ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 48, No. 1—January 2020

ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Vol. 48, No. 1—January 2020

THE ANNALS OF PROBABILITY Vol. 48, No. 1, pp. 1–525 January 2020

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The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 48, Number 1, January 2020. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

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The Annals of Probability
2020, Vol. 48, No. 1, 1–52
https://doi.org/10.1214/18-AOP1326
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DIMERS AND IMAGINARY GEOMETRY

BY NATHANAËL BERESTYCKI[1], BENOÎT LASLIER[2] AND GOURAB RAY[3]

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We show that the winding of the branches in a uniform spanning tree on a planar graph converge in the limit of fine mesh size to a Gaussian free field. The result holds assuming only convergence of simple random walk to Brownian motion and a Russo–Seymour–Welsh type crossing estimate, thereby establishing a strong form of universality. As an application, we prove universality of the fluctuations of the height function associated to the dimer model, in several situations.

The proof relies on a connection to imaginary geometry, where the scaling limit of a uniform spanning tree is viewed as a set of flow lines associated to a Gaussian free field. In particular, we obtain an explicit construction of the a.s. unique Gaussian free field coupled to a continuum uniform spanning tree in this way, which is of independent interest.

REFERENCES

[1] BERESTYCKI, N. (2015). Diffusion in planar Liouville quantum gravity. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 947–964. MR3365969 https://doi.org/10.1214/14-AIHP605

[2] BERESTYCKI, N. (2015). Introduction to the Gaussian free field and Liouville quantum gravity. Lecture notes, available on author's webpage.

[3] BERESTYCKI, N., LASLIER, B. and RAY, G. (2016). A note on dimers and T-graphs. Available at https://arxiv.org/abs/1610.07994.

[4] BERESTYCKI, N., LASLIER, B. and RAY, G. (2020). Supplement to "Dimers and imaginary geometry." https://doi.org/10.1214/18-AOP1326SUPP.

[5] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[6] BUFETOV, A. and GORIN, V. (2018). Fluctuations of particle systems determined by Schur generating functions. *Adv. Math.* **338** 702–781. MR3861715 https://doi.org/10.1016/j.aim.2018.07.009

[7] CHELKAK, D. and SMIRNOV, S. (2011). Discrete complex analysis on isoradial graphs. *Adv. Math.* **228** 1590–1630. MR2824564 https://doi.org/10.1016/j.aim.2011.06.025

[8] COHN, H., KENYON, R. and PROPP, J. (2001). A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** 297–346. MR1815214 https://doi.org/10.1090/S0894-0347-00-00355-6

[9] DE TILIÈRE, B. (2007). Scaling limit of isoradial dimer models and the case of triangular quadri-tilings. *Ann. Inst. Henri Poincaré Probab. Stat.* **43** 729–750. MR3252428

[10] DUBÉDAT, J. (2009). SLE and the free field: Partition functions and couplings. *J. Amer. Math. Soc.* **22** 995–1054. MR2525778 https://doi.org/10.1090/S0894-0347-09-00636-5

[11] FIELD, L. S. and LAWLER, G. F. (2013). Reversed radial SLE and the Brownian loop measure. *J. Stat. Phys.* **150** 1030–1062. MR3038676 https://doi.org/10.1007/s10955-013-0729-5

[12] GARBAN, C. (2013). Quantum gravity and the KPZ formula [after Duplantier–Sheffield]. *Astérisque* **352** 315–354. MR3087350

[13] GARBAN, C., RHODES, R. and VARGAS, V. (2016). Liouville Brownian motion. *Ann. Probab.* **44** 3076–3110. MR3531686 https://doi.org/10.1214/15-AOP1042

[14] GIULIANI, A., MASTROPIETRO, V. and TONINELLI, F. L. (2017). Haldane relation for interacting dimers. *J. Stat. Mech. Theory Exp.* **3** 034002. MR3637384 https://doi.org/10.1088/1742-5468/aa5d1f

[15] GIULIANI, A., MASTROPIETRO, V. and TONINELLI, F. L. (2017). Height fluctuations in interacting dimers. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 98–168. MR3606736 https://doi.org/10.1214/15-AIHP710

MSC2010 subject classifications. 60B05, 60B99.
Key words and phrases. Dimer model, imaginary geometry, uniform spanning tree, SLE, Gaussian free field.

[16] JOCKUSCH, W., PROPP, J. and SHOR, P. (1998). Random domino tilings and the arctic circle theorem. arXiv preprint. Available at arXiv:math/9801068.

[17] KASTELEYN, P. W. (1961). The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice. *Physica* **27** 1209–1225.

[18] KENYON, R. (2000). Conformal invariance of domino tiling. *Ann. Probab.* **28** 759–795. MR1782431 https://doi.org/10.1214/aop/1019160260

[19] KENYON, R. (2001). Dominos and the Gaussian free field. *Ann. Probab.* **29** 1128–1137. MR1872739 https://doi.org/10.1214/aop/1015345599

[20] KENYON, R. (2002). The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.* **150** 409–439. MR1933589 https://doi.org/10.1007/s00222-002-0249-4

[21] KENYON, R. (2008). Height fluctuations in the honeycomb dimer model. *Comm. Math. Phys.* **281** 675–709. MR2415464 https://doi.org/10.1007/s00220-008-0511-8

[22] KENYON, R. (2009). Lectures on dimers. In *Statistical Mechanics. IAS/Park City Math. Ser.* **16** 191–230. Amer. Math. Soc., Providence, RI. MR2523460

[23] KENYON, R., MILLER, J., SHEFFIELD, S. and WILSON, D. B. (2017). Six-vertex model and Schramm–Loewner evolution. *Phys. Rev. E* **95** 052146.

[24] KENYON, R. and OKOUNKOV, A. (2006). Planar dimers and Harnack curves. *Duke Math. J.* **131** 499–524. MR2219249 https://doi.org/10.1215/S0012-7094-06-13134-4

[25] KENYON, R., OKOUNKOV, A. and SHEFFIELD, S. (2006). Dimers and amoebae. *Ann. of Math.* (2) **163** 1019–1056. MR2215138 https://doi.org/10.4007/annals.2006.163.1019

[26] KENYON, R. W., PROPP, J. G. and WILSON, D. B. (2000). Trees and matchings. *Electron. J. Combin.* **7** 25. MR1756162

[27] KENYON, R. W. and SHEFFIELD, S. (2004). Dimers, tilings and trees. *J. Combin. Theory Ser. B* **92** 295–317. MR2099145 https://doi.org/10.1016/j.jctb.2004.07.001

[28] LASLIER, B. (2013). Central limit theorem for T-graphs. arXiv preprint. Available at arXiv:1312.3177.

[29] LAWLER, G. F. (2005). *Conformally Invariant Processes in the Plane. Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588

[30] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2004). Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32** 939–995. MR2044671 https://doi.org/10.1214/aop/1079021469

[31] LI, Z. (2017). Conformal invariance of dimer heights on isoradial double graphs. *Ann. Inst. Henri Poincaré D* **4** 273–307. MR3713018 https://doi.org/10.4171/AIHPD/41

[32] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[33] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[34] MILLER, J. and SHEFFIELD, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2

[35] PEMANTLE, R. (1991). Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.* **19** 1559–1574. MR1127715

[36] PETROV, L. (2015). Asymptotics of uniformly random lozenge tilings of polygons. Gaussian free field. *Ann. Probab.* **43** 1–43. MR3298467 https://doi.org/10.1214/12-AOP823

[37] POMMERENKE, C. (1992). *Boundary Behaviour of Conformal Maps. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **299**. Springer, Berlin. MR1217706 https://doi.org/10.1007/978-3-662-02770-7

[38] ROMIK, D. (2012). Arctic circles, domino tilings and square Young tableaux. *Ann. Probab.* **40** 611–647. MR2952086 https://doi.org/10.1214/10-AOP628

[39] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084 https://doi.org/10.1007/BF02803524

[40] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[41] TEMPERLEY, H. N. V. and FISHER, M. E. (1961). Dimer problem in statistical mechanics—an exact result. *Philos. Mag.* **6** 1061–1063. MR0136398

[42] THURSTON, W. P. (1990). Conway's tiling groups. *Amer. Math. Monthly* **97** 757–773. MR1072815 https://doi.org/10.2307/2324578

[43] YADIN, A. and YEHUDAYOFF, A. (2011). Loop-erased random walk and Poisson kernel on planar graphs. *Ann. Probab.* **39** 1243–1285. MR2857239 https://doi.org/10.1214/10-AOP579

The Annals of Probability
2020, Vol. 48, No. 1, 53–93
https://doi.org/10.1214/19-AOP1345
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ON A PERTURBATION THEORY AND ON STRONG CONVERGENCE RATES FOR STOCHASTIC ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS WITH NONGLOBALLY MONOTONE COEFFICIENTS

BY MARTIN HUTZENTHALER[1] AND ARNULF JENTZEN[2]

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We develop a perturbation theory for stochastic differential equations (SDEs) by which we mean both stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs). In particular, we estimate the L^p-distance between the solution process of an SDE and an arbitrary Itô process, which we view as a perturbation of the solution process of the SDE, by the L^q-distances of the differences of the local characteristics for suitable $p, q > 0$. As one application of the developed perturbation theory, we establish strong convergence rates for numerical approximations of a class of SODEs with nonglobally monotone coefficients. As another application of the developed perturbation theory, we prove strong convergence rates for spatial spectral Galerkin approximations of solutions of semilinear SPDEs with nonglobally monotone nonlinearities including Cahn–Hilliard–Cook-type equations and stochastic Burgers equations. Further applications of the developed perturbation theory include regularity analyses of solutions of SDEs with respect to their initial values as well as small-noise analyses for ordinary and partial differential equations.

REFERENCES

[1] ALABERT, A. and GYÖNGY, I. (2006). On numerical approximation of stochastic Burgers' equation. In *From Stochastic Calculus to Mathematical Finance* 1–15. Springer, Berlin. MR2233532 https://doi.org/10.1007/978-3-540-30788-4_1

[2] ALBEVERIO, S. and RÖCKNER, M. (1991). Stochastic differential equations in infinite dimensions: Solutions via Dirichlet forms. *Probab. Theory Related Fields* **89** 347–386. MR1113223 https://doi.org/10.1007/BF01198791

[3] ALFONSI, A. (2013). Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process. *Statist. Probab. Lett.* **83** 602–607. MR3006996 https://doi.org/10.1016/j.spl.2012.10.034

[4] BLÖMKER, D., KAMRANI, M. and HOSSEINI, S. M. (2013). Full discretization of the stochastic Burgers equation with correlated noise. *IMA J. Numer. Anal.* **33** 825–848. MR3081485 https://doi.org/10.1093/imanum/drs035

[5] BOU-RABEE, N. and HAIRER, M. (2013). Nonasymptotic mixing of the MALA algorithm. *IMA J. Numer. Anal.* **33** 80–110. MR3020951 https://doi.org/10.1093/imanum/drs003

[6] BRZEŹNIAK, Z., CARELLI, E. and PROHL, A. (2013). Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing. *IMA J. Numer. Anal.* **33** 771–824. MR3081484 https://doi.org/10.1093/imanum/drs032

[7] CARELLI, E. and PROHL, A. (2012). Rates of convergence for discretizations of the stochastic incompressible Navier–Stokes equations. *SIAM J. Numer. Anal.* **50** 2467–2496. MR3022227 https://doi.org/10.1137/110845008

[8] CERRAI, S. (1998). Differentiability with respect to initial datum for solutions of SPDE's with no Fréchet differentiable drift term. *Commun. Appl. Anal.* **2** 249–270. MR1614630

[9] CERRAI, S. (2001). *Second Order PDE's in Finite and Infinite Dimension: A Probabilistic Approach. Lecture Notes in Math.* **1762**. Springer, Berlin. MR1840644 https://doi.org/10.1007/b80743

MSC2010 subject classifications. 65C30.
Key words and phrases. Perturbation, stochastic differential equation, convergence rate, nonglobally monotone, small-noise analysis, Cahn–Hilliard–Cook equation, stochastic Burgers equation.

[10] CERRAI, S. (2003). Stochastic reaction–diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Probab. Theory Related Fields* **125** 271–304. MR1961346 https://doi.org/10.1007/s00440-002-0230-6

[11] CONUS, D., JENTZEN, A. and KURNIAWAN, R. (2019). Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. *Ann. Appl. Probab.* **29** 653–716. MR3910014 https://doi.org/10.1214/17-AAP1352

[12] COX, S., HUTZENTHALER, M., JENTZEN, A., VAN NEERVEN, J. and WELTI, T. (2016). Convergence in Hölder norms with applications to Monte Carlo methods in infinite dimensions. *IMA J. Numer. Anal.*. To appear. Available at arXiv:1605.00856.

[13] COX, S. G., HUTZENTHALER, M. and JENTZEN, A. (2014). Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. Revision requested from *Mem. Amer. Math. Soc.*. Available at arXiv:1309.5595v2.

[14] DA PRATO, G. and DEBUSSCHE, A. (1996). Stochastic Cahn–Hilliard equation. *Nonlinear Anal.* **26** 241–263. MR1359472 https://doi.org/10.1016/0362-546X(94)00277-O

[15] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[16] DATTA, S. and BHATTACHARJEE, J. K. (2001). Effect of stochastic forcing on the Duffing oscillator. *Phys. Lett. A* **283** 323–326. MR1837846 https://doi.org/10.1016/S0375-9601(01)00258-4

[17] DAVIE, A. M. and GAINES, J. G. (2001). Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Math. Comp.* **70** 121–134. MR1803132 https://doi.org/10.1090/S0025-5718-00-01224-2

[18] DEREICH, S., NEUENKIRCH, A. and SZPRUCH, L. (2012). An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **468** 1105–1115. MR2898556 https://doi.org/10.1098/rspa.2011.0505

[19] DÖRSEK, P. (2012). Semigroup splitting and cubature approximations for the stochastic Navier–Stokes equations. *SIAM J. Numer. Anal.* **50** 729–746. MR2914284 https://doi.org/10.1137/110833841

[20] ES-SARHIR, A. and STANNAT, W. (2010). Improved moment estimates for invariant measures of semilinear diffusions in Hilbert spaces and applications. *J. Funct. Anal.* **259** 1248–1272. MR2652188 https://doi.org/10.1016/j.jfa.2010.02.017

[21] FANG, S., IMKELLER, P. and ZHANG, T. (2007). Global flows for stochastic differential equations without global Lipschitz conditions. *Ann. Probab.* **35** 180–205. MR2303947 https://doi.org/10.1214/009117906000000412

[22] FREIDLIN, M. I. and WENTZELL, A. D. (2012). *Random Perturbations of Dynamical Systems*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. Springer, Heidelberg. MR2953753 https://doi.org/10.1007/978-3-642-25847-3

[23] GILES, M. (2008). Improved multilevel Monte Carlo convergence using the Milstein scheme. In *Monte Carlo and Quasi-Monte Carlo Methods* 2006 343–358. Springer, Berlin. MR2479233 https://doi.org/10.1007/978-3-540-74496-2_20

[24] GILES, M. B. (2008). Multilevel Monte Carlo path simulation. *Oper. Res.* **56** 607–617. MR2436856 https://doi.org/10.1287/opre.1070.0496

[25] GYÖNGY, I. and MILLET, A. (2005). On discretization schemes for stochastic evolution equations. *Potential Anal.* **23** 99–134. MR2139212 https://doi.org/10.1007/s11118-004-5393-6

[26] GYÖNGY, I. and RÁSONYI, M. (2011). A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. *Stochastic Process. Appl.* **121** 2189–2200. MR2822773 https://doi.org/10.1016/j.spa.2011.06.008

[27] HAIRER, M., HUTZENTHALER, M. and JENTZEN, A. (2015). Loss of regularity for Kolmogorov equations. *Ann. Probab.* **43** 468–527. MR3305998 https://doi.org/10.1214/13-AOP838

[28] HAIRER, M. and MATTINGLY, J. C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993–1032. MR2259251 https://doi.org/10.4007/annals.2006.164.993

[29] HEINRICH, S. (1998). Monte Carlo complexity of global solution of integral equations. *J. Complexity* **14** 151–175. MR1629093 https://doi.org/10.1006/jcom.1998.0471

[30] HEINRICH, S. (2001). Multilevel Monte Carlo methods. In *Large-Scale Scientific Computing*. *Lecture Notes Comput. Sci.* **2179** 58–67. Springer, Berlin.

[31] HIEBER, M. and STANNAT, W. (2013). Stochastic stability of the Ekman spiral. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **12** 189–208. MR3088441

[32] HIGHAM, D. J., MAO, X. and STUART, A. M. (2002). Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.* **40** 1041–1063. MR1949404 https://doi.org/10.1137/S0036142901389530

[33] HU, Y. (1996). Semi-implicit Euler–Maruyama scheme for stiff stochastic equations. In *Stochastic Analysis and Related Topics*, *V* (*Silivri*, 1994). *Progress in Probability* **38** 183–202. Birkhäuser, Boston, MA. MR1396331

[34] HUTZENTHALER, M. and JENTZEN, A. (2015). Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem*. *Amer. Math*. *Soc*. **236** v + 99. MR3364862 https://doi.org/10.1090/memo/1112

[35] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2011). Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc*. *R*. *Soc*. *Lond*. *Ser. A Math*. *Phys*. *Eng*. *Sci*. **467** 1563–1576. MR2795791 https://doi.org/10.1098/rspa.2010.0348

[36] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2012). Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann*. *Appl*. *Probab.* **22** 1611–1641. MR2985171 https://doi.org/10.1214/11-AAP803

[37] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2013). Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations. *Ann*. *Appl*. *Probab.* **23** 1913–1966. MR3134726 https://doi.org/10.1214/12-AAP890

[38] HUTZENTHALER, M., JENTZEN, A. and WANG, X. (2018). Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations. *Math*. *Comp*. **87** 1353–1413. MR3766391 https://doi.org/10.1090/mcom/3146

[39] ICHIKAWA, A. (1984). Semilinear stochastic evolution equations: Boundedness, stability and invariant measures. *Stochastics* **12** 1–39. MR0738933 https://doi.org/10.1080/17442508408833293

[40] JACOBE DE NAUROIS, L., JENTZEN, A. and WELTI, T. (2018). Lower bounds for weak approximation errors for spatial spectral Galerkin approximations of stochastic wave equations. In *Stochastic Partial Differential Equations and Related Fields*. *Springer Proc*. *Math*. *Stat*. **229** 237–248. Springer, Cham. MR3828171

[41] JENTZEN, A. and PUŠNIK, P. (2015). Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. *IMA J. Numer. Anal*. To appear. Available at arXiv:1504.03523.

[42] KAMRANI, M. and BLÖMKER, D. (2017). Pathwise convergence of a numerical method for stochastic partial differential equations with correlated noise and local Lipschitz condition. *J. Comput*. *Appl*. *Math*. **323** 123–135. MR3649320 https://doi.org/10.1016/j.cam.2017.04.012

[43] KEBAIER, A. (2005). Statistical Romberg extrapolation: A new variance reduction method and applications to option pricing. *Ann*. *Appl*. *Probab.* **15** 2681–2705. MR2187308 https://doi.org/10.1214/105051605000000511

[44] KLOEDEN, P. and NEUENKIRCH, A. (2013). Convergence of numerical methods for stochastic differential equations in mathematical finance. In *Recent Developments in Computational Finance*. *Interdiscip*. *Math*. *Sci*. **14** 49–80. World Scientific, Hackensack, NJ. MR3288634 https://doi.org/10.1142/9789814436434_0002

[45] KOVÁCS, M., LARSSON, S. and LINDGREN, F. (2015). On the backward Euler approximation of the stochastic Allen–Cahn equation. *J. Appl*. *Probab.* **52** 323–338. MR3372078 https://doi.org/10.1239/jap/1437658601

[46] KOVÁCS, M., LARSSON, S. and MESFORUSH, A. (2011). Finite element approximation of the Cahn–Hilliard–Cook equation. *SIAM J. Numer. Anal*. **49** 2407–2429. MR2854602 https://doi.org/10.1137/110828150

[47] KÜHN, C. (2004). Stochastische Analysis mit Finanzmathematik.

[48] LEHA, G. and RITTER, G. (1994). Lyapunov-type conditions for stationary distributions of diffusion processes on Hilbert spaces. *Stoch*. *Stoch*. *Rep*. **48** 195–225. MR1782748 https://doi.org/10.1080/17442509408833906

[49] LEHA, G. and RITTER, G. (2003). Lyapunov functions and stationary distributions of stochastic evolution equations. *Stoch*. *Anal*. *Appl*. **21** 763–799. MR1988794 https://doi.org/10.1081/SAP-120022862

[50] LI, X.-M. (1994). Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Related Fields* **100** 485–511. MR1305784 https://doi.org/10.1007/BF01268991

[51] LIU, D. (2003). Convergence of the spectral method for stochastic Ginzburg–Landau equation driven by space–time white noise. *Commun*. *Math*. *Sci*. **1** 361–375. MR1980481

[52] MAO, X. and SZPRUCH, L. (2013). Strong convergence rates for backward Euler–Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients. *Stochastics* **85** 144–171. MR3011916 https://doi.org/10.1080/17442508.2011.651213

[53] MASLOWSKI, B. (1986). On some stability properties of stochastic differential equations of Itô's type. *Čas*. *Pěst*. *Mat*. **111** 404–423, 435. MR0871716

[54] MINTY, G. J. (1962). Monotone (nonlinear) operators in Hilbert space. *Duke Math. J.* **29** 341–346. MR0169064

[55] MINTY, G. J. (1963). On a "monotonicity" method for the solution of non-linear equations in Banach spaces. *Proc. Natl. Acad. Sci. USA* **50** 1038–1041. MR0162159 https://doi.org/10.1073/pnas.50.6.1038

[56] MÜLLER-GRONBACH, T. and RITTER, K. (2007). Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Comput. Math.* **7** 135–181. MR2324415 https://doi.org/10.1007/s10208-005-0166-6

[57] MÜLLER-GRONBACH, T., RITTER, K. and WAGNER, T. (2008). Optimal pointwise approximation of a linear stochastic heat equation with additive space–time white noise. In *Monte Carlo and Quasi-Monte Carlo Methods* 2006 577–589. Springer, Berlin. MR2479247 https://doi.org/10.1007/978-3-540-74496-2_34

[58] MÜLLER-GRONBACH, T., RITTER, K. and WAGNER, T. (2008). Optimal pointwise approximation of infinite-dimensional Ornstein–Uhlenbeck processes. *Stoch. Dyn.* **8** 519–541. MR2444516 https://doi.org/10.1142/S0219493708002433

[59] NEUENKIRCH, A. and SZPRUCH, L. (2014). First order strong approximations of scalar SDEs defined in a domain. *Numer. Math.* **128** 103–136. MR3248050 https://doi.org/10.1007/s00211-014-0606-4

[60] PARDOUX, E. (1975). Équations aux dérivées partielles stochastiques de type monotone. In *Séminaire sur les Équations aux Dérivées Partielles* (1974–1975), *III*, *Exp. No.* 2 1–10. MR0651582

[61] PRÉVÔT, C. and RÖCKNER, M. (2007). *A Concise Course on Stochastic Partial Differential Equations*. *Lecture Notes in Math.* **1905**. Springer, Berlin. MR2329435

[62] PRINTEMS, J. (2001). On the discretization in time of parabolic stochastic partial differential equations. *ESAIM Math. Model. Numer. Anal.* **35** 1055–1078. MR1873517 https://doi.org/10.1051/m2an:2001148

[63] SABANIS, S. (2013). A note on tamed Euler approximations. *Electron. Commun. Probab.* **18** Art. ID 47. MR3070913 https://doi.org/10.1214/ECP.v18-2824

[64] SABANIS, S. (2016). Euler approximations with varying coefficients: The case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab.* **26** 2083–2105. MR3543890 https://doi.org/10.1214/15-AAP1140

[65] SAUER, M. and STANNAT, W. (2015). Lattice approximation for stochastic reaction diffusion equations with one-sided Lipschitz condition. *Math. Comp.* **84** 743–766. MR3290962 https://doi.org/10.1090/S0025-5718-2014-02873-1

[66] SCHENK-HOPPÉ, K. R. (1996). Deterministic and stochastic Duffing–van der Pol oscillators are non-explosive. *Z. Angew. Math. Phys.* **47** 740–759. MR1420853 https://doi.org/10.1007/BF00915273

[67] SELL, G. R. and YOU, Y. (2002). *Dynamics of Evolutionary Equations*. *Applied Mathematical Sciences* **143**. Springer, New York. MR1873467 https://doi.org/10.1007/978-1-4757-5037-9

[68] SZPRUCH, L. (2013). V-stable tamed Euler schemes. Available at arXiv:1310.0785.

[69] TRETYAKOV, M. V. and ZHANG, Z. (2013). A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM J. Numer. Anal.* **51** 3135–3162. MR3129758 https://doi.org/10.1137/120902318

[70] ZHANG, X. (2010). Stochastic flows and Bismut formulas for stochastic Hamiltonian systems. *Stochastic Process. Appl.* **120** 1929–1949. MR2673982 https://doi.org/10.1016/j.spa.2010.05.015

[71] ZHOU, X. and E, W. (2010). Study of noise-induced transitions in the Lorenz system using the minimum action method. *Commun. Math. Sci.* **8** 341–355. MR2664454

The Annals of Probability
2020, Vol. 48, No. 1, 94–146
https://doi.org/10.1214/19-AOP1347
© Institute of Mathematical Statistics, 2020

QUENCHED INVARIANCE PRINCIPLES FOR THE MAXIMAL PARTICLE IN BRANCHING RANDOM WALK IN RANDOM ENVIRONMENT AND THE PARABOLIC ANDERSON MODEL

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We consider branching random walk in spatial random branching environment (BRWRE) in dimension one, as well as related differential equations: the Fisher–KPP equation with random branching and its linearized version, the parabolic Anderson model (PAM). When the random environment is bounded, we show that after recentering and scaling, the position of the maximal particle of the BRWRE, the front of the solution of the PAM, as well as the front of the solution of the randomized Fisher–KPP equation fulfill quenched invariance principles. In addition, we prove that at time t the distance between the median of the maximal particle of the BRWRE and the front of the solution of the PAM is in $O(\ln t)$. This partially transfers classical results of Bramson (*Comm. Pure Appl. Math.* **31** (1978) 531–581) to the setting of BRWRE.

REFERENCES

[1] ADDARIO-BERRY, L. and REED, B. (2009). Minima in branching random walks. *Ann. Probab.* **37** 1044–1079. MR2537549 https://doi.org/10.1214/08-AOP428

[2] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680 https://doi.org/10.1214/12-AOP750

[3] ATHREYA, S., DREWITZ, A. and SUN, R. (2017). Subdiffusivity of a random walk among a Poisson system of moving traps on $\mathbb{Z}$. *Math. Phys. Anal. Geom.* **20** Art. 1, 22. MR3590740 https://doi.org/10.1007/s11040-016-9227-8

[4] BACHMANN, M. (2000). Limit theorems for the minimal position in a branching random walk with independent logconcave displacements. *Adv. in Appl. Probab.* **32** 159–176. MR1765165 https://doi.org/10.1239/aap/1013540028

[5] BERESTYCKI, J., BRUNET, É., HARRIS, J. W., HARRIS, S. C. and ROBERTS, M. I. (2015). Growth rates of the population in a branching Brownian motion with an inhomogeneous breeding potential. *Stochastic Process. Appl.* **125** 2096–2145. MR3315624 https://doi.org/10.1016/j.spa.2014.12.008

[6] BHATTACHARYA, R. N. and RANGA RAO, R. (1976). *Normal Approximation and Asymptotic Expansions. Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR0436272

[7] BIGGINS, J. D. (1976). The first- and last-birth problems for a multitype age-dependent branching process. *Adv. in Appl. Probab.* **8** 446–459. MR0420890 https://doi.org/10.2307/1426138

[8] BOVIER, A. (2017). *Gaussian Processes on Trees: From Spin Glasses to Branching Brownian Motion. Cambridge Studies in Advanced Mathematics* **163**. Cambridge Univ. Press, Cambridge. MR3618123 https://doi.org/10.1017/9781316675779

[9] BOVIER, A. and HARTUNG, L. (2014). The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.* **19** no. 18, 28. MR3164771 https://doi.org/10.1214/EJP.v19-2982

[10] BOVIER, A. and HARTUNG, L. (2015). Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 261–291. MR3351476

[11] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44** iv+190. MR0705746 https://doi.org/10.1090/memo/0285

[12] BRAMSON, M. and ZEITOUNI, O. (2009). Tightness for a family of recursion equations. *Ann. Probab.* **37** 615–653. MR2510018 https://doi.org/10.1214/08-AOP414

MSC2010 subject classifications. 60J80, 60G70, 82B44.
Key words and phrases. Branching random walk, random environment, parabolic Anderson model, invariance principles.

[13] BRAMSON, M. and ZEITOUNI, O. (2012). Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **65** 1–20. MR2846636 https://doi.org/10.1002/cpa.20390

[14] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541 https://doi.org/10.1002/cpa.3160310502

[15] CHAUVIN, B. and DRMOTA, M. (2006). The random multisection problem, travelling waves and the distribution of the height of m-ary search trees. *Algorithmica* **46** 299–327. MR2291958 https://doi.org/10.1007/s00453-006-0107-7

[16] COMETS, F. and POPOV, S. (2007). On multidimensional branching random walks in random environment. *Ann. Probab.* **35** 68–114. MR2303944 https://doi.org/10.1214/009117906000000926

[17] COMETS, F. and POPOV, S. (2007). Shape and local growth for multidimensional branching random walks in random environment. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 273–299. MR2365644

[18] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036 https://doi.org/10.1007/978-1-4612-5320-4

[19] DREWITZ, A. (2008). Lyapunov exponents for the one-dimensional parabolic Anderson model with drift. *Electron. J. Probab.* **13** 2283–2336. MR2469612 https://doi.org/10.1214/EJP.v13-586

[20] DRMOTA, M. (2003). An analytic approach to the height of binary search trees. II. *J. ACM* **50** 333–374. MR2146358 https://doi.org/10.1145/765568.765572

[21] FANG, M. and ZEITOUNI, O. (2012). Branching random walks in time inhomogeneous environments. *Electron. J. Probab.* **17** no. 67, 18. MR2968674 https://doi.org/10.1214/EJP.v17-2253

[22] FANG, M. and ZEITOUNI, O. (2012). Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.* **149** 1–9. MR2981635 https://doi.org/10.1007/s10955-012-0581-z

[23] FREIDLIN, M. (1985). *Functional Integration and Partial Differential Equations*. *Annals of Mathematics Studies* **109**. Princeton Univ. Press, Princeton, NJ. MR0833742 https://doi.org/10.1515/9781400881598

[24] GANTERT, N., MÜLLER, S., POPOV, S. and VACHKOVSKAIA, M. (2010). Survival of branching random walks in random environment. *J. Theoret. Probab.* **23** 1002–1014. MR2735734 https://doi.org/10.1007/s10959-009-0227-5

[25] GÄRTNER, J. and FREIDLIN, M. I. (1979). The propagation of concentration waves in periodic and random media. *Dokl. Akad. Nauk SSSR* **249** 521–525. MR0553200

[26] GÄRTNER, J. and KÖNIG, W. (2005). The parabolic Anderson model. In *Interacting Stochastic Systems* 153–179. Springer, Berlin. MR2118574 https://doi.org/10.1007/3-540-27110-4_8

[27] GÄRTNER, J. and MOLCHANOV, S. A. (1990). Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.* **132** 613–655. MR1069840

[28] GIKHMAN, I. I. and SKOROKHOD, A. V. (1969). *Introduction to the Theory of Random Processes*. *Translated from the Russian by Scripta Technica*, *Inc*. W. B. Saunders Co., Philadelphia. MR0247660

[29] GREVEN, A. and DEN HOLLANDER, F. (1992). Branching random walk in random environment: Phase transitions for local and global growth rates. *Probab. Theory Related Fields* **91** 195–249. MR1147615 https://doi.org/10.1007/BF01291424

[30] GÜN, O., KÖNIG, W. and SEKULOVIĆ, O. (2013). Moment asymptotics for branching random walks in random environment. *Electron. J. Probab.* **18** no. 63, 18. MR3078022 https://doi.org/10.1214/ejp.v18-2212

[31] HAMEL, F., NOLEN, J., ROQUEJOFFRE, J.-M. and RYZHIK, L. (2016). The logarithmic delay of KPP fronts in a periodic medium. *J. Eur. Math. Soc.* (*JEMS*) **18** 465–505. MR3463416 https://doi.org/10.4171/JEMS/595

[32] HAMMERSLEY, J. M. (1974). Postulates for subadditive processes. *Ann. Probab.* **2** 652–680. MR0370721 https://doi.org/10.1214/aop/1176996611

[33] HARRIS, S. C. and ROBERTS, M. I. (2017). The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 226–242. MR3606740 https://doi.org/10.1214/15-AIHP714

[34] HEIL, H., NAKASHIMA, M. and YOSHIDA, N. (2011). Branching random walks in random environment are diffusive in the regular growth phase. *Electron. J. Probab.* **16** 1316–1340. MR2827461 https://doi.org/10.1214/EJP.v16-922

[35] HU, Y. and SHI, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.* **37** 742–789. MR2510023 https://doi.org/10.1214/08-AOP419

[36] HU, Y. and YOSHIDA, N. (2009). Localization for branching random walks in random environment. *Stochastic Process. Appl.* **119** 1632–1651. MR2513122 https://doi.org/10.1016/j.spa.2008.08.005

[37] HUANG, C. and LIU, Q. (2014). Branching random walk with a random environment in time. Preprint. Available at arXiv:1407.7623.

[38] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968). Branching Markov processes. I. *J. Math. Kyoto Univ.* **8** 233–278. MR0232439 https://doi.org/10.1215/kjm/1250524137

[39] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968). Branching Markov processes. II. *J. Math. Kyoto Univ.* **8** 365–410. MR0238401 https://doi.org/10.1215/kjm/1250524059

[40] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1969). Branching Markov processes. III. *J. Math. Kyoto Univ.* **9** 95–160. MR0246376 https://doi.org/10.1215/kjm/1250524013

[41] KEVEI, P. and MASON, D. M. (2011). A note on a maximal Bernstein inequality. *Bernoulli* **17** 1054–1062. MR2817617 https://doi.org/10.3150/10-BEJ304

[42] KINGMAN, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.* **3** 790–801. MR0400438 https://doi.org/10.1214/aop/1176996266

[43] KOLMOGOROV, A., PRETROVSKII, I. and PISKUNOV, N. (1937). Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique. *Bull. Univ. État Moscou, Sér. Int., Sect. A: Math. et Mécan.* 1, *Fasc.* **6** 1–25.

[44] KÖNIG, W. (2016). *The Parabolic Anderson Model. Pathways in Mathematics*. Birkhäuser/Springer, Cham. MR3526112 https://doi.org/10.1007/978-3-319-33596-4

[45] LALLEY, S. and SELLKE, T. (1988). Traveling waves in inhomogeneous branching Brownian motions. I. *Ann. Probab.* **16** 1051–1062. MR0942755

[46] LALLEY, S. and SELLKE, T. (1989). Travelling waves in inhomogeneous branching Brownian motions. II. *Ann. Probab.* **17** 116–127. MR0972775

[47] LIGGETT, T. M. (1985). An improved subadditive ergodic theorem. *Ann. Probab.* **13** 1279–1285. MR0806224

[48] MACHADO, F. P. and POPOV, S. Y. (2000). One-dimensional branching random walks in a Markovian random environment. *J. Appl. Probab.* **37** 1157–1163. MR1808881 https://doi.org/10.1017/s0021900200018350

[49] MAILLARD, P. and ZEITOUNI, O. (2016). Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1144–1160. MR3531703 https://doi.org/10.1214/15-AIHP675

[50] MALLEIN, B. (2015). Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Process. Appl.* **125** 3958–4019. MR3373310 https://doi.org/10.1016/j.spa.2015.05.011

[51] MALLEIN, B. and MIŁOŚ, P. (2018). Brownian motion and random walk above quenched random wall. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 1877–1916. MR3865661 https://doi.org/10.1214/17-AIHP859

[52] MALLEIN, B. and MIŁOŚ, P. (2019). Maximal displacement of a supercritical branching random walk in a time-inhomogeneous random environment. *Stochastic Process. Appl.* **129** 3239–3260. MR3985561 https://doi.org/10.1016/j.spa.2018.09.008

[53] MCDIARMID, C. (1995). Minimal positions in a branching random walk. *Ann. Appl. Probab.* **5** 128–139. MR1325045

[54] MCKEAN, H. P. (1975). Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov. *Comm. Pure Appl. Math.* **28** 323–331. MR0400428 https://doi.org/10.1002/cpa.3160280302

[55] MERLEVÈDE, F., PELIGRAD, M. and UTEV, S. (2006). Recent advances in invariance principles for stationary sequences. *Probab. Surv.* **3** 1–36. MR2206313 https://doi.org/10.1214/154957806100000202

[56] NAKASHIMA, M. (2011). Almost sure central limit theorem for branching random walks in random environment. *Ann. Appl. Probab.* **21** 351–373. MR2759206 https://doi.org/10.1214/10-AAP699

[57] NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675. MR0624694

[58] NOLEN, J. (2011). A central limit theorem for pulled fronts in a random medium. *Netw. Heterog. Media* **6** 167–194. MR2806072 https://doi.org/10.3934/nhm.2011.6.167

[59] NOLEN, J., ROQUEJOFFRE, J.-M. and RYZHIK, L. (2015). Power-like delay in time inhomogeneous Fisher–KPP equations. *Comm. Partial Differential Equations* **40** 475–505. MR3285242 https://doi.org/10.1080/03605302.2014.972744

[60] ORTGIESE, M. and ROBERTS, M. I. (2016). Intermittency for branching random walk in Pareto environment. *Ann. Probab.* **44** 2198–2263. MR3502604 https://doi.org/10.1214/15-AOP1021

[61] REED, B. (2003). The height of a random binary search tree. *J. ACM* **50** 306–332. MR2146357 https://doi.org/10.1145/765568.765571

[62] RIO, E. (2013). Inequalities and limit theorems for weakly dependent sequences. Lecture.

[63] SHI, Z. (2015). *Branching Random Walks. Lecture Notes in Math.* **2151**. Springer, Cham. MR3444654 https://doi.org/10.1007/978-3-319-25372-5

The Annals of Probability
2020, Vol. 48, No. 1, 147–177
https://doi.org/10.1214/19-AOP1357
© Institute of Mathematical Statistics, 2020

CONTINUOUS BREUER–MAJOR THEOREM: TIGHTNESS AND NONSTATIONARITY

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Let $Y = (Y(t))_{t\geq 0}$ be a zero-mean Gaussian stationary process with covariance function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\rho(0) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a square-integrable function with respect to the standard Gaussian measure, and suppose the Hermite rank of f is $d \geq 1$. If $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$, then the celebrated Breuer–Major theorem (in its continuous version) asserts that the finite-dimensional distributions of $Z_\varepsilon := \sqrt{\varepsilon} \int_0^{\cdot/\varepsilon} f(Y(s))\, ds$ converge to those of σW as $\varepsilon \to 0$, where W is a standard Brownian motion and σ is some explicit constant. Since its first appearance in 1983, this theorem has become a crucial probabilistic tool in different areas, for instance in signal processing or in statistical inference for fractional Gaussian processes.

The goal of this paper is twofold. First, we investigate the tightness in the Breuer–Major theorem. Surprisingly, this problem did not receive a lot of attention until now, and the best available condition due to Ben Hariz [*J. Multivariate Anal.* **80** (2002) 191–216] is neither arguably very natural, nor easy-to-check in practice. In contrast, our condition very simple, as it only requires that $|f|^p$ must be integrable with respect to the standard Gaussian measure for some p *strictly* bigger than 2. It is obtained by means of the Malliavin calculus, in particular Meyer inequalities.

Second, and motivated by a problem of geometrical nature, we extend the continuous Breuer–Major theorem to the notoriously difficult case of self-similar Gaussian processes which are *not* necessarily stationary. An application to the fluctuations associated with the length process of a regularized version of the bifractional Brownian motion concludes the paper.

REFERENCES

[1] BEN HARIZ, S. (2002). Limit theorems for the non-linear functional of stationary Gaussian processes. *J. Multivariate Anal.* **80** 191–216. MR1889773 https://doi.org/10.1006/jmva.2001.1986
[2] BRETON, J.-C. and NOURDIN, I. (2008). Error bounds on the non-normal approximation of Hermite power variations of fractional Brownian motion. *Electron. Commun. Probab.* **13** 482–493. MR2447835 https://doi.org/10.1214/ECP.v13-1415
[3] BREUER, P. and MAJOR, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. MR0716933 https://doi.org/10.1016/0047-259X(83)90019-2
[4] CHAMBERS, D. and SLUD, E. (1989). Central limit theorems for nonlinear functionals of stationary Gaussian processes. *Probab. Theory Related Fields* **80** 323–346. MR0976529 https://doi.org/10.1007/BF01794427
[5] DARSES, S., NOURDIN, I. and NUALART, D. (2010). Limit theorems for nonlinear functionals of Volterra processes via white noise analysis. *Bernoulli* **16** 1262–1293. MR2759179 https://doi.org/10.3150/10-BEJ258
[6] DOBRUSHIN, R. L. and MAJOR, P. (1979). Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27–52. MR0550122 https://doi.org/10.1007/BF00535673
[7] HARNETT, D. and NUALART, D. (2018). Central limit theorem for functionals of a generalized self-similar Gaussian process. *Stochastic Process. Appl.* **128** 404–425. MR3739502 https://doi.org/10.1016/j.spa.2017.04.014

MSC2010 subject classifications. Primary 60G15, 60F17, 60H07; secondary 60G05.
Key words and phrases. Breuer–Major theorem, functional convergence, tightness, self-similar Gaussian process, bifractional Brownian motion.

[8] HU, Y. and NUALART, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33** 948–983. MR2135309 https://doi.org/10.1214/009117905000000017

[9] JARAMILLO, A. and NUALART, D. (2019). Functional limit theorem for the self-intersection local time of the fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 480–527. MR3901653 https://doi.org/10.1214/18-aihp889

[10] LEI, P. and NUALART, D. (2009). A decomposition of the bifractional Brownian motion and some applications. *Statist. Probab. Lett.* **79** 619–624. MR2499385 https://doi.org/10.1016/j.spl.2008.10.009

[11] NOURDIN, I. (2012). *Selected Aspects of Fractional Brownian Motion*. *Bocconi & Springer Series* **4**. Springer, Milan. MR3076266 https://doi.org/10.1007/978-88-470-2823-4

[12] NOURDIN, I., NUALART, D. and TUDOR, C. A. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1055–1079. MR2744886 https://doi.org/10.1214/09-AIHP342

[13] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge Univ. Press, Cambridge. MR2962301 https://doi.org/10.1017/CBO9781139084659

[14] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[15] NUALART, D. and NUALART, E. (2018). *Introduction to Malliavin Calculus*. *Institute of Mathematical Statistics Textbooks* **9**. Cambridge Univ. Press, Cambridge. MR3838464 https://doi.org/10.1017/9781139856485

[16] NUALART, D. and PECCATI, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863 https://doi.org/10.1214/009117904000000621

[17] PECCATI, G. and TUDOR, C. A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII*. *Lecture Notes in Math.* **1857** 247–262. Springer, Berlin. MR2126978 https://doi.org/10.1007/978-3-540-31449-3_17

[18] TAQQU, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* **50** 53–83. MR0550123 https://doi.org/10.1007/BF00535674

The Annals of Probability
2020, Vol. 48, No. 1, 178–210
https://doi.org/10.1214/19-AOP1358
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STRONG EXISTENCE AND UNIQUENESS FOR STABLE STOCHASTIC DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL DRIFT

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We consider the stochastic differential equation

$$dX_t = b(X_t)\,dt + dL_t,$$

where the drift b is a generalized function and L is a symmetric one dimensional α-stable Lévy processes, $\alpha \in (1, 2)$. We define the notion of solution to this equation and establish strong existence and uniqueness whenever b belongs to the Besov–Hölder space $\mathcal{C}^\beta$ for $\beta > 1/2 - \alpha/2$.

REFERENCES

[1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446 https://doi.org/10.1214/aop/1176995579

[2] APPLEBAUM, D. (2009). *Lévy Processes and Stochastic Calculus*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **116**. Cambridge Univ. Press, Cambridge. MR2512800 https://doi.org/10.1017/CBO9780511809781

[3] ATHREYA, S., BUTKOVSKY, O. and MYTNIK, L. (2020). Supplement to "Strong existence and uniqueness for stable stochastic differential equations with distributional drift." https://doi.org/10.1214/19-AOP1358SUPP.

[4] BANDINI, E. and RUSSO, F. (2017). Weak Dirichlet processes with jumps. *Stochastic Process. Appl.* **127** 4139–4189. MR3718109 https://doi.org/10.1016/j.spa.2017.04.001

[5] BAÑOS, D.R., ORTIZ-LATORRE, S., PILIPENKO, A. and PROSKE, F. (2017). Strong solutions of SDE's with generalized drift and multidimensional fractional Brownian initial noise. arXiv preprint, arXiv:1705.01616.

[6] BASS, R. F. and CHEN, Z.-Q. (2001). Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields* **121** 422–446. MR1867429 https://doi.org/10.1007/s004400100151

[7] BASS, R. F. and CHEN, Z.-Q. (2003). Brownian motion with singular drift. *Ann. Probab.* **31** 791–817. MR1964949 https://doi.org/10.1214/aop/1048516536

[8] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[9] BLUMENTHAL, R. M. and GETOOR, R. K. (1960). Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95** 263–273. MR0119247 https://doi.org/10.2307/1993291

[10] BOGACHEV, V. I. and PILIPENKO, A. YU. (2016). Strong solutions to stochastic equations with Lévy noise and a discontinuous drift coefficient. *Dokl. Akad. Nauk* **463** 509–513. MR3444003 https://doi.org/10.1134/S1064562415040213

[11] CATELLIER, R. and GUBINELLI, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.* **126** 2323–2366. MR3505229 https://doi.org/10.1016/j.spa.2016.02.002

[12] CHEN, Z.-Q., SONG, R. and ZHANG, X. (2018). Stochastic flows for Lévy processes with Hölder drifts. *Rev. Mat. Iberoam.* **34** 1755–1788. MR3896248 https://doi.org/10.4171/rmi/1042

[13] CHEN, Z.-Q. and WANG, L. (2016). Uniqueness of stable processes with drift. *Proc. Amer. Math. Soc.* **144** 2661–2675. MR3477084 https://doi.org/10.1090/proc/12909

[14] COQUET, F., JAKUBOWSKI, A., MÉMIN, J. and SŁOMIŃSKI, L. (2006). Natural decomposition of processes and weak Dirichlet processes. In *In Memoriam Paul-André Meyer*: *Séminaire de Probabilités*

MSC2010 subject classifications. 60H10, 60G52.
Key words and phrases. Stochastic differential equations, strong solution, regularization by noise, stable processes, Zvonkin transformation.

XXXIX. *Lecture Notes in Math*. **1874** 81–116. Springer, Berlin. MR2276891 https://doi.org/10.1007/978-3-540-35513-7_8

[15] DAVIE, A. M. (2007). Uniqueness of solutions of stochastic differential equations. *Int*. *Math*. *Res*. *Not*. *IMRN* **24** Art. ID rnm124, 26. MR2377011 https://doi.org/10.1093/imrn/rnm124

[16] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[17] FLANDOLI, F., GUBINELLI, M. and PRIOLA, E. (2010). Well-posedness of the transport equation by stochastic perturbation. *Invent*. *Math*. **180** 1–53. MR2593276 https://doi.org/10.1007/s00222-009-0224-4

[18] FLANDOLI, F., ISSOGLIO, E. and RUSSO, F. (2017). Multidimensional stochastic differential equations with distributional drift. *Trans*. *Amer*. *Math*. *Soc*. **369** 1665–1688. MR3581216 https://doi.org/10.1090/tran/6729

[19] FLANDOLI, F., RUSSO, F. and WOLF, J. (2003). Some SDEs with distributional drift. I. General calculus. *Osaka J*. *Math*. **40** 493–542. MR1988703

[20] FLANDOLI, F., RUSSO, F. and WOLF, J. (2004). Some SDEs with distributional drift. II. Lyons–Zheng structure, Itô's formula and semimartingale characterization. *Random Oper*. *Stoch*. *Equ*. **12** 145–184. MR2065168 https://doi.org/10.1163/156939704323074700

[21] FÖLLMER, H. (1981). Dirichlet processes. In *Stochastic Integrals* (*Proc*. *Sympos*., *Univ*. *Durham*, *Durham*, 1980). *Lecture Notes in Math*. **851** 476–478. Springer, Berlin. MR0621001

[22] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math*. *Pi* **3** e6, 75. MR3406823 https://doi.org/10.1017/fmp.2015.2

[23] GUBINELLI, M. and PERKOWSKI, N. (2015). *Lectures on Singular Stochastic PDEs*. *Ensaios Matemáticos* [*Mathematical Surveys*] **29**. Sociedade Brasileira de Matemática, Rio de Janeiro. MR3445609

[24] HAADEM, S. and PROSKE, F. (2014). On the construction and Malliavin differentiability of solutions of Lévy noise driven SDE's with singular coefficients. *J*. *Funct*. *Anal*. **266** 5321–5359. MR3177338 https://doi.org/10.1016/j.jfa.2014.02.009

[25] HAIRER, M. (2014). A theory of regularity structures. *Invent*. *Math*. **198** 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4

[26] HARRISON, J. M. and SHEPP, L. A. (1981). On skew Brownian motion. *Ann*. *Probab*. **9** 309–313. MR0606993

[27] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[28] KIM, P. and SONG, R. (2014). Stable process with singular drift. *Stochastic Process*. *Appl*. **124** 2479–2516. MR3192504 https://doi.org/10.1016/j.spa.2014.03.006

[29] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab*. *Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z

[30] KULIK, A. M. (2019). On weak uniqueness and distributional properties of a solution to an SDE with α-stable noise. *Stochastic Process*. *Appl*. **129** 473–506. MR3907007 https://doi.org/10.1016/j.spa.2018.03.010

[31] KURTZ, T. G. (2014). Weak and strong solutions of general stochastic models. *Electron*. *Commun*. *Probab*. **19** no. 58, 16. MR3254737 https://doi.org/10.1214/ECP.v19-2833

[32] KURTZ, T. G. and PROTTER, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann*. *Probab*. **19** 1035–1070. MR1112406

[33] KWAŚNICKI, M. (2017). Ten equivalent definitions of the fractional Laplace operator. *Fract*. *Calc*. *Appl*. *Anal*. **20** 7–51. MR3613319 https://doi.org/10.1515/fca-2017-0002

[34] LE GALL, J.-F. (1984). One-dimensional stochastic differential equations involving the local times of the unknown process. In *Stochastic Analysis and Applications* (*Swansea*, 1983). *Lecture Notes in Math*. **1095** 51–82. Springer, Berlin. MR0777514 https://doi.org/10.1007/BFb0099122

[35] LI, Z. and MYTNIK, L. (2011). Strong solutions for stochastic differential equations with jumps. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **47** 1055–1067. MR2884224 https://doi.org/10.1214/10-AIHP389

[36] LI, Z. and PU, F. (2012). Strong solutions of jump-type stochastic equations. *Electron*. *Commun*. *Probab*. **17** no. 33, 13. MR2965746 https://doi.org/10.1214/ECP.v17-1915

[37] PERKOWSKI, N. (2017). Paracontrolled distributions and singular diffusions. Preprint, http://www.mathematik.hu-berlin.de/~perkowsk/files/teaching/paracontrolled-bonn.pdf.

[38] PORTENKO, N. I. (1979). Diffusion processes with a generalized drift coefficient. *Teor*. *Veroyatn*. *Primen*. **24** 62–77. MR0522237

[39] PORTENKO, N. I. (1990). *Generalized Diffusion Processes*. *Translations of Mathematical Monographs* **83**. Amer. Math. Soc., Providence, RI. Translated from the Russian by H. H. McFaden. MR1104660

[40] PRIOLA, E. (2012). Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka J. Math.* **49** 421–447. MR2945756

[41] PRIOLA, E. (2018). Davie's type uniqueness for a class of SDEs with jumps. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 694–725. MR3795063 https://doi.org/10.1214/16-AIHP818

[42] TANAKA, H., TSUCHIYA, M. and WATANABE, S. (1974). Perturbation of drift-type for Lévy processes. *J. Math. Kyoto Univ.* **14** 73–92. MR0368146 https://doi.org/10.1215/kjm/1250523280

[43] TRIEBEL, H. (2010). *Theory of Function Spaces*. *Modern Birkhäuser Classics*. Birkhäuser/Springer Basel AG, Basel. Reprint of 1983 edition [MR0730762], Also published in 1983 by Birkhäuser Verlag [MR0781540]. MR3024598

[44] VERETENNIKOV, A. JU. (1980). Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb.* **111(153)** 434–452, 480. MR0568986

[45] ZHANG, X. and ZHAO, G. (2018). Heat kernel and ergodicity of SDEs with distributional drifts. arXiv preprint, arXiv:1710.10537.

[46] ZVONKIN, A. K. (1974). A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb.* **93(135)** 129–149, 152. MR0336813

The Annals of Probability
2020, Vol. 48, No. 1, 211–263
https://doi.org/10.1214/19-AOP1359
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FROM THE MASTER EQUATION TO MEAN FIELD GAME LIMIT THEORY: LARGE DEVIATIONS AND CONCENTRATION OF MEASURE

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We study a sequence of symmetric n-player stochastic differential games driven by both idiosyncratic and common sources of noise, in which players interact with each other through their empirical distribution. The unique Nash equilibrium empirical measure of the n-player game is known to converge, as n goes to infinity, to the unique equilibrium of an associated mean field game. Under suitable regularity conditions, in the absence of common noise, we complement this law of large numbers result with nonasymptotic concentration bounds for the Wasserstein distance between the n-player Nash equilibrium empirical measure and the mean field equilibrium. We also show that the sequence of Nash equilibrium empirical measures satisfies a weak large deviation principle, which can be strengthened to a full large deviation principle only in the absence of common noise. For both sets of results, we first use the master equation, an infinite-dimensional partial differential equation that characterizes the value function of the mean field game, to construct an associated McKean–Vlasov interacting n-particle system that is exponentially close to the Nash equilibrium dynamics of the n-player game for large n, by refining estimates obtained in our companion paper. Then we establish a weak large deviation principle for McKean–Vlasov systems in the presence of common noise. In the absence of common noise, we upgrade this to a full large deviation principle and obtain new concentration estimates for McKean–Vlasov systems. Finally, in two specific examples that do not satisfy the assumptions of our main theorems, we show how to adapt our methodology to establish large deviations and concentration results.

REFERENCES

[1] BAYRAKTAR, E. and COHEN, A. (2018). Analysis of a finite state many player game using its master equation. *SIAM J. Control Optim.* **56** 3538–3568. MR3860894 https://doi.org/10.1137/17M113887X
[2] BEN AROUS, G. and BRUNAUD, M. (1990). Méthode de Laplace: étude variationnelle des fluctuations de diffusions de type "champ moyen". *Stoch. Stoch. Rep.* **31** 79–144. MR1080535 https://doi.org/10.1080/03610919008833649
[3] BENSOUSSAN, A. and FREHSE, J. (1983). *Nonlinear Elliptic Systems in Stochastic Game Theory*, Universität Bonn. SFB 72. Approximation und Optimierung, 1983.
[4] BENSOUSSAN, A., FREHSE, J. and YAM, S. C. P. (2015). The master equation in mean field theory. *J. Math. Pures Appl.* (9) **103** 1441–1474. MR3343705 https://doi.org/10.1016/j.matpur.2014.11.005
[5] BENSOUSSAN, A., FREHSE, J. and YAM, S. C. P. (2017). On the interpretation of the Master Equation. *Stochastic Process. Appl.* **127** 2093–2137. MR3652408 https://doi.org/10.1016/j.spa.2016.10.004
[6] BOBKOV, S. G. and GÖTZE, F. (1999). Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.* **163** 1–28. MR1682772 https://doi.org/10.1006/jfan.1998.3326

MSC2010 subject classifications. Primary 60F10, 60E15, 60H10, 91A13, 91A15; secondary 91G80, 60K35.
Key words and phrases. Mean field games, master equation, McKean–Vlasov limit, interacting particle systems, common noise, large deviation principle, concentration of measure, transport inequalities, linear-quadratic systems, systemic risk.

[7] Bolley, F., Guillin, A. and Malrieu, F. (2010). Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov–Fokker–Planck equation. *ESAIM Math. Model. Numer. Anal.* **44** 867–884. MR2731396 https://doi.org/10.1051/m2an/2010045

[8] Bolley, F., Guillin, A. and Villani, C. (2007). Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields* **137** 541–593. MR2280433 https://doi.org/10.1007/s00440-006-0004-7

[9] Budhiraja, A., Dupuis, P. and Fischer, M. (2012). Large deviation properties of weakly interacting processes via weak convergence methods. *Ann. Probab.* **40** 74–102. MR2917767 https://doi.org/10.1214/10-AOP616

[10] Cardaliaguet, P., Delarue, F., Lasry, J.-M. and Lions, P.-L. (2019) *The Master Equation and the Convergence Problem in Mean Field Games. Annals of Mathematics Studies* **12** Princeton Univ. Press.

[11] Carmona, R. and Delarue, F. (2014). The master equation for large population equilibriums. In *Stochastic Analysis and Applications* 2014. *Springer Proc. Math. Stat.* **100** 77–128. Springer, Cham. MR3332710 https://doi.org/10.1007/978-3-319-11292-3_4

[12] Carmona, R. and Delarue, F. (2017). *Probabilistic Theory of Mean Field Games: Vol. I, Mean Field FBSDEs, Control, and Games. Stochastic Analysis and Applications*, Springer, Berlin.

[13] Carmona, R. and Delarue, F. (2017). *Probabilistic Theory of Mean Field Games: Vol. II, Mean Field Games with Common Noise and Master Equations. Stochastic Analysis and Applications*, Springer, Berlin.

[14] Carmona, R., Fouque, J.-P. and Sun, L.-H. (2015). Mean field games and systemic risk. *Commun. Math. Sci.* **13** 911–933. MR3325083 https://doi.org/10.4310/CMS.2015.v13.n4.a4

[15] Carmona, R. and Zhu, X. (2016). A probabilistic approach to mean field games with major and minor players. *Ann. Appl. Probab.* **26** 1535–1580. MR3513598 https://doi.org/10.1214/15-AAP1125

[16] Cecchin, A. and Fischer, M. (2017). Probabilistic approach to finite state mean field games. Preprint. Available at arXiv:1704.00984.

[17] Cecchin, A. and Pelino, G. (2019). Convergence, fluctuations and large deviations for finite state mean field games via the master equation. *Stochastic Process. Appl.* **129** 4510–4555.

[18] Chassagneux, J.-F., Crisan, D. and Delarue, F. (2014). A probabilistic approach to classical solutions of the master equation for large population equilibria. Preprint. Available at arXiv:1411.3009.

[19] Dawson, D. A. and Gärtner, J. (1987). Large deviations from the McKean–Vlasov limit for weakly interacting diffusions. *Stochastics* **20** 247–308. MR0885876 https://doi.org/10.1080/17442508708833446

[20] Delarue, F., Lacker, D. and Ramanan, K. (2018). From the master equation to mean field game limit theory: A central limit theorem. *Electron. J. Probab.* **24** 1–54.

[21] Dembo, A. and Zeitouni, O. (2010). *Large Deviations Techniques and Applications. Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. MR2571413 https://doi.org/10.1007/978-3-642-03311-7

[22] Djellout, H., Guillin, A. and Wu, L. (2004). Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann. Probab.* **32** 2702–2732. MR2078555 https://doi.org/10.1214/009117904000000531

[23] Fischer, M. (2014). On the form of the large deviation rate function for the empirical measures of weakly interacting systems. *Bernoulli* **20** 1765–1801. MR3263089 https://doi.org/10.3150/13-BEJ540

[24] Fischer, M. (2017). On the connection between symmetric N-player games and mean field games. *Ann. Appl. Probab.* **27** 757–810. MR3655853 https://doi.org/10.1214/16-AAP1215

[25] Fournier, N. and Guillin, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[26] Gangbo, W. and Święch, A. (2015). Existence of a solution to an equation arising from the theory of mean field games. *J. Differential Equations* **259** 6573–6643. MR3397332 https://doi.org/10.1016/j.jde.2015.08.001

[27] Gozlan, N. (2009). A characterization of dimension free concentration in terms of transportation inequalities. *Ann. Probab.* **37** 2480–2498. MR2573565 https://doi.org/10.1214/09-AOP470

[28] Gozlan, N. and Léonard, C. (2010). Transport inequalities. A survey. *Markov Process. Related Fields* **16** 635–736. MR2895086

[29] Horowitz, J. and Karandikar, R. L. (1994). Mean rates of convergence of empirical measures in the Wasserstein metric. *J. Comput. Appl. Math.* **55** 261–273. MR1329874 https://doi.org/10.1016/0377-0427(94)90033-7

[30] Lacker, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014 https://doi.org/10.1007/s00440-015-0641-9

[31] LACKER, D. and RAMANAN, K. (2019). Rare Nash equilibria and the price of anarchy in large static games. *Math*. *Oper. Res*. **44** 400–422.

[32] LACKER, D. and ZARIPHOPOULOU, T. (2017). Mean field and n-agent games for optimal investment under relative performance criteria. Preprint. Available at arXiv:1703.07685.

[33] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn*. *J*. *Math*. **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8

[34] LEDOUX, M. (2001). *The Concentration of Measure Phenomenon*. *Mathematical Surveys and Monographs* **89**. Amer. Math. Soc., Providence, RI. MR1849347

[35] STRÖMBERG, T. (2009). A note on the differentiability of conjugate functions. *Arch*. *Math*. (*Basel*) **93** 481–485. MR2563594 https://doi.org/10.1007/s00013-009-0050-3

[36] STROOCK, D. W. and VARADHAN, S. R. S. (1972). On the support of diffusion processes with applications to the strong maximum principle. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif*., 1970/1971), *Vol*. *III*: *Probability theory* Univ. California Press, Berkeley, CA. 333–359. MR0400425

[37] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[38] ÜSTÜNEL, A. S. (2012). Transportation cost inequalities for diffusions under uniform distance. In *Stochastic Analysis and Related Topics*. *Springer Proc*. *Math*. *Stat*. **22** 203–214. Springer, Heidelberg. MR3236093 https://doi.org/10.1007/978-3-642-29982-7_9

[39] WANG, R., WANG, X. and WU, L. (2010). Sanov's theorem in the Wasserstein distance: A necessary and sufficient condition. *Statist*. *Probab*. *Lett*. **80** 505–512. MR2593592 https://doi.org/10.1016/j.spl.2009.12.003

The Annals of Probability
2020, Vol. 48, No. 1, 264–295
https://doi.org/10.1214/19-AOP1360
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CONVERGENCE OF TRANSPORT NOISE TO ORNSTEIN–UHLENBECK FOR 2D EULER EQUATIONS UNDER THE ENSTROPHY MEASURE

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We consider the vorticity form of the 2D Euler equations which is perturbed by a suitable transport type noise and has white noise initial condition. It is shown that stationary solutions of this equation converge to the unique stationary solution of the 2D Navier–Stokes equation driven by the space-time white noise.

REFERENCES

[1] ALBEVERIO, S., BARBU, V. and FERRARIO, B. (2008). Uniqueness of the generators of the 2D Euler and Navier–Stokes flows. *Stochastic Process*. *Appl*. **118** 2071–2084. MR2462289 https://doi.org/10.1016/j.spa.2007.12.003

[2] ALBEVERIO, S. and CRUZEIRO, A. B. (1990). Global flows with invariant (Gibbs) measures for Euler and Navier–Stokes two-dimensional fluids. *Comm*. *Math*. *Phys*. **129** 431–444. MR1051499

[3] ALBEVERIO, S. and FERRARIO, B. (2002). Uniqueness results for the generators of the two-dimensional Euler and Navier–Stokes flows. The case of Gaussian invariant measures. *J*. *Funct*. *Anal*. **193** 77–93. MR1923629 https://doi.org/10.1006/jfan.2001.3927

[4] ALBEVERIO, S. and FERRARIO, B. (2004). Uniqueness of solutions of the stochastic Navier–Stokes equation with invariant measure given by the enstrophy. *Ann*. *Probab*. **32** 1632–1649. MR2060312 https://doi.org/10.1214/009117904000000379

[5] ALBEVERIO, S. and FERRARIO, B. (2008). Some methods of infinite dimensional analysis in hydrodynamics: An introduction. In *SPDE in Hydrodynamic*: *Recent Progress and Prospects*. *Lecture Notes in Math*. **1942** 1–50. Springer, Berlin. MR2459084 https://doi.org/10.1007/978-3-540-78493-7_1

[6] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[7] CAILLERIE, N. and VOVELLE, J. (2017). Diffusion-approximation for a kinetic equation with perturbed velocity redistribution process. Available at arXiv:1712.10173v1.

[8] COGHI, M. and FLANDOLI, F. (2016). Propagation of chaos for interacting particles subject to environmental noise. *Ann*. *Appl*. *Probab*. **26** 1407–1442. MR3513594 https://doi.org/10.1214/15-AAP1120

[9] DA PRATO, G. and DEBUSSCHE, A. (2002). Two-dimensional Navier–Stokes equations driven by a space-time white noise. *J*. *Funct*. *Anal*. **196** 180–210. MR1941997 https://doi.org/10.1006/jfan.2002.3919

[10] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[11] DEBUSSCHE, A. (2003). The 2D-Navier–Stokes equations perturbed by a delta correlated noise. In *Probabilistic Methods in Fluids* 115–129. World Sci. Publ., River Edge, NJ. MR2083368 https://doi.org/10.1142/9789812703989_0008

[12] DEBUSSCHE, A. and VOVELLE, J. (2017). Diffusion-approximation in stochastically forced kinetic equations. Available at arXiv:1707.07874v3.

[13] FLANDOLI, F. (2018). Weak vorticity formulation of 2D Euler equations with white noise initial condition. *Comm*. *Partial Differential Equations* **43** 1102–1149. MR3910197 https://doi.org/10.1080/03605302.2018.1467448

[14] FLANDOLI, F. and LUO, D. (2019). ρ-white noise solution to 2D stochastic Euler equations. *Probab. Theory Related Fields*. https://doi.org/10.1007/s00440-019-00902-8

MSC2010 subject classifications. Primary 35Q35; secondary 60H40.

Key words and phrases. Navier–Stokes equations, Euler equations, space-time white noise, vorticity formulation, weak convergence.

[15] FLANDOLI, F. and LUO, D. (2019). Kolmogorov equations associated to the stochastic two dimensional Euler equations. *SIAM J. Math. Anal.* **51** 1761–1791. MR3947288 https://doi.org/10.1137/18M1175756

[16] GALEATI, L. (2019). On the convergence of stochastic transport equations to a deterministic parabolic one. Available at arXiv:1902.06960.

[17] KUKSIN, S. and SHIRIKYAN, A. (2003). Some limiting properties of randomly forced two-dimensional Navier–Stokes equations. *Proc. Roy. Soc. Edinburgh Sect. A* **133** 875–891. MR2006207 https://doi.org/10.1017/S0308210500002729

[18] KUKSIN, S. and SHIRIKYAN, A. (2004). Randomly forced CGL equation: Stationary measures and the inviscid limit. *J. Phys. A* **37** 3805–3822. MR2039838 https://doi.org/10.1088/0305-4470/37/12/006

[19] KURTZ, T. G. (2007). The Yamada–Watanabe–Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.* **12** 951–965. MR2336594 https://doi.org/10.1214/EJP.v12-431

[20] RÖCKNER, M., ZHU, R. and ZHU, X. (2017). Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications. *J. Funct. Anal.* **272** 4263–4303. MR3626040 https://doi.org/10.1016/j.jfa.2017.01.023

[21] SAUER, M. (2016). L^1-uniqueness of Kolmogorov operators associated with two-dimensional stochastic Navier–Stokes Coriolis equations with space-time white noise. *J. Theoret. Probab.* **29** 569–589. MR3500411 https://doi.org/10.1007/s10959-014-0582-8

[22] SIMON, J. (1987). Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* (4) **146** 65–96. MR0916688 https://doi.org/10.1007/BF01762360

[23] STANNAT, W. (2003). L^1-uniqueness of regularized 2D-Euler and stochastic Navier–Stokes equations. *J. Funct. Anal.* **200** 101–117. MR1974090 https://doi.org/10.1016/S0022-1236(02)00182-9

[24] STANNAT, W. (2007). A new a priori estimate for the Kolmogorov operator of a 2D-stochastic Navier–Stokes equation. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** 483–497. MR2376438 https://doi.org/10.1142/S0219025707002920

[25] ZHU, R. and ZHU, X. (2017). Strong-Feller property for Navier-Stokes equations driven by space-time white noise. Available at arXiv:1709.09306.

The Annals of Probability
2020, Vol. 48, No. 1, 296–316
https://doi.org/10.1214/19-AOP1361
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QUENCHED INVARIANCE PRINCIPLE FOR RANDOM WALKS AMONG RANDOM DEGENERATE CONDUCTANCES

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We consider the random conductance model in a stationary and ergodic environment. Under suitable moment conditions on the conductances and their inverse, we prove a quenched invariance principle for the random walk among the random conductances. The moment conditions improve earlier results of Andres, Deuschel and Slowik (*Ann. Probab.* **43** (2015) 1866–1891) and are the minimal requirement to ensure that the corrector is sublinear everywhere. The key ingredient is an essentially optimal deterministic local boundedness result for finite difference equations in divergence form.

REFERENCES

[1] ANDRES, S., BARLOW, M. T., DEUSCHEL, J.-D. and HAMBLY, B. M. (2013). Invariance principle for the random conductance model. *Probab. Theory Related Fields* **156** 535–580. MR3078279 https://doi.org/10.1007/s00440-012-0435-2

[2] ANDRES, S., CHIARINI, A., DEUSCHEL, J.-D. and SLOWIK, M. (2018). Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *Ann. Probab.* **46** 302–336. MR3758732 https://doi.org/10.1214/17-AOP1186

[3] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2015). Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43** 1866–1891. MR3353817 https://doi.org/10.1214/14-AOP921

[4] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2016). Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probab. Theory Related Fields* **164** 931–977. MR3477784 https://doi.org/10.1007/s00440-015-0623-y

[5] ANDRES, S. and NEUKAMM, S. (2018). Berry–Esseen theorem and quantitative homogenization for the random conductance model with degenerate conductances. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** 240–296. MR3949105 https://doi.org/10.1007/s40072-018-0127-8

[6] ARMSTRONG, S. and DARIO, P. (2018). Elliptic regularity and quantitative homogenization on percolation clusters. *Comm. Pure Appl. Math.* **71** 1717–1849. MR3847767 https://doi.org/10.1002/cpa.21726

[7] BA, M. and MATHIEU, P. (2015). A Sobolev inequality and the individual invariance principle for diffusions in a periodic potential. *SIAM J. Math. Anal.* **47** 2022–2043. MR3348123 https://doi.org/10.1137/130949683

[8] BARLOW, M., BURDZY, K. and TIMÁR, Á. (2016). Comparison of quenched and annealed invariance principles for random conductance model. *Probab. Theory Related Fields* **164** 741–770. MR3477779 https://doi.org/10.1007/s00440-015-0618-8

[9] BARLOW, M. T. and DEUSCHEL, J.-D. (2010). Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.* **38** 234–276. MR2599199 https://doi.org/10.1214/09-AOP481

[10] BELLA, P., FEHRMAN, B. and OTTO, F. (2018). A Liouville theorem for elliptic systems with degenerate ergodic coefficients. *Ann. Appl. Probab.* **28** 1379–1422. MR3809467 https://doi.org/10.1214/17-AAP1332

[11] BELLA, P. and SCHÄFFNER, M. (2019). Local boundedness and Harnack inequality for solutions of linear non-uniformly elliptic equations. Preprint. Available at arXiv:1901.07958.

[12] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453 https://doi.org/10.1007/s00440-006-0498-z

MSC2010 subject classifications. 60K37, 60F17, 82C41.
Key words and phrases. Random conductance model, invariance principle, stochastic homogenization, nonuniformly elliptic equations.

[13] BISKUP, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373. MR2861133 https://doi.org/10.1214/11-PS190

[14] BISKUP, M. and KUMAGAI, T. (2014). Quenched invariance principle for a class of random conductance models with long-range jumps. Preprint. Available at arXiv:1412.0175.

[15] BISKUP, M. and PRESCOTT, T. M. (2007). Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** 1323–1348. MR2354160 https://doi.org/10.1214/EJP.v12-456

[16] BISKUP, M. and RODRIGUEZ, P.-F. (2018). Limit theory for random walks in degenerate time-dependent random environments. *J. Funct. Anal.* **274** 985–1046. MR3743189 https://doi.org/10.1016/j.jfa.2017.12.002

[17] BOIVIN, D. (1993). Weak convergence for reversible random walks in a random environment. *Ann. Probab.* **21** 1427–1440. MR1235423

[18] BREZIS, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext.* Springer, New York. MR2759829

[19] CHIARINI, A. and DEUSCHEL, J.-D. (2016). Invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1535–1563. MR3573286 https://doi.org/10.1214/15-AIHP688

[20] DE MASI, A., FERRARI, P. A., GOLDSTEIN, S. and WICK, W. D. (1989). An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.* **55** 787–855. MR1003538 https://doi.org/10.1007/BF01041608

[21] DEUSCHEL, J.-D., NGUYEN, T. A. and SLOWIK, M. (2018). Quenched invariance principles for the random conductance model on a random graph with degenerate ergodic weights. *Probab. Theory Related Fields* **170** 363–386. MR3748327 https://doi.org/10.1007/s00440-017-0759-z

[22] FANNJIANG, A. and KOMOROWSKI, T. (1997). A martingale approach to homogenization of unbounded random flows. *Ann. Probab.* **25** 1872–1894. MR1487440 https://doi.org/10.1214/aop/1023481115

[23] FRANCHI, B., SERAPIONI, R. and SERRA CASSANO, F. (1998). Irregular solutions of linear degenerate elliptic equations. *Potential Anal.* **9** 201–216. MR1666899 https://doi.org/10.1023/A:1008684127989

[24] HELLAND, I. S. (1982). Central limit theorems for martingales with discrete or continuous time. *Scand. J. Stat.* **9** 79–94. MR0668684

[25] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. MR0834478

[26] KRENGEL, U. (1985). *Ergodic Theorems. De Gruyter Studies in Mathematics* **6**. de Gruyter, Berlin. MR0797411 https://doi.org/10.1515/9783110844641

[27] KUMAGAI, T. (2014). Random walks on disordered media and their scaling limits. *Lecture Notes from the* 40*th Probability Summer School Held in Saint-Flour* 2010. *Lecture Notes in Math.* Springer, Cham. MR3156983 https://doi.org/10.1007/978-3-319-03152-1

[28] MATHIEU, P. (2008). Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.* **130** 1025–1046. MR2384074 https://doi.org/10.1007/s10955-007-9465-z

[29] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 2287–2307. MR2345229 https://doi.org/10.1098/rspa.2007.1876

[30] MOSER, J. (1960). A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.* **13** 457–468. MR0170091 https://doi.org/10.1002/cpa.3160130308

[31] MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** 577–591. MR0159138 https://doi.org/10.1002/cpa.3160140329

[32] MOURRAT, J.-C. and OTTO, F. (2016). Anchored Nash inequalities and heat kernel bounds for static and dynamic degenerate environments. *J. Funct. Anal.* **270** 201–228. MR3419760 https://doi.org/10.1016/j.jfa.2015.09.020

[33] MURTHY, M. K. V. and STAMPACCHIA, G. (1968). Boundary value problems for some degenerate-elliptic operators. *Ann. Mat. Pura Appl.* (4) **80** 1–122. MR0249828 https://doi.org/10.1007/BF02413623

[34] NEUKAMM, S., SCHÄFFNER, M. and SCHLÖMERKEMPER, A. (2017). Stochastic homogenization of nonconvex discrete energies with degenerate growth. *SIAM J. Math. Anal.* **49** 1761–1809. MR3650427 https://doi.org/10.1137/16M1097705

[35] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields* **129** 219–244. MR2063376 https://doi.org/10.1007/s00440-004-0336-0

[36] TRUDINGER, N. S. (1971). On the regularity of generalized solutions of linear, non-uniformly elliptic equations. *Arch. Ration. Mech. Anal.* **42** 50–62. MR0344656 https://doi.org/10.1007/BF00282317

[37] TRUDINGER, N. S. (1973). Linear elliptic operators with measurable coefficients. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (3) **27** 265–308. MR0369884

The Annals of Probability
2020, Vol. 48, No. 1, 317–342
https://doi.org/10.1214/19-AOP1362
© Institute of Mathematical Statistics, 2020

EXACT ASYMPTOTICS FOR DUARTE AND SUPERCRITICAL ROOTED KINETICALLY CONSTRAINED MODELS

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Kinetically constrained models (KCM) are a class of interacting particle systems which represent a natural stochastic (and nonmonotone) counterpart of the family of cellular automata known as $\mathcal{U}$-bootstrap percolation. A key issue for KCM is to identify the divergence of the characteristic time scales when the equilibrium density of empty sites, q, goes to zero. In (*Ann. Probab.* **47** (2019) 324–361; *Comm. Math. Phys.* **369** (2019) 761–809), a general scheme was devised to determine a sharp upper bound for these time scales. Our paper is devoted to developing a (very different) technique which allows to prove matching lower bounds. We analyse the class of two-dimensional *supercritical rooted KCM* and the *Duarte KCM*. We prove that the relaxation time and the mean infection time diverge for supercritical rooted KCM as $e^{\Theta((\log q)^2)}$ and for Duarte KCM as $e^{\Theta((\log q)^4/q^2)}$ when $q \downarrow 0$. These results prove the conjectures put forward in (*European J. Combin.* **66** (2017) 250–263; *Comm. Math. Phys.* **369** (2019) 761–809) for these models, and establish that the time scales for these KCM diverge much faster than for the corresponding $\mathcal{U}$-bootstrap processes, the main reason being the occurrence of energy barriers which determine the dominant behaviour for KCM, but which do not matter for the bootstrap dynamics.

REFERENCES

[1] ALDOUS, D. and DIACONIS, P. (2002). The asymmetric one-dimensional constrained Ising model: Rigorous results. *J. Stat. Phys.* **107** 945–975. MR1901508 https://doi.org/10.1023/A:1015170205728

[2] ASSELAH, A. and DAI PRA, P. (2001). Quasi-stationary measures for conservative dynamics in the infinite lattice. *Ann. Probab.* **29** 1733–1754. MR1880240 https://doi.org/10.1214/aop/1015345770

[3] BALISTER, P., BOLLOBÁS, B., PRZYKUCKI, M. and SMITH, P. (2016). Subcritical $\mathcal{U}$-bootstrap percolation models have non-trivial phase transitions. *Trans. Amer. Math. Soc.* **368** 7385–7411. MR3471095 https://doi.org/10.1090/tran/6586

[4] BERTHIER, L. and BIROLI, G. (2011). Theoretical perspective on the glass transition and amorphous materials. *Rev. Modern Phys.* **83** 587–645.

[5] BOLLOBÁS, B., DUMINIL-COPIN, H., MORRIS, R. and SMITH, P. (2016). Universality of two-dimensional critical cellular automata. *Proc. London Math. Soc.* To appear. arXiv.org:1406.6680.

[6] BOLLOBÁS, B., DUMINIL-COPIN, H., MORRIS, R. and SMITH, P. (2017). The sharp threshold for the Duarte model. *Ann. Probab.* **45** 4222–4272. MR3737910 https://doi.org/10.1214/16-AOP1163

[7] BOLLOBÁS, B., SMITH, P. and UZZELL, A. (2015). Monotone cellular automata in a random environment. *Combin. Probab. Comput.* **24** 687–722. MR3350030 https://doi.org/10.1017/S0963548315000012

[8] CANCRINI, N., MARTINELLI, F., ROBERTO, C. and TONINELLI, C. (2008). Kinetically constrained spin models. *Probab. Theory Related Fields* **140** 459–504. MR2365481 https://doi.org/10.1007/s00440-007-0072-3

[9] CHLEBOUN, P., FAGGIONATO, A. and MARTINELLI, F. (2014). Time scale separation and dynamic heterogeneity in the low temperature East model. *Comm. Math. Phys.* **328** 955–993. MR3201217 https://doi.org/10.1007/s00220-014-1985-1

MSC2010 subject classifications. Primary 60K35; secondary 60J27.

Key words and phrases. Glauber dynamics, kinetically constrained models, spectral gap, bootstrap percolation, Duarte model.

[10] CHLEBOUN, P., FAGGIONATO, A. and MARTINELLI, F. (2016). Relaxation to equilibrium of generalized East processes on $\mathbb{Z}^d$: Renormalization group analysis and energy-entropy competition. *Ann. Probab.* **44** 1817–1863. MR3502595 https://doi.org/10.1214/15-AOP1011

[11] CHUNG, F., DIACONIS, P. and GRAHAM, R. (2001). Combinatorics for the East model. *Adv. in Appl. Math.* **27** 192–206.

[12] DUARTE, J. A. M. S. (1989). Simulation of a cellular automat with an oriented bootstrap rule. *Phys. A* **157** 1075–1079.

[13] FAGGIONATO, A., MARTINELLI, F., ROBERTO, C. and TONINELLI, C. (2013). The East model: Recent results and new progresses. *Markov Process. Related Fields* **19** 407–452. MR3156959

[14] GARRAHAN, J. P., SOLLICH, P. and TONINELLI, C. (2011). Kinetically constrained models. In *Dynamical Heterogeneities in Glasses, Colloids, and Granular Media* (L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti and W. van Saarloos, eds.) Oxford Univ. Press, Oxford.

[15] JÄCKLE, J. and EISINGER, S. (1991). A hierarchically constrained kinetic Ising model. *Z. Phys. B, Condens. Matter* **84** 115–124.

[16] KEYS, A. S., GARRAHAN, J. P. and CHANDLER, D. (2013). Calorimetric glass transition explained by hierarchical dynamic facilitation. *Proc. Natl. Acad. Sci. USA* **110** 4482–4487.

[17] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[18] MARÊCHÉ, L. (2020). Combinatorics for general kinetically constrained spin models. *SIAM J. Discrete Math.* **34** 370–384. MR4062795

[19] MARTINELLI, F., MORRIS, R. and TONINELLI, C. (2019). Universality results for kinetically constrained spin models in two dimensions. *Comm. Math. Phys.* **369** 761–809.

[20] MARTINELLI, F. and TONINELLI, C. (2019). Towards a universality picture for the relaxation to equilibrium of kinetically constrained models. *Ann. Probab.* **47** 324–361. MR3909971 https://doi.org/10.1214/18-AOP1262

[21] MORRIS, R. (2017). Bootstrap percolation, and other automata. *European J. Combin.* **66** 250–263. MR3692148 https://doi.org/10.1016/j.ejc.2017.06.024

[22] MOUNTFORD, T. S. (1995). Critical length for semi-oriented bootstrap percolation. *Stochastic Process. Appl.* **56** 185–205. MR1325219 https://doi.org/10.1016/0304-4149(94)00061-W

[23] REED, M. and SIMON, B. (1973). *Methods of Modern Mathematical Physics: Functional Analysis*. Academic Press, New York.

[24] RITORT, F. and SOLLICH, P. (2003). Glassy dynamics of kinetically constrained models. *Adv. Phys.* **52** 219–342.

[25] SOLLICH, P. and EVANS, M. R. (2003). Glassy dynamics in the asymmetrically constrained kinetic Ising chain. *Phys. Rev. E* 031504.

The Annals of Probability
2020, Vol. 48, No. 1, 343–379
https://doi.org/10.1214/19-AOP1363
© Institute of Mathematical Statistics, 2020

MALLOWS PERMUTATIONS AND FINITE DEPENDENCE

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We use the Mallows permutation model to construct a new family of stationary finitely dependent proper colorings of the integers. We prove that these colorings can be expressed as finitary factors of i.i.d. processes with finite mean coding radii. They are the first colorings known to have these properties. Moreover, we prove that the coding radii have exponential tails, and that the colorings can also be expressed as functions of countable-state Markov chains. We deduce analogous existence statements concerning shifts of finite type and higher-dimensional colorings.

REFERENCES

[1] AARONSON, J., GILAT, D. and KEANE, M. (1992). On the structure of 1-dependent Markov chains. *J. Theoret. Probab.* **5** 545–561. MR1176437 https://doi.org/10.1007/BF01060435

[2] AARONSON, J., GILAT, D., KEANE, M. and DE VALK, V. (1989). An algebraic construction of a class of one-dependent processes. *Ann. Probab.* **17** 128–143. MR0972778

[3] ALON, N. and FELDHEIM, O. N. (2014). A note on general sliding window processes. *Electron. Commun. Probab.* **19** 66. MR3262072 https://doi.org/10.1214/ECP.v19-3341

[4] ALON, N. and SPENCER, J. H. (2016). *The Probabilistic Method*, 4th ed. *Wiley Series in Discrete Mathematics and Optimization*. Wiley, Hoboken, NJ. MR3524748

[5] ANGEL, O., HOLROYD, A. E., HUTCHCROFT, T. and LEVY, A. (2018). Mallows permutations as stable matchings.

[6] AUGER, N., BOUVEL, M., NICAUD, C. and PIVOTEAU, C. (2016). Analysis of algorithms for permutations biased by their number of records. In *Proceedings of the* 27*th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms—AofA'*16 12. Jagiellonian Univ., Dept. Theor. Comput. Sci., Kraków. MR3817514

[7] BASU, R. and BHATNAGAR, N. (2017). Limit theorems for longest monotone subsequences in random Mallows permutations. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1934–1951. MR3729641 https://doi.org/10.1214/16-AIHP777

[8] BECKENBACH, E. F., PÓLYA, G., LEHMER, D. H. and PHISTER, M. (1964). *Applied Combinatorial Mathematics*. *University of California Engineering and Physical Sciences Extension Series*. Wiley, New York. MR0174486

[9] BENJAMINI, I., BERGER, N., HOFFMAN, C. and MOSSEL, E. (2005). Mixing times of the biased card shuffling and the asymmetric exclusion process. *Trans. Amer. Math. Soc.* **357** 3013–3029. MR2135733 https://doi.org/10.1090/S0002-9947-05-03610-X

[10] BHATNAGAR, N. and PELED, R. (2015). Lengths of monotone subsequences in a Mallows permutation. *Probab. Theory Related Fields* **161** 719–780. MR3334280 https://doi.org/10.1007/s00440-014-0559-7

[11] BÓNA, M. (2012). *Combinatorics of Permutations*, 2nd ed. *Discrete Mathematics and Its Applications* (*Boca Raton*). CRC Press, Boca Raton, FL. MR2919720 https://doi.org/10.1201/b12210

[12] BORODIN, A., DIACONIS, P. and FULMAN, J. (2010). On adding a list of numbers (and other one-dependent determinantal processes). *Bull. Amer. Math. Soc.* (*N.S.*) **47** 639–670. MR2721041 https://doi.org/10.1090/S0273-0979-2010-01306-9

[13] BRAVERMAN, M. and MOSSEL, E. (2009). Sorting from noisy information. CoRR. Available at arXiv:0910.1191.

[14] BROMAN, E. I. (2005). One-dependent trigonometric determinantal processes are two-block-factors. *Ann. Probab.* **33** 601–609. MR2123203 https://doi.org/10.1214/009117904000000595

MSC2010 subject classifications. Primary 60G10; secondary 05C15, 05A05.
Key words and phrases. Proper coloring, finite dependence, random permutation.

[15] Burton, R. M., Goulet, M. and Meester, R. (1993). On 1-dependent processes and k-block factors. *Ann. Probab.* **21** 2157–2168. MR1245304

[16] Chern, H.-H., Hwang, H.-K. and Yeh, Y.-N. (2000). Distribution of the number of consecutive records. In *Proceedings of the Ninth International Conference "Random Structures and Algorithms"* (*Poznan*, 1999) **17** 169–196. MR1801131 https://doi.org/10.1002/1098-2418(200010/12)17:3/4<169::AID-RSA1>3.0.CO;2-K

[17] de Valk, V. (1988). The maximal and minimal 2-correlation of a class of 1-dependent 0–1 valued processes. *Israel J. Math.* **62** 181–205. MR0947821 https://doi.org/10.1007/BF02787121

[18] de Valk, V. (1989). A problem on 0–1 matrices. *Compos. Math.* **71** 139–179. MR1012638

[19] de Valk, V. (1993). Hilbert space representations of m-dependent processes. *Ann. Probab.* **21** 1550–1570. MR1235428

[20] de Valk, V. (1994). *One-Dependent Processes*: *Two-Block Factors and Non-two-Block Factors*. *CWI Tract* **85**. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam. MR1269540

[21] Diaconis, P. and Ram, A. (2000). Analysis of systematic scan Metropolis algorithms using Iwahori–Hecke algebra techniques Dept. Statistics, Stanford Univ., Stanford, CA.

[22] Erdős, P. and Lovász, L. (1975). Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets* (*Colloq.*, *Keszthely*, 1973; *Dedicated to P. Erdős on His* 60*th Birthday*), *Vol. II* 609–627. MR0382050

[23] Fill, J. A. (1996). On the distribution of binary search trees under the random permutation model. *Random Structures Algorithms* **8** 1–25. MR1368848 https://doi.org/10.1002/(SICI)1098-2418(199601)8:1<1::AID-RSA1>3.0.CO;2-1

[24] Gandolfi, A., Keane, M. and de Valk, V. (1989). Extremal two-correlations of two-valued stationary one-dependent processes. *Probab. Theory Related Fields* **80** 475–480. MR0976537 https://doi.org/10.1007/BF01794435

[25] Garban, C. (2013). Quantum gravity and the KPZ formula. *Séminaire Bourbaki* **64** 1052.

[26] Glick, N. (1978). Breaking records and breaking boards. *Amer. Math. Monthly* **85** 2–26. MR0461727 https://doi.org/10.2307/2978044

[27] Gnedin, A. and Olshanski, G. (2012). The two-sided infinite extension of the Mallows model for random permutations. *Adv. in Appl. Math.* **48** 615–639. MR2920835 https://doi.org/10.1016/j.aam.2012.01.001

[28] Heinrich, L. (1990). Asymptotic expansions in the central limit theorem for a special class of m-dependent random fields. II. Lattice case. *Math. Nachr.* **145** 309–327. MR1069038 https://doi.org/10.1002/mana.19901450124

[29] Hoeffding, W. and Robbins, H. (1948). The central limit theorem for dependent random variables. *Duke Math. J.* **15** 773–780. MR0026771

[30] Holroyd, A. E. (2017). One-dependent coloring by finitary factors. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 753–765. MR3634273 https://doi.org/10.1214/15-AIHP735

[31] Holroyd, A. E. and Liggett, T. M. (2015). Symmetric 1-dependent colorings of the integers. *Electron. Commun. Probab.* **20** 31. MR3327870 https://doi.org/10.1214/ECP.v20-4070

[32] Holroyd, A. E. and Liggett, T. M. (2016). Finitely dependent coloring. *Forum Math. Pi* **4** e9. MR3570073 https://doi.org/10.1017/fmp.2016.7

[33] Holroyd, A. E., Schramm, O. and Wilson, D. B. (2017). Finitary coloring. *Ann. Probab.* **45** 2867–2898. MR3706734 https://doi.org/10.1214/16-AOP1127

[34] Ibragimov, I. A. and Linnik, J. V. (1965). *Independent and Stationarily Connected Variables*. Izdat. "Nauka," Moscow.

[35] Janson, S. (1983). Renewal theory for M-dependent variables. *Ann. Probab.* **11** 558–568. MR0704542

[36] Janson, S. (1984). Runs in m-dependent sequences. *Ann. Probab.* **12** 805–818. MR0744235

[37] Janson, S. (2015). On degenerate sums of m-dependent variables. *J. Appl. Probab.* **52** 1146–1155. MR3439177 https://doi.org/10.1239/jap/1450802758

[38] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

[39] Knuth, D. E. (1998). *The Art of Computer Programming*: *sorting and searching*. *Vol.* 3. Addison-Wesley, Reading, MA. MR3077154

[40] Kortchemski, I. (2009). Asymptotic behavior of permutation records. *J. Combin. Theory Ser. A* **116** 1154–1166. MR2523585 https://doi.org/10.1016/j.jcta.2009.03.003

[41] Lawler, G. F. (2005). *Conformally Invariant Processes in the Plane*. *Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588

[42] Levy, A. (2017). Finitely dependent insertion processes. *Electron. J. Probab.* **22** 32. MR3646058 https://doi.org/10.1214/17-EJP35

[43] LIGGETT, T. M., SCHONMANN, R. H. and STACEY, A. M. (1997). Domination by product measures. *Ann. Probab.* **25** 71–95. MR1428500 https://doi.org/10.1214/aop/1024404279

[44] LINIAL, N. (1987). Distributive graph algorithms global solutions from local data. In *Proceedings of the* 28*th Annual Symposium on Foundations of Computer Science*. *SFCS* '87 331–335. IEEE Computer Society, Washington, DC.

[45] MALLOWS, C. L. (1957). Non-null ranking models. I. *Biometrika* **44** 114–130. MR0087267 https://doi.org/10.1093/biomet/44.1-2.114

[46] MATÚŠ, F. (1996). On two-block-factor sequences and one-dependence. *Proc. Amer. Math. Soc.* **124** 1237–1242. MR1301518 https://doi.org/10.1090/S0002-9939-96-03094-8

[47] MATÚŠ, F. (1998). Combining m-dependence with Markovness. *Ann. Inst. Henri Poincaré Probab. Stat.* **34** 407–423. MR1632845 https://doi.org/10.1016/S0246-0203(98)80023-X

[48] MUELLER, C. and STARR, S. (2013). The length of the longest increasing subsequence of a random Mallows permutation. *J. Theoret. Probab.* **26** 514–540. MR3055815 https://doi.org/10.1007/s10959-011-0364-5

[49] NAOR, M. (1991). A lower bound on probabilistic algorithms for distributive ring coloring. *SIAM J. Discrete Math.* **4** 409–412. MR1105946 https://doi.org/10.1137/0404036

[50] O'BRIEN, G. (1980). Scaling transformations for $\{0, 1\}$-valued sequences. *Z. Wahrsch. Verw. Gebiete* **53** 35–49.

[51] PROPP, J. and WILSON, D. (1998). Coupling from the past: A user's guide. In *Microsurveys in Discrete Probability* (*Princeton*, *NJ*, 1997). *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **41** 181–192. Amer. Math. Soc., Providence, RI. MR1630414

[52] RÜSCHENDORF, L. and DE VALK, V. (1993). On regression representations of stochastic processes. *Stochastic Process. Appl.* **46** 183–198. MR1226406 https://doi.org/10.1016/0304-4149(93)90001-K

[53] SMORODINSKY, M. (1992). Finitary isomorphism of m-dependent processes. In *Symbolic Dynamics and Its Applications* (*New Haven*, *CT*, 1991). *Contemp. Math.* **135** 373–376. Amer. Math. Soc., Providence, RI. MR1185104 https://doi.org/10.1090/conm/135/1185104

[54] SPINKA, Y. (2019). Finitely-dependent processes are finitary.

[55] STANLEY, R. P. (1997). *Enumerative Combinatorics*. *Vol.* 1. *Cambridge Studies in Advanced Mathematics* **49**. Cambridge Univ. Press, Cambridge. MR1442260 https://doi.org/10.1017/CBO9780511805967

[56] STARR, S. (2009). Thermodynamic limit for the Mallows model on S_n. *J. Math. Phys.* **50** 095208. MR2566888 https://doi.org/10.1063/1.3156746

[57] STARR, S. and WALTERS, M. (2018). Phase uniqueness for the Mallows measure on permutations. *J. Math. Phys.* **59** 063301. MR3817550 https://doi.org/10.1063/1.5017924

[58] STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Stat.* **36** 423–439. MR0177430 https://doi.org/10.1214/aoms/1177700153

The Annals of Probability
2020, Vol. 48, No. 1, 380–403
https://doi.org/10.1214/19-AOP1364
© Institute of Mathematical Statistics, 2020

GEOMETRIC ERGODICITY IN A WEIGHTED SOBOLEV SPACE

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For a discrete-time Markov chain $X = \{X(t)\}$ evolving on $\mathbb{R}^\ell$ with transition kernel P, natural, general conditions are developed under which the following are established:

(i) The transition kernel P has a purely discrete spectrum, when viewed as a linear operator on a weighted Sobolev space $L_\infty^{v,1}$ of functions with norm,

$$\|f\|_{v,1} = \sup_{x\in\mathbb{R}^\ell} \frac{1}{v(x)} \max\{|f(x)|, |\partial_1 f(x)|, \dots, |\partial_\ell f(x)|\},$$

where $v\colon \mathbb{R}^\ell \to [1,\infty)$ is a Lyapunov function and $\partial_i := \partial/\partial x_i$.

(ii) The Markov chain is geometrically ergodic in $L_\infty^{v,1}$: There is a unique invariant probability measure π and constants $B < \infty$ and $\delta > 0$ such that, for each $f \in L_\infty^{v,1}$, any initial condition $X(0) = x$, and all $t \geq 0$:

$$\big|\mathsf{E}_x[f(X(t))] - \pi(f)\big| \leq B\|f\|_{v,1} e^{-\delta t} v(x),$$

$$\big\|\nabla \mathsf{E}_x[f(X(t))]\big\|_2 \leq B\|f\|_{v,1} e^{-\delta t} v(x),$$

where $\pi(f) = \int f\,d\pi$.

(iii) For any function $f \in L_\infty^{v,1}$ there is a function $h \in L_\infty^{v,1}$ solving Poisson's equation:

$$h - Ph = f - \pi(f).$$

Part of the analysis is based on an operator-theoretic treatment of the sensitivity process that appears in the theory of Lyapunov exponents. Relationships with topological coupling, in terms of the Wasserstein metric, are also explored.

REFERENCES

[1] ARNOLD, L. and CRAUEL, H. (1991). Random dynamical systems. In *Lyapunov Exponents* (*Oberwolfach*, 1990). *Lecture Notes in Math.* **1486** 1–22. Springer, Berlin. MR1178943 https://doi.org/10.1007/BFb0086654

[2] ASMUSSEN, S. and GLYNN, P. W. (2007). *Stochastic Simulation*: *Algorithms and Analysis*. *Stochastic Modelling and Applied Probability* **57**. Springer, New York. MR2331321

[3] ATAR, R. and ZEITOUNI, O. (1997). Lyapunov exponents for finite state nonlinear filtering. *SIAM J. Control Optim*. **35** 36–55. MR1430282 https://doi.org/10.1137/S0363012994272046

[4] BALAJI, S. and MEYN, S. P. (2000). Multiplicative ergodicity and large deviations for an irreducible Markov chain. *Stochastic Process*. *Appl*. **90** 123–144. MR1787128 https://doi.org/10.1016/S0304-4149(00)00032-6

[5] BAXENDALE, P. H. (1989). Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. *Probab. Theory Related Fields* **81** 521–554. MR0995809 https://doi.org/10.1007/BF00367301

[6] BUTKOVSKY, O. (2014). Subgeometric rates of convergence of Markov processes in the Wasserstein metric. *Ann*. *Appl*. *Probab*. **24** 526–552. MR3178490 https://doi.org/10.1214/13-AAP922

[7] BUTZER, P. L. (1953). On two-dimensional Bernstein polynomials. *Canad*. *J*. *Math*. **5** 107–113. MR0052573 https://doi.org/10.4153/cjm-1953-014-2

MSC2010 subject classifications. 60J05, 60J35, 37A30, 47H20.
Key words and phrases. Markov chain, stochastic Lyapunov function, discrete spectrum, sensitivity process, weighted Sobolev space, Lyapunov exponent.

[8] CAO, X.-R. and CHEN, H.-F. (1997). Perturbation realization, potentials, and sensitivity analysis of Markov processes. *IEEE Trans. Automat. Control* **42** 1382–1393. MR1472856 https://doi.org/10.1109/9.633827

[9] CATTIAUX, P., CHAFAÏ, D. and GUILLIN, A. (2012). Central limit theorems for additive functionals of ergodic Markov diffusions processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 337–382. MR3069369

[10] CATTIAUX, P. and GUILLIN, A. (2017). Hitting times, functional inequalities, Lyapunov conditions and uniform ergodicity. *J. Funct. Anal.* **272** 2361–2391. MR3603301 https://doi.org/10.1016/j.jfa.2016.10.003

[11] CATTIAUX, P., GUILLIN, A., WANG, F.-Y. and WU, L. (2009). Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.* **256** 1821–1841. MR2498560 https://doi.org/10.1016/j.jfa.2009.01.003

[12] DEVRAJ, A., KONTOYIANNIS, I. and MEYN, S. (2018). Differential temporal difference learning. arXiv preprint. Available at arXiv:1812.11137.

[13] DEVRAJ, A. M. and MEYN, S. P. (2016). Differential TD learning for value function approximation. In *55th Conference on Decision and Control* 6347–6354.

[14] DJELLOUT, H., GUILLIN, A. and WU, L. (2004). Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann. Probab.* **32** 2702–2732. MR2078555 https://doi.org/10.1214/009117904000000531

[15] DONSKER, M. D. and VARADHAN, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time. I. II. *Comm. Pure Appl. Math.* **28** 1–47; ibid. **28**, 279–301. MR0386024 https://doi.org/10.1002/cpa.3160280102

[16] DONSKER, M. D. and VARADHAN, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time. III. *Comm. Pure Appl. Math.* **29** 389–461. MR0428471 https://doi.org/10.1002/cpa.3160290405

[17] DONSKER, M. D. and VARADHAN, S. R. S. (1983). Asymptotic evaluation of certain Markov process expectations for large time. IV. *Comm. Pure Appl. Math.* **36** 183–212. MR0690656 https://doi.org/10.1002/cpa.3160360204

[18] DOWN, D., MEYN, S. P. and TWEEDIE, R. L. (1995). Exponential and uniform ergodicity of Markov processes. *Ann. Probab.* **23** 1671–1691. MR1379163

[19] DURMUS, A. and MOULINES, É. (2017). Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann. Appl. Probab.* **27** 1551–1587. MR3678479 https://doi.org/10.1214/16-AAP1238

[20] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans. Amer. Math. Soc.* **371** 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[21] EDWARDS, D. A. (2011). On the Kantorovich–Rubinstein theorem. *Expo. Math.* **29** 387–398. MR2861765 https://doi.org/10.1016/j.exmath.2011.06.005

[22] FENG, J. (1999). Martingale problems for large deviations of Markov processes. *Stochastic Process. Appl.* **81** 165–216. MR1694569 https://doi.org/10.1016/S0304-4149(98)00104-5

[23] FENG, J. and KURTZ, T. G. (2006). *Large Deviations for Stochastic Processes. Mathematical Surveys and Monographs* **131**. Amer. Math. Soc., Providence, RI. MR2260560 https://doi.org/10.1090/surv/131

[24] FLEMING, W. H. (1977/78). Exit probabilities and optimal stochastic control. *Appl. Math. Optim.* **4** 329–346. MR0512217 https://doi.org/10.1007/BF01442148

[25] GLYNN, P. W. and MEYN, S. P. (1996). A Liapounov bound for solutions of the Poisson equation. *Ann. Probab.* **24** 916–931. MR1404536 https://doi.org/10.1214/aop/1039639370

[26] GONG, F. and WU, L. (2006). Spectral gap of positive operators and applications. *J. Math. Pures Appl.* (9) **85** 151–191. MR2199011 https://doi.org/10.1016/j.matpur.2004.11.004

[27] HAIRER, M. and MATTINGLY, J. C. (2008). Spectral gaps in Wasserstein distances and the 2D stochastic Navier–Stokes equations. *Ann. Probab.* **36** 2050–2091. MR2478676 https://doi.org/10.1214/08-AOP392

[28] HAIRER, M., MATTINGLY, J. C. and SCHEUTZOW, M. (2011). Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030 https://doi.org/10.1007/s00440-009-0250-6

[29] HAUSSMANN, W. and POTTINGER, P. (1976). On multivariate approximation by continuous linear operators. In *Constructive Theory of Functions of Several Variables: Proceedings of a Conference Held at Oberwolfach April 25–May 1* (W. Schempp and K. Zeller, eds.) 101–108. Springer, Berlin.

[30] HUANG, J., KONTOYIANNIS, I. and MEYN, S. P. (2002). The ODE method and spectral theory of Markov operators. In *Proceedings of the Workshop Held at the University of Kansas, Lawrence, Kansas, October 18–20, 2001* (T. E. Duncan and B. Pasik-Duncan, eds.) *Lecture Notes in Control and Information Sciences* **280**. Springer, Berlin.

[31] JAMISON, B. (1970). Irreducible Markov operators on $C(S)$. *Proc. Amer. Math. Soc.* **24** 366–370. MR0250046 https://doi.org/10.2307/2036363

[32] KINGSLEY, E. H. (1951). Bernstein polynomials for functions of two variables of class $C^{(k)}$. *Proc. Amer. Math. Soc.* **2** 64–71. MR0042548 https://doi.org/10.2307/2032622
[33] KONTOYIANNIS, I. and MEYN, S. P. (2003). Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Probab.* **13** 304–362. MR1952001 https://doi.org/10.1214/aoap/1042765670
[34] KONTOYIANNIS, I. and MEYN, S. P. (2005). Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electron. J. Probab.* **10** 61–123. MR2120240 https://doi.org/10.1214/EJP.v10-231
[35] KONTOYIANNIS, I. and MEYN, S. P. (2012). Geometric ergodicity and the spectral gap of non-reversible Markov chains. *Probab. Theory Related Fields* **154** 327–339. MR2981426 https://doi.org/10.1007/s00440-011-0373-4
[36] KONTOYIANNIS, I. and MEYN, S. P. (2017). Approximating a diffusion by a finite-state hidden Markov model. *Stochastic Process. Appl.* **127** 2482–2507. MR3660880 https://doi.org/10.1016/j.spa.2016.11.004
[37] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR1070361
[38] LAUGESEN, R. S., MEHTA, P. G., MEYN, S. P. and RAGINSKY, M. (2015). Poisson's equation in nonlinear filtering. *SIAM J. Control Optim.* **53** 501–525. MR3311867 https://doi.org/10.1137/13094743X
[39] MAJER, P. (2012). Multivariate Bernstein polynomials for approximation of derivatives (answer). *MathOverflow*. Available at https://mathoverflow.net/q/111257.
[40] MEYN, S. (2008). *Control Techniques for Complex Networks*. Cambridge Univ. Press, Cambridge. MR2372453
[41] MEYN, S. and TWEEDIE, R. L. (2009). *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge Univ. Press, Cambridge. MR2509253 https://doi.org/10.1017/CBO9780511626630
[42] PARDOUX, E. and VERETENNIKOV, A. Y. (2001). On the Poisson equation and diffusion approximation. I. *Ann. Probab.* **29** 1061–1085. MR1872736 https://doi.org/10.1214/aop/1015345596
[43] QIN, Q. and HOBERT, J. P. (2019). Geometric convergence bounds for Markov chains in Wasserstein distance based on generalized drift and contraction conditions. ArXiv E-prints. Available at arXiv:1902.02964.
[44] RADHAKRISHNAN, A., DEVRAJ, A. and MEYN, S. (2016). Learning techniques for feedback particle filter design. In 55*th Conference on Decision and Control* 5453–5459.
[45] RHEE, C.-H. and GLYNN, P. (2017). Lyapunov conditions for differentiability of Markov chain expectations: The absolutely continuous case. ArXiv e-prints.
[46] RIESZ, F. and NAGY, B. S. (1955). *Functional Analysis*. Ungar, New York. Translated from the 2d French by Leo F. Boron.
[47] SCHWEITZER, P. J. (1968). Perturbation theory and finite Markov chains. *J. Appl. Probab.* **5** 401–413. MR0234527 https://doi.org/10.2307/3212261
[48] WU, L. (2000). Uniformly integrable operators and large deviations for Markov processes. *J. Funct. Anal.* **172** 301–376. MR1753178 https://doi.org/10.1006/jfan.1999.3544
[49] WU, L. (2001). Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process. Appl.* **91** 205–238. MR1807683 https://doi.org/10.1016/S0304-4149(00)00061-2
[50] WU, L. M. (1995). Large deviations for Markov processes under superboundedness. *C. R. Acad. Sci. Paris Sér. I Math.* **321** 777–782. MR1354725

The Annals of Probability
2020, Vol. 48, No. 1, 404–444
https://doi.org/10.1214/19-AOP1365
© Institute of Mathematical Statistics, 2020

HYDRODYNAMICS IN A CONDENSATION REGIME: THE DISORDERED ASYMMETRIC ZERO-RANGE PROCESS

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We study asymmetric zero-range processes on $\mathbb{Z}$ with nearest-neighbour jumps and site disorder. The jump rate of particles is an arbitrary but bounded nondecreasing function of the number of particles. For any given environment satisfying suitable averaging properties, we establish a hydrodynamic limit given by a scalar conservation law *including* the domain above critical density, where the flux is shown to be constant.

REFERENCES

[1] ANDJEL, E. D. (1982). Invariant measures for the zero range processes. *Ann. Probab.* **10** 525–547. MR0659526

[2] ANDJEL, E. D., FERRARI, P. A., GUIOL, H. and LANDIM, C. (2000). Convergence to the maximal invariant measure for a zero-range process with random rates. *Stochastic Process. Appl.* **90** 67–81. MR1787125 https://doi.org/10.1016/S0304-4149(00)00037-5

[3] BAHADORAN, C. (1998). Hydrodynamical limit for spatially heterogeneous simple exclusion processes. *Probab. Theory Related Fields* **110** 287–331. MR1616563 https://doi.org/10.1007/s004400050150

[4] BAHADORAN, C. and BODINEAU, T. (2018). Quantitative estimates for the flux of TASEP with dilute site disorder. *Electron. J. Probab.* **23** Paper No. 44. MR3806412 https://doi.org/10.1214/18-EJP137

[5] BAHADORAN, C., GUIOL, H., RAVISHANKAR, K. and SAADA, E. (2002). A constructive approach to Euler hydrodynamics for attractive processes. Application to k-step exclusion. *Stochastic Process. Appl.* **99** 1–30. MR1894249 https://doi.org/10.1016/S0304-4149(01)00158-2

[6] BAHADORAN, C., GUIOL, H., RAVISHANKAR, K. and SAADA, E. (2006). Euler hydrodynamics of one-dimensional attractive particle systems. *Ann. Probab.* **34** 1339–1369. MR2257649 https://doi.org/10.1214/009117906000000115

[7] BAHADORAN, C., GUIOL, H., RAVISHANKAR, K. and SAADA, E. (2010). Strong hydrodynamic limit for attractive particle systems on $\mathbb{Z}$. *Electron. J. Probab.* **15** 1–43. MR2578381 https://doi.org/10.1214/EJP.v15-728

[8] BAHADORAN, C., GUIOL, H., RAVISHANKAR, K. and SAADA, E. (2014). Euler hydrodynamics for attractive particle systems in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 403–424. MR3189077 https://doi.org/10.1214/12-AIHP510

[9] BAHADORAN, C., GUIOL, H., RAVISHANKAR, K. and SAADA, E. (2019). Constructive Euler hydrodynamics for one-dimensional attractive particle systems. In *Sojourns in Probability and Statistical Physics* (V. Sidoravicius, ed.). Springer, Berlin. Preprint. Available at arXiv:1701.07994.

[10] BAHADORAN, C., MOUNTFORD, T., RAVISHANKAR, K. and SAADA, E. (2015). Supercriticality conditions for asymmetric zero-range process with sitewise disorder. *Braz. J. Probab. Stat.* **29** 313–335. MR3336869 https://doi.org/10.1214/14-BJPS273

[11] BAHADORAN, C., MOUNTFORD, T., RAVISHANKAR, K. and SAADA, E. (2017). Supercritical behavior of asymmetric zero-range process with sitewise disorder. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 766–801. MR3634274 https://doi.org/10.1214/15-AIHP736

[12] BAHADORAN, C., MOUNTFORD, T. S., RAVISHANKAR, K. and SAADA, E. (2019). Quenched convergence and strong local equilibrium for asymmetric zero-range process with sitewise disorder. *Probab. Theory Related Fields*. https://doi.org/10.1007/s00440-019-00916-2

MSC2010 subject classifications. 60K35, 82C22.

Key words and phrases. Asymmetric zero-range process, site disorder, phase transition, condensation, hydrodynamic limit.

[13] BALAZS, M. and SEPPÄLÄINEN, T. (2007). A convexity property of expectations under exponential weights. Available at arXiv.org/abs/0707.4273.

[14] BENJAMINI, I., FERRARI, P. A. and LANDIM, C. (1996). Asymmetric conservative processes with random rates. *Stochastic Process. Appl.* **61** 181–204. MR1386172 https://doi.org/10.1016/0304-4149(95)00077-1

[15] CHEN, G.-Q., EVEN, N. and KLINGENBERG, C. (2008). Hyperbolic conservation laws with discontinuous fluxes and hydrodynamic limit for particle systems. *J. Differential Equations* **245** 3095–3126. MR2460022 https://doi.org/10.1016/j.jde.2008.07.036

[16] CHLEBOUN, P. and GROSSKINSKY, S. (2014). Condensation in stochastic particle systems with stationary product measures. *J. Stat. Phys.* **154** 432–465. MR3162548 https://doi.org/10.1007/s10955-013-0844-3

[17] COVERT, P. and REZAKHANLOU, F. (1997). Hydrodynamic limit for particle systems with nonconstant speed parameter. *J. Stat. Phys.* **88** 383–426. MR1468390 https://doi.org/10.1007/BF02508477

[18] EVANS, M. R. (1996). Bose–Einstein condensation in disordered exclusion models and relation to traffic flow. *Europhys. Lett.* **36** 13.

[19] FERRARI, P. and KRUG, J. (1996). Phase transitions in driven diffusive systems with random rates. *J. Phys. A* **29** L:465–471.

[20] FERRARI, P. A. and SISKO, V. V. (2007). Escape of mass in zero-range processes with random rates. In *Asymptotics: Particles, Processes and Inverse Problems. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **55** 108–120. IMS, Beachwood, OH. MR2459934 https://doi.org/10.1214/074921707000000300

[21] GOBRON, T. and SAADA, E. (2010). Couplings, attractiveness and hydrodynamics for conservative particle systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1132–1177. MR2744889 https://doi.org/10.1214/09-AIHP347

[22] HARRIS, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Adv. Math.* **9** 66–89. MR0307392 https://doi.org/10.1016/0001-8708(72)90030-8

[23] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[24] KRUG, J. (2000). Phase separation in disordered exclusion models. *Braz. J. Phys.* **30** 97–104.

[25] LANDIM, C. (1996). Hydrodynamical limit for space inhomogeneous one-dimensional totally asymmetric zero-range processes. *Ann. Probab.* **24** 599–638. MR1404522 https://doi.org/10.1214/aop/1039639356

[26] LANDIM, C. (2014). Metastability for a non-reversible dynamics: The evolution of the condensate in totally asymmetric zero range processes. *Comm. Math. Phys.* **330** 1–32. MR3215575 https://doi.org/10.1007/s00220-014-2072-3

[27] LIGGETT, T. M. (1976). Coupling the simple exclusion process. *Ann. Probab.* **4** 339–356. MR0418291 https://doi.org/10.1214/aop/1176996084

[28] LIGGETT, T. M. (2005). *Interacting Particle Systems. Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374

[29] REZAKHANLOU, F. (1991). Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Comm. Math. Phys.* **140** 417–448. MR1130693

[30] REZAKHANLOU, F. (2002). Continuum limit for some growth models. *Stochastic Process. Appl.* **101** 1–41. MR1921440 https://doi.org/10.1016/S0304-4149(02)00100-X

[31] SCHÜTZ, G. M. and HARRIS, R. J. (2007). Hydrodynamics of the zero-range process in the condensation regime. *J. Stat. Phys.* **127** 419–430. MR2314354 https://doi.org/10.1007/s10955-007-9280-6

[32] SEPPÄLÄINEN, T. and KRUG, J. (1999). Hydrodynamics and platoon formation for a totally asymmetric exclusion model with particlewise disorder. *J. Stat. Phys.* **95** 525–567. MR1700871

The Annals of Probability
2020, Vol. 48, No. 1, 445–457
https://doi.org/10.1214/19-AOP1366
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A SIMPLE PROOF OF THE DPRZ THEOREM FOR 2D COVER TIMES

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We give a simple proof of the theorem by Dembo, Peres, Rosen and Zeitouni (DPRZ) regarding the time Brownian motion needs to cover every ε ball on the two-dimensional unit torus in the $\varepsilon \searrow 0$ limit.

REFERENCES

[1] BELIUS, D. and KISTLER, N. (2017). The subleading order of two dimensional cover times. *Probab. Theory Related Fields* **167** 461–552. MR3602852 https://doi.org/10.1007/s00440-015-0689-6

[2] DEMBO, A. (2006). Simple random covering disconnection, late and favorite points. In *International Congress of Mathematicians*. *Vol. III* 535–558. Eur. Math. Soc., Zürich. MR2275695

[3] DEMBO, A., PERES, Y., ROSEN, J. and ZEITOUNI, O. (2004). Cover times for Brownian motion and random walks in two dimensions. *Ann. of Math.* (2) **160** 433–464. MR2123929 https://doi.org/10.4007/annals.2004.160.433

[4] FITZSIMMONS, P. J. and PITMAN, J. (1999). Kac's moment formula and the Feynman–Kac formula for additive functionals of a Markov process. *Stochastic Process. Appl.* **79** 117–134. MR1670526 https://doi.org/10.1016/S0304-4149(98)00081-7

[5] KISTLER, N. (2015). Derrida's random energy models. From spin glasses to the extremes of correlated random fields. In *Correlated Random Systems: Five Different Methods*. *Lecture Notes in Math.* **2143** 71–120. Springer, Cham. MR3380419 https://doi.org/10.1007/978-3-319-17674-1_3

[6] RAY, D. (1963). Sojourn times and the exact Hausdorff measure of the sample path for planar Brownian motion. *Trans. Amer. Math. Soc.* **106** 436–444. MR0145599 https://doi.org/10.2307/1993753

MSC2010 subject classifications. 60J65, 60G50, 60G70, 60K35.
Key words and phrases. Brownian motion, multiscale analysis, cover time, hitting time.

The Annals of Probability
2020, Vol. 48, No. 1, 458–492
https://doi.org/10.1214/19-AOP1369
© Institute of Mathematical Statistics, 2020

ITÔ'S FORMULA FOR GAUSSIAN PROCESSES WITH STOCHASTIC DISCONTINUITIES

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We introduce a Skorokhod type integral and prove an Itô formula for a wide class of Gaussian processes which may exhibit stochastic discontinuities. Our Itô formula unifies and extends the classical one for general (i.e., possibly discontinuous) Gaussian martingales in the sense of Itô integration and the one for stochastically continuous Gaussian non-martingales in the Skorokhod sense, which was first derived in Alòs et al. (*Ann. Probab.* **29** (2001) 766–801).

REFERENCES

[1] ALÒS, E., MAZET, O. and NUALART, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29** 766–801. MR1849177 https://doi.org/10.1214/aop/1008956692

[2] ALPAY, D. and KIPNIS, A. (2013). A generalized white noise space approach to stochastic integration for a class of Gaussian stationary increment processes. *Opuscula Math.* **33** 395–417. MR3046404 https://doi.org/10.7494/OpMath.2013.33.3.395

[3] BENDER, C. (2003). An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Process. Appl.* **104** 81–106. MR1956473 https://doi.org/10.1016/S0304-4149(02)00212-0

[4] BENDER, C. (2003). An S-transform approach to integration with respect to a fractional Brownian motion. *Bernoulli* **9** 955–983. MR2046814 https://doi.org/10.3150/bj/1072215197

[5] BENDER, C. (2014). Backward SDEs driven by Gaussian processes. *Stochastic Process. Appl.* **124** 2892–2916. MR3217428 https://doi.org/10.1016/j.spa.2014.03.013

[6] CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2003). Stochastic volatility for Lévy processes. *Math. Finance* **13** 345–382. MR1995283 https://doi.org/10.1111/1467-9965.00020

[7] CHERIDITO, P. and NUALART, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 1049–1081. MR2172209 https://doi.org/10.1016/j.anihpb.2004.09.004

[8] DĘBICKI, K., HASHORVA, E. and JI, L. (2014). Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. *Extremes* **17** 411–429. MR3252819 https://doi.org/10.1007/s10687-014-0186-9

[9] DECREUSEFOND, L. and ÜSTÜNEL, A. S. (1999). Stochastic analysis of the fractional Brownian motion. *Potential Anal.* **10** 177–214. MR1677455 https://doi.org/10.1023/A:1008634027843

[10] DUDLEY, R. M. and NORVAIŠA, R. (1999). *Differentiability of Six Operators on Nonsmooth Functions and p-Variation*. *Lecture Notes in Math.* **1703**. Springer, Berlin. MR1705318 https://doi.org/10.1007/BFb0100744

[11] FRIZ, P. K., GESS, B., GULISASHVILI, A. and RIEDEL, S. (2016). The Jain–Monrad criterion for rough paths and applications to random Fourier series and non-Markovian Hörmander theory. *Ann. Probab.* **44** 684–738. MR3456349 https://doi.org/10.1214/14-AOP986

[12] GANTI, V., SINGH, A., PASSALACQUA, P. and FOUFOULA-GEORGIOU, E. (2009). Subordinated Brownian motion model for sediment transport. *Phys. Rev. E* **80** 011111.

[13] HU, Y., JOLIS, M. and TINDEL, S. (2013). On Stratonovich and Skorohod stochastic calculus for Gaussian processes. *Ann. Probab.* **41** 1656–1693. MR3098687 https://doi.org/10.1214/12-AOP751

[14] JAIN, N. C. and MONRAD, D. (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete* **59** 139–159. MR0650607 https://doi.org/10.1007/BF00531739

MSC2010 subject classifications. 60H07, 60H05, 60G15.
Key words and phrases. Gaussian processes, Itô's formula, stochastic discontinuities, stochastic integrals, S-transform.

[15] JANSON, S. (1997). *Gaussian Hilbert Spaces*. *Cambridge Tracts in Mathematics* **129**. Cambridge Univ. Press, Cambridge. MR1474726 https://doi.org/10.1017/CBO9780511526169

[16] KOZUBOWSKI, T. J., MEERSCHAERT, M. M. and PODGÓRSKI, K. (2006). Fractional Laplace motion. *Adv. in Appl. Probab.* **38** 451–464. MR2264952 https://doi.org/10.1239/aap/1151337079

[17] KRUK, I., RUSSO, F. and TUDOR, C. A. (2007). Wiener integrals, Malliavin calculus and covariance measure structure. *J. Funct. Anal.* **249** 92–142. MR2338856 https://doi.org/10.1016/j.jfa.2007.03.031

[18] KUBO, I. (1983). Itô formula for generalized Brownian functionals. In *Theory and Application of Random Fields* (*Bangalore*, 1982). *Lect. Notes Control Inf. Sci.* **49** 156–166. Springer, Berlin. MR0799940 https://doi.org/10.1007/BFb0044689

[19] KUMAR, A., MEERSCHAERT, M. M. and VELLAISAMY, P. (2011). Fractional normal inverse Gaussian diffusion. *Statist. Probab. Lett.* **81** 146–152. MR2740078 https://doi.org/10.1016/j.spl.2010.10.007

[20] KUMAR, A. and VELLAISAMY, P. (2012). Fractional normal inverse Gaussian process. *Methodol. Comput. Appl. Probab.* **14** 263–283. MR2912339 https://doi.org/10.1007/s11009-010-9201-z

[21] LEBOVITS, J. (2019). Stochastic calculus with respect to Gaussian processes. *Potential Anal.* **50** 1–42. MR3900844 https://doi.org/10.1007/s11118-017-9671-5

[22] LEBOVITS, J. and VÉHEL, J. L. (2014). White noise-based stochastic calculus with respect to multifractional Brownian motion. *Stochastics* **86** 87–124. MR3176508 https://doi.org/10.1080/17442508.2012.758727

[23] Lei and Nualart LEI, P. and NUALART, D. (2012). Stochastic calculus for Gaussian processes and application to hitting times. *Commun. Stoch. Anal.* **6** 379–402. MR2988698

[24] MEERSCHAERT, M. M., KOZUBOWSKI, T. J., MOLZ, F. J. and LU, S. (2004). Fractional Laplace model for hydraulic conductivity. *Geophys. Res. Lett.* **31** L08501.

[25] MOCIOALCA, O. and VIENS, F. (2005). Skorohod integration and stochastic calculus beyond the fractional Brownian scale. *J. Funct. Anal.* **222** 385–434. MR2132395 https://doi.org/10.1016/j.jfa.2004.07.013

[26] NORVAIŠA, R. (2002). Chain rules and p-variation. *Studia Math.* **149** 197–238. MR1890731 https://doi.org/10.4064/sm149-3-1

[27] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[28] NUALART, D. and TAQQU, M. S. (2006). Wick–Itô formula for Gaussian processes. *Stoch. Anal. Appl.* **24** 599–614. MR2220074 https://doi.org/10.1080/07362990600629348

[29] NUALART, D. and TAQQU, M. S. (2008). Wick–Itô formula for regular processes and applications to the Black and Scholes formula. *Stochastics* **80** 477–487. MR2456333 https://doi.org/10.1080/17442500801928788

[30] PIPIRAS, V. and TAQQU, M. S. (2001). Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli* **7** 873–897. MR1873833 https://doi.org/10.2307/3318624

[31] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. Second edition. Version 2.1, Corrected third printing. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[32] SAMORODNITSKY, G. (1988). Continuity of Gaussian processes. *Ann. Probab.* **16** 1019–1033. MR0942752

[33] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author. MR1739520

[34] WYŁOMAŃSKA, A., KUMAR, A., POŁOCZAŃSKI, R. and VELLAISAMY, P. (2016). Inverse Gaussian and its inverse process as the subordinators of fractional Brownian motion. *Phys. Rev. E* **94** 042128. https://doi.org/10.1103/PhysRevE.94.042128

[35] YOSIDA, K. (1995). *Functional Analysis*. *Classics in Mathematics*. Springer, Berlin. Reprint of the sixth (1980) edition. MR1336382 https://doi.org/10.1007/978-3-642-61859-8

The Annals of Probability
2020, Vol. 48, No. 1, 493–525
https://doi.org/10.1214/19-AOP1371
© Institute of Mathematical Statistics, 2020

ON THE PROBABILITY OF NONEXISTENCE IN BINOMIAL SUBSETS

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Given a hypergraph $\Gamma = (\Omega, \mathcal{X})$ and a sequence $\mathbf{p} = (p_\omega)_{\omega \in \Omega}$ of values in $(0, 1)$, let $\Omega_{\mathbf{p}}$ be the random subset of Ω obtained by keeping every vertex ω independently with probability p_ω. We investigate the general question of deriving fine (asymptotic) estimates for the probability that $\Omega_{\mathbf{p}}$ is an independent set in Γ, which is an omnipresent problem in probabilistic combinatorics. Our main result provides a sequence of upper and lower bounds on this probability, each of which can be evaluated explicitly in terms of the joint cumulants of small sets of edge indicator random variables. Under certain natural conditions, these upper and lower bounds coincide asymptotically, thus giving the precise asymptotics of the probability in question. We demonstrate the applicability of our results with two concrete examples: subgraph containment in random (hyper)graphs and arithmetic progressions in random subsets of the integers.

REFERENCES

[1] AHLBACH, C., USATINE, J. and PIPPENGER, N. A combinatorial interpretation of the joint cumulant. Preprint. Available at arXiv:1211.0652.

[2] ALON, N. and SPENCER, J. H. (2016). *The Probabilistic Method*, 4th ed. *Wiley Series in Discrete Mathematics and Optimization*. Wiley, Hoboken, NJ. MR3524748

[3] BALOGH, J., MORRIS, R. and SAMOTIJ, W. (2015). Independent sets in hypergraphs. *J. Amer. Math. Soc.* **28** 669–709. MR3327533 https://doi.org/10.1090/S0894-0347-2014-00816-X

[4] BALOGH, J., MORRIS, R., SAMOTIJ, W. and WARNKE, L. (2016). The typical structure of sparse K_{r+1}-free graphs. *Trans. Amer. Math. Soc.* **368** 6439–6485. MR3461039 https://doi.org/10.1090/tran/6552

[5] BOLLOBÁS, B. (1981). Threshold functions for small subgraphs. *Math. Proc. Cambridge Philos. Soc.* **90** 197–206. MR0620729 https://doi.org/10.1017/S0305004100058655

[6] BOLLOBÁS, B. (1988). The chromatic number of random graphs. *Combinatorica* **8** 49–55. MR0951992 https://doi.org/10.1007/BF02122551

[7] BOPPONA, R. and SPENCER, J. (1989). A useful elementary correlation inequality. *J. Combin. Theory Ser. A* **50** 305–307. MR0989201 https://doi.org/10.1016/0097-3165(89)90022-8

[8] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magy. Tud. Akad. Mat. Kut. Intéz. Közl.* **5** 17–61. MR0125031

[9] FRIEZE, A. (1992). On small subgraphs of random graphs. In *Random Graphs, Vol.* 2 (*Poznań*, 1989). *Wiley-Intersci. Publ.* 67–90. Wiley, New York. MR1166608

[10] HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Camb. Philos. Soc.* **56** 13–20. MR0115221

[11] HUNDACK, C., PRÖMEL, H. J. and STEGER, A. (1993). Extremal graph problems for graphs with a color-critical vertex. *Combin. Probab. Comput.* **2** 465–477. MR1264719 https://doi.org/10.1017/S0963548300000833

[12] JANSON, S. (1990). Poisson approximation for large deviations. *Random Structures Algorithms* **1** 221–229. MR1138428 https://doi.org/10.1002/rsa.3240010209

[13] JANSON, S. (1998). New versions of Suen's correlation inequality. In *Proceedings of the Eighth International Conference "Random Structures and Algorithms"* (*Poznan*, 1997) **13** 467–483. MR1662796 https://doi.org/10.1002/(SICI)1098-2418(199810/12)13:3/4<467::AID-RSA15>3.3.CO;2-E

MSC2010 subject classifications. 60C05, 05C65, 05C69, 05C80.
Key words and phrases. Janson's inequality, Harris's inequality, joint cumulants.

[14] JANSON, S., ŁUCZAK, T. and RUCIŃSKI, A. (1990). An exponential bound for the probability of nonexistence of a specified subgraph in a random graph. In *Random Graphs '87* (*Poznań*, 1987) 73–87. Wiley, Chichester. MR1094125

[15] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs*. *Wiley-Interscience Series in Discrete Mathematics and Optimization*. Wiley-Interscience, New York. MR1782847 https://doi.org/10.1002/9781118032718

[16] KAROŃSKI, M. and RUCIŃSKI, A. (1983). On the number of strictly balanced subgraphs of a random graph. In *Graph Theory* (*Łagów*, 1981). *Lecture Notes in Math*. **1018** 79–83. Springer, Berlin. MR0730636 https://doi.org/10.1007/BFb0071616

[17] OSTHUS, D., PRÖMEL, H. J. and TARAZ, A. (2003). For which densities are random triangle-free graphs almost surely bipartite? **23** 105–150. MR1996629 https://doi.org/10.1007/s00493-003-0016-1

[18] PRÖMEL, H. J. and STEGER, A. (1992). The asymptotic number of graphs not containing a fixed color-critical subgraph. *Combinatorica* **12** 463–473. MR1194736 https://doi.org/10.1007/BF01305238

[19] PRÖMEL, H. J. and STEGER, A. (1996). On the asymptotic structure of sparse triangle free graphs. *J. Graph Theory* **21** 137–151. MR1368739 https://doi.org/10.1002/(SICI)1097-0118(199606)22:2<137::AID-JGT4>3.0.CO;2-O

[20] RIORDAN, O. and WARNKE, L. (2015). The Janson inequalities for general up-sets. *Random Structures Algorithms* **46** 391–395. MR3302903 https://doi.org/10.1002/rsa.20506

[21] SAXTON, D. and THOMASON, A. (2015). Hypergraph containers. *Invent*. *Math*. **201** 925–992. MR3385638 https://doi.org/10.1007/s00222-014-0562-8

[22] SCHÜRGER, K. (1979). Limit theorems for complete subgraphs of random graphs. *Period*. *Math*. *Hungar*. **10** 47–53. MR0505825 https://doi.org/10.1007/BF02018372

[23] STARK, D. and WORMALD, N. (2018). The probability of non-existence of a subgraph in a moderately sparse random graph. *Combin*. *Probab*. *Comput*. **27** 672–715. MR3816064 https://doi.org/10.1017/S0963548318000202

[24] SUEN, W.-C. S. (1990). A correlation inequality and a Poisson limit theorem for nonoverlapping balanced subgraphs of a random graph. *Random Structures Algorithms* **1** 231–242. MR1138429 https://doi.org/10.1002/rsa.3240010210

[25] WORMALD, N. C. (1996). The perturbation method and triangle-free random graphs. In *Proceedings of the Seventh International Conference on Random Structures and Algorithms* (*Atlanta*, *GA*, 1995) **9** 253–270. MR1611697 https://doi.org/10.1002/(SICI)1098-2418(199608/09)9:1/2<253::AID-RSA15>3.3.CO;2-F

The Annals of Probability

Vol. 48 March 2020 No. 2

Articles