, the random assignment problem, the Sherrington–Kirkpatrick model of spin glasses, first-passage percolation and random matrices. A long list of open problems is provided at the end.

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STEIN KERNELS AND MOMENT MAPS

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We describe a construction of Stein kernels using moment maps, which are solutions to a variant of the Monge–Ampère equation. As a consequence, we show how regularity bounds in certain weighted Sobolev spaces on these maps control the rate of convergence in the classical central limit theorem, and derive new rates in Kantorovitch–Wasserstein distance in the log-concave situation, with explicit polynomial dependence on the dimension.

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LARGE DEVIATIONS AND WANDERING EXPONENT FOR RANDOM WALK IN A DYNAMIC BETA ENVIRONMENT[1]

BY MÁRTON BALÁZS[2], FIRAS RASSOUL-AGHA[3] AND TIMO SEPPÄLÄINEN[4]

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Random walk in a dynamic i.i.d. beta random environment, conditioned to escape at an atypical velocity, converges to a Doob transform of the original walk. The Doob-transformed environment is correlated in time, i.i.d. in space and its marginal density function is a product of a beta density and a hypergeometric function. Under its averaged distribution, the transformed walk obeys the wandering exponent $2/3$ that agrees with Kardar–Parisi–Zhang universality. The harmonic function in the Doob transform comes from a Busemann-type limit and appears as an extremal in a variational problem for the quenched large deviation rate function.

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THOULESS–ANDERSON–PALMER EQUATIONS FOR GENERIC p-SPIN GLASSES

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We study the Thouless–Anderson–Palmer (TAP) equations for spin glasses on the hypercube. First, using a random, approximately ultrametric decomposition of the hypercube, we decompose the Gibbs measure, $\langle\cdot\rangle_N$, into a mixture of conditional laws, $\langle\cdot\rangle_{\alpha,N}$. We show that the TAP equations hold for the spin at any site with respect to $\langle\cdot\rangle_{\alpha,N}$ simultaneously for all α. This result holds for generic models provided that the Parisi measure of the model has a jump at the top of its support.

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THE STRUCTURE OF EXTREME LEVEL SETS IN BRANCHING BROWNIAN MOTION

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We study the structure of extreme level sets of a standard one-dimensional branching Brownian motion, namely the sets of particles whose height is within a fixed distance from the order of the global maximum. It is well known that such particles congregate at large times in clusters of order-one genealogical diameter around local maxima which form a Cox process in the limit. We add to these results by finding the asymptotic size of extreme level sets and the typical height of the local maxima whose clusters carry such level sets. We also find the right tail decay of the distribution of the distance between the two highest particles. These results confirm two conjectures of Brunet and Derrida (*J. Stat. Phys.* **143** (2011) 420–446). The proofs rely on a careful study of the cluster distribution.

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METRIC GLUING OF BROWNIAN AND $\sqrt{8/3}$-LIOUVILLE QUANTUM GRAVITY SURFACES

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In a recent series of works, Miller and Sheffield constructed a metric on $\sqrt{8/3}$-Liouville quantum gravity (LQG) under which $\sqrt{8/3}$-LQG surfaces (e.g., the LQG sphere, wedge, cone and disk) are isometric to their Brownian surface counterparts (e.g., the Brownian map, half-plane, plane and disk).

We identify the metric gluings of certain collections of independent $\sqrt{8/3}$-LQG surfaces with boundaries identified together according to LQG length along their boundaries. Our results imply in particular that the metric gluing of two independent instances of the Brownian half-plane along their positive boundaries is isometric to a certain LQG wedge decorated by an independent chordal $\mathrm{SLE}_{8/3}$ curve. If one identifies the two sides of the boundary of a single Brownian half-plane, one obtains a certain LQG cone decorated by an independent whole-plane $\mathrm{SLE}_{8/3}$. If one identifies the entire boundaries of two Brownian half-planes, one obtains a different LQG cone and the interface between them is a two-sided variant of whole-plane $\mathrm{SLE}_{8/3}$.

Combined with another work of the authors, the present work identifies the scaling limit of self-avoiding walk on random quadrangulations with $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG.

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THE CIRCULAR LAW FOR SPARSE NON-HERMITIAN MATRICES

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For a class of sparse random matrices of the form $A_n = (\xi_{i,j}\delta_{i,j})_{i,j=1}^n$, where $\{\xi_{i,j}\}$ are i.i.d. centered sub-Gaussian random variables of unit variance, and $\{\delta_{i,j}\}$ are i.i.d. Bernoulli random variables taking value 1 with probability p_n, we prove that the empirical spectral distribution of $A_n/\sqrt{np_n}$ converges weakly to the circular law, in probability, for all p_n such that $p_n = \omega(\log^2 n/n)$. Additionally if p_n satisfies the inequality $np_n > \exp(c\sqrt{\log n})$ for some constant c, then the above convergence is shown to hold almost surely. The key to this is a new bound on the smallest singular value of complex shifts of real valued sparse random matrices. The circular law limit also extends to the adjacency matrix of a directed Erdős–Rényi graph with edge connectivity probability p_n.

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STRONG CONVERGENCE OF EIGENANGLES AND EIGENVECTORS FOR THE CIRCULAR UNITARY ENSEMBLE

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It is known that a unitary matrix can be decomposed into a product of complex reflections, one for each dimension, and that these reflections are independent and uniformly distributed on the space where they live if the initial matrix is Haar-distributed. If we take an infinite sequence of such reflections, and consider their successive products, then we get an infinite sequence of unitary matrices of increasing dimension, all of them following the circular unitary ensemble.

In this coupling, we show that the eigenvalues of the matrices converge almost surely to the eigenvalues of the flow, which are distributed according to a sine-kernel point process, and we get some estimates of the rate of convergence. Moreover, we also prove that the eigenvectors of the matrices converge almost surely to vectors which are distributed as Gaussian random fields on a countable set.

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ON MACROSCOPIC HOLES IN SOME SUPERCRITICAL STRONGLY DEPENDENT PERCOLATION MODELS

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We consider $\mathbb{Z}^d$, $d \geq 3$. We investigate the vacant set $\mathcal{V}^u$ of random interlacements in the strongly percolative regime, the vacant set $\mathcal{V}$ of the simple random walk and the excursion set $E^{\geq \alpha}$ of the Gaussian free field in the strongly percolative regime. We consider the large deviation probability that the adequately thickened component of the boundary of a large box centered at the origin in the respective vacant sets or excursion set leaves in the box a macroscopic volume in its complement. We derive asymptotic upper and lower exponential bounds for theses large deviation probabilities. We also derive geometric information on the shape of the left-out volume. It is plausible, but open at the moment, that certain critical levels coincide, both in the case of random interlacements and of the Gaussian free field. If this holds true, the asymptotic upper and lower bounds that we obtain are matching in principal order for all three models, and the macroscopic holes are nearly spherical. We heavily rely on the recent work by Maximilian Nitzschner (2018) and the author for the coarse graining procedure, which we employ in the derivation of the upper bounds.

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INVARIANT MEASURE FOR RANDOM WALKS ON ERGODIC ENVIRONMENTS ON A STRIP

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Environment viewed from the particle is a powerful method of analyzing random walks (RW) in random environment (RE). It is well known that in this setting the environment process is a Markov chain on the set of environments. We study the fundamental question of existence of the density of the invariant measure of this Markov chain with respect to the measure on the set of environments for RW on a strip. We first describe all positive subexponentially growing solutions of the corresponding invariant density equation in the deterministic setting and then derive necessary and sufficient conditions for the existence of the density when the environment is ergodic in both the transient and the recurrent regimes. We also provide applications of our analysis to the question of positive and null recurrence, the study of the Green functions and to random walks on orbits of a dynamical system.

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EXTREMAL THEORY FOR LONG RANGE DEPENDENT INFINITELY DIVISIBLE PROCESSES

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We prove limit theorems of an entirely new type for certain long memory regularly varying stationary infinitely divisible random processes. These theorems involve multiple phase transitions governed by how long the memory is. Apart from one regime, our results exhibit limits that are not among the classical extreme value distributions. Restricted to the one-dimensional case, the distributions we obtain interpolate, in the appropriate parameter range, the α-Fréchet distribution and the skewed α-stable distribution. In general, the limit is a new family of stationary and self-similar random sup-measures with parameters $\alpha \in (0, \infty)$ and $\beta \in (0, 1)$, with representations based on intersections of independent β-stable regenerative sets. The tail of the limit random sup-measure on each interval with finite positive length is regularly varying with index $-\alpha$. The intriguing structure of these random sup-measures is due to intersections of independent β-stable regenerative sets and the fact that the number of such sets intersecting simultaneously increases to infinity as β increases to one. The results in this paper extend substantially previous investigations where only $\alpha \in (0, 2)$ and $\beta \in (0, 1/2)$ have been considered.

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DENSITY OF THE SET OF PROBABILITY MEASURES WITH THE MARTINGALE REPRESENTATION PROPERTY

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Let ψ be a multidimensional random variable. We show that the set of probability measures $\mathbb{Q}$ such that the $\mathbb{Q}$-martingale $S_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$ has the Martingale Representation Property (MRP) is either empty or dense in $\mathcal{L}_\infty$-norm. The proof is based on a related result involving analytic fields of terminal conditions $(\psi(x))_{x \in U}$ and probability measures $(\mathbb{Q}(x))_{x \in U}$ over an open set U. Namely, we show that the set of points $x \in U$ such that $S_t(x) = \mathbb{E}^{\mathbb{Q}(x)}[\psi(x)|\mathcal{F}_t]$ does not have the MRP, either coincides with U or has Lebesgue measure zero. Our study is motivated by the problem of endogenous completeness in financial economics.

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ON THE DIMENSION OF BERNOULLI CONVOLUTIONS

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The Bernoulli convolution with parameter $\lambda \in (0, 1)$ is the probability measure μ_λ that is the law of the random variable $\sum_{n\geq 0} \pm\lambda^n$, where the signs are independent unbiased coin tosses.

We prove that each parameter $\lambda \in (1/2, 1)$ with $\dim \mu_\lambda < 1$ can be approximated by algebraic parameters $\eta \in (1/2, 1)$ within an error of order $\exp(-\deg(\eta)^A)$ such that $\dim \mu_\eta < 1$, for any number A. As a corollary, we conclude that $\dim \mu_\lambda = 1$ for each of $\lambda = \ln 2, e^{-1/2}, \pi/4$. These are the first explicit examples of such transcendental parameters. Moreover, we show that Lehmer's conjecture implies the existence of a constant $a < 1$ such that $\dim \mu_\lambda = 1$ for all $\lambda \in (a, 1)$.

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