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AN ALMOST SURE KPZ RELATION FOR SLE AND BROWNIAN MOTION

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The peanosphere construction of Duplantier, Miller and Sheffield provides a means of representing a γ-Liouville quantum gravity (LQG) surface, $\gamma \in (0, 2)$, decorated with a space-filling form of Schramm's SLE_κ, $\kappa = 16/\gamma^2 \in (4, \infty)$, η as a gluing of a pair of trees which are encoded by a correlated two-dimensional Brownian motion Z. We prove a KPZ-type formula which relates the Hausdorff dimension of any Borel subset A of the range of η, which can be defined as a function of η (modulo time parameterization) to the Hausdorff dimension of the corresponding time set $\eta^{-1}(A)$. This result serves to reduce the problem of computing the Hausdorff dimension of any set associated with an SLE, CLE or related processes in the interior of a domain to the problem of computing the Hausdorff dimension of a certain set associated with a Brownian motion. For many natural examples, the associated Brownian motion set is well known. As corollaries, we obtain new proofs of the Hausdorff dimensions of the SLE_κ curve for $\kappa \neq 4$; the double points and cut points of SLE_κ for $\kappa > 4$; and the intersection of two flow lines of a Gaussian free field. We obtain the Hausdorff dimension of the set of m-tuple points of space-filling SLE_κ for $\kappa > 4$ and $m \geq 3$ by computing the Hausdorff dimension of the so-called $(m-2)$-tuple $\pi/2$-cone times of a correlated planar Brownian motion.

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LOCALIZATION IN RANDOM GEOMETRIC GRAPHS WITH TOO MANY EDGES

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We consider a random geometric graph $G(\chi_n, r_n)$, given by connecting two vertices of a Poisson point process χ_n of intensity n on the d-dimensional unit torus whenever their distance is smaller than the parameter r_n. The model is conditioned on the rare event that the number of edges observed, $|E|$, is greater than $(1+\delta)\mathbb{E}(|E|)$, for some fixed $\delta > 0$. This article proves that upon conditioning, with high probability there exists a ball of diameter r_n which contains a clique of at least $\sqrt{2\delta\mathbb{E}(|E|)}(1-\varepsilon)$ vertices, for any given $\varepsilon > 0$. Intuitively, this region contains all the "excess" edges the graph is forced to contain by the conditioning event, up to lower order corrections. As a consequence of this result, we prove a large deviations principle for the upper tail of the edge count of the random geometric graph. The rate function of this large deviation principle turns out to be nonconvex.

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THE MAXIMAL FLOW FROM A COMPACT CONVEX SUBSET TO INFINITY IN FIRST PASSAGE PERCOLATION ON $\mathbb{Z}^d$

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We consider the standard first passage percolation model on $\mathbb{Z}^d$ with a distribution G on $\mathbb{R}^+$ that admits an exponential moment. We study the maximal flow between a compact convex subset A of $\mathbb{R}^d$ and infinity. The study of maximal flow is associated with the study of sets of edges of minimal capacity that cut A from infinity. We prove that the rescaled maximal flow between nA and infinity $\phi(nA)/n^{d-1}$ almost surely converges toward a deterministic constant depending on A. This constant corresponds to the capacity of the boundary ∂A of A and is the integral of a deterministic function over ∂A. This result was shown in dimension 2 and conjectured for higher dimensions by Garet in (*Annals of Applied Probability* **19** (2009) 641–660).

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HITTING PROBABILITIES OF A BROWNIAN FLOW WITH RADIAL DRIFT

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We consider a stochastic flow $\phi_t(x, \omega)$ in $\mathbb{R}^n$ with initial point $\phi_0(x, \omega) = x$, driven by a single n-dimensional Brownian motion, and with an outward radial drift of magnitude $\frac{F(\|\phi_t(x)\|)}{\|\phi_t(x)\|}$, with F nonnegative, bounded and Lipschitz. We consider initial points x lying in a set of positive distance from the origin. We show that there exist constants $C^*, c^* > 0$ not depending on n, such that if $F > C^* n$ then the image of the initial set under the flow has probability 0 of hitting the origin. If $0 \leq F \leq c^* n^{3/4}$, and if the initial set has a nonempty interior, then the image of the set has positive probability of hitting the origin.

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RANDOM MOMENT PROBLEMS UNDER CONSTRAINTS

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We investigate moment sequences of probability measures on subsets of the real line under constraints of certain moments being fixed. This corresponds to studying sections of nth moment spaces, that is, the spaces of moment sequences of order n. By equipping these sections with the uniform or more general probability distributions, we manage to give for large n precise results on the (probabilistic) barycenters of moment space sections and the fluctuations of random moments around these barycenters. The measures associated to the barycenters belong to the Bernstein–Szegő class and show strong universal behavior. We prove Gaussian fluctuations and moderate and large deviations principles. Furthermore, we demonstrate how fixing moments by a constraint leads to breaking the connection between random moments and random matrices.

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THE MAXIMUM OF THE FOUR-DIMENSIONAL MEMBRANE MODEL

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We show that the centred maximum of the four-dimensional membrane model on a box of sidelength N converges in distribution. To do so, we use a criterion of Ding, Roy and Zeitouni (*Ann. Probab.* **45** (2017) 3886–3928) and prove sharp estimates for the Green's function of the discrete Bilaplacian. These estimates are the main contribution of this work and might also be of independent interest. To derive them, we use estimates for the approximation quality of finite difference schemes as well as results for the Green's function of the continuous Bilaplacian.

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CUTOFF FOR THE MEAN-FIELD ZERO-RANGE PROCESS WITH BOUNDED MONOTONE RATES

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We consider the zero-range process with arbitrary bounded monotone rates on the complete graph, in the regime where the number of sites diverges while the density of particles per site converges. We determine the asymptotics of the mixing time from any initial configuration, and establish the cutoff phenomenon. The intuitive picture is that the system separates into a slowly evolving solid phase and a quickly relaxing liquid phase: as time passes, the solid phase dissolves into the liquid phase, and the mixing time is essentially the time at which the system becomes completely liquid. Our proof uses the path coupling technique of Bubley and Dyer, and the analysis of a suitable hydrodynamic limit. To the best of our knowledge, even the order of magnitude of the mixing time was unknown, except in the special case of constant rates.

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TRANSLATION-INVARIANT GIBBS STATES OF THE ISING MODEL: GENERAL SETTING

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We prove that at any inverse temperature β and on any transitive amenable graph, the automorphism-invariant Gibbs states of the ferromagnetic Ising model are convex combinations of the plus and minus states. The theorem is equivalent with the differentiability of the free energy with respect to the temperature at any temperature. This is obtained for a general class of interactions, that is automorphism-invariant and irreducible coupling constants. The proof uses the random current representation of the Ising model. The result is novel when the graph is not $\mathbb{Z}^d$, or when the graph is $\mathbb{Z}^d$ but endowed with infinite-range interactions, or even $\mathbb{Z}^2$ with finite-range interactions.

Among the other corollaries of this result, we can list continuity of the magnetization at any noncritical temperature and the uniqueness of FK-Ising infinite-volume measures at any temperature.

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BUSEMANN FUNCTIONS AND GIBBS MEASURES IN DIRECTED POLYMER MODELS ON $\mathbb{Z}^2$

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We consider random walk in a space-time random potential, also known as directed random polymer measures, on the planar square lattice with nearest-neighbor steps and general i.i.d. weights on the vertices. We construct covariant cocycles and use them to prove new results on existence, uniqueness/nonuniqueness, and asymptotic directions of semi-infinite polymer measures (solutions to the Dobrushin–Lanford–Ruelle equations). We also prove nonexistence of covariant or deterministically directed bi-infinite polymer measures. Along the way, we prove almost sure existence of Busemann function limits in directions where the limiting free energy has some regularity.

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THE ENDPOINT DISTRIBUTION OF DIRECTED POLYMERS

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Probabilistic models of directed polymers in random environment have received considerable attention in recent years. Much of this attention has focused on integrable models. In this paper, we introduce some new computational tools that do not require integrability. We begin by defining a new kind of abstract limit object, called "partitioned subprobability measure," to describe the limits of endpoint distributions of directed polymers. Inspired by a recent work of Mukherjee and Varadhan on large deviations of the occupation measure of Brownian motion, we define a suitable topology on the space of partitioned subprobability measures and prove that this topology is compact. Then using a variant of the cavity method from the theory of spin glasses, we show that any limit law of a sequence of endpoint distributions must satisfy a fixed point equation on this abstract space, and that the limiting free energy of the model can be expressed as the solution of a variational problem over the set of fixed points. As a first application of the theory, we prove that in an environment with finite exponential moment, the endpoint distribution is asymptotically purely atomic if and only if the system is in the low temperature phase. The analogous result for a heavy-tailed environment was proved by Vargas in 2007. As a second application, we prove a subsequential version of the longstanding conjecture that in the low temperature phase, the endpoint distribution is asymptotically localized in a region of stochastically bounded diameter. All our results hold in arbitrary dimensions, and make no use of integrability.

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THE DISTRIBUTION OF GAUSSIAN MULTIPLICATIVE CHAOS ON THE UNIT INTERVAL

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We consider a subcritical Gaussian multiplicative chaos (GMC) measure defined on the unit interval $[0, 1]$ and prove an exact formula for the fractional moments of the total mass of this measure. Our formula includes the case where log-singularities (also called insertion points) are added in 0 and 1, the most general case predicted by the Selberg integral. The idea to perform this computation is to introduce certain auxiliary functions resembling holomorphic observables of conformal field theory that will be solutions of hypergeometric equations. Solving these equations then provides nontrivial relations that completely determine the moments we wish to compute. We also include a detailed discussion of the so-called reflection coefficients appearing in tail expansions of GMC measures and in Liouville theory. Our theorem provides an exact value for one of these coefficients. Lastly, we mention some additional applications to small deviations for GMC measures, to the behavior of the maximum of the log-correlated field on the interval and to random hermitian matrices.

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TRANSITION FROM TRACY–WIDOM TO GAUSSIAN FLUCTUATIONS OF EXTREMAL EIGENVALUES OF SPARSE ERDŐS–RÉNYI GRAPHS

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We consider the statistics of the extreme eigenvalues of sparse random matrices, a class of random matrices that includes the normalized adjacency matrices of the Erdős–Rényi graph $G(N, p)$. Tracy–Widom fluctuations of the extreme eigenvalues for $p \gg N^{-2/3}$ was proved in (*Probab. Theory Related Fields* **171** (2018) 543–616; *Comm. Math. Phys.* **314** (2012) 587–640). We prove that there is a crossover in the behavior of the extreme eigenvalues at $p \sim N^{-2/3}$. In the case that $N^{-7/9} \ll p \ll N^{-2/3}$, we prove that the extreme eigenvalues have asymptotically Gaussian fluctuations. Under a mean zero condition and when $p = CN^{-2/3}$, we find that the fluctuations of the extreme eigenvalues are given by a combination of the Gaussian and the Tracy–Widom distribution. These results show that the eigenvalues at the edge of the spectrum of sparse Erdős–Rényi graphs are less rigid than those of random d-regular graphs (Bauerschmidt et al. (2019)) of the same average degree.

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CORRELATED RANDOM MATRICES: BAND RIGIDITY AND EDGE UNIVERSALITY

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We prove edge universality for a general class of correlated real symmetric or complex Hermitian Wigner matrices with arbitrary expectation. Our theorem also applies to internal edges of the self-consistent density of states. In particular, we establish a strong form of band rigidity which excludes mismatches between location and label of eigenvalues close to internal edges in these general models.

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ON THE NATURE OF THE SWISS CHEESE IN DIMENSION 3

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We study scenarii linked with the *Swiss cheese picture* in dimension 3 obtained when two random walks are forced to meet often, or when one random walk is forced to squeeze its range. In the case of two random walks, we show that they most likely meet in a region of *optimal density*. In the case of one random walk, we show that a small range is reached by a strategy uniform in time. Both results rely on an original inequality estimating the cost of visiting sparse sites, and in the case of one random walk on the precise large deviation principle of van den Berg, Bolthausen and den Hollander (*Ann. of Math. (2)* **153** (2001) 355–406), including their sharp estimates of the rate functions in the neighborhood of the origin.

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CONSTRUCTING A SOLUTION OF THE $(2+1)$-DIMENSIONAL KPZ EQUATION

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The $(d+1)$-dimensional KPZ equation is the canonical model for the growth of rough d-dimensional random surfaces. A deep mathematical understanding of the KPZ equation for $d=1$ has been achieved in recent years, and the case $d \geq 3$ has also seen some progress. The most physically relevant case of $d=2$, however, is not very well understood mathematically, largely due to the renormalization that is required: in the language of renormalization group analysis, the $d=2$ case is neither ultraviolet superrenormalizable like the $d=1$ case nor infrared superrenormalizable like the $d \geq 3$ case. Moreover, unlike in $d=1$, the Cole–Hopf transform is not directly usable in $d=2$ because solutions to the multiplicative stochastic heat equation are distributions rather than functions. In this article, we show the existence of subsequential scaling limits as $\varepsilon \to 0$ of Cole–Hopf solutions of the $(2+1)$-dimensional KPZ equation with white noise mollified to spatial scale ε and nonlinearity multiplied by the vanishing factor $|\log \varepsilon|^{-\frac{1}{2}}$. We also show that the scaling limits obtained in this way do not coincide with solutions to the linearized equation, meaning that the nonlinearity has a nonvanishing effect. We thus propose our scaling limit as a notion of KPZ evolution in $2+1$ dimensions.

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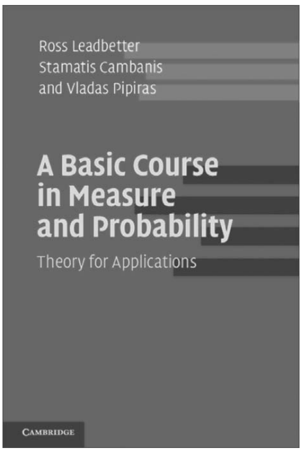

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