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PLANAR BROWNIAN MOTION AND GAUSSIAN MULTIPLICATIVE CHAOS

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We construct the analogue of Gaussian multiplicative chaos measures for the local times of planar Brownian motion by exponentiating the square root of the local times of small circles. We also consider a flat measure supported on points whose local time is within a constant of the desired thickness level and show a simple relation between the two objects. Our results extend those of (*Ann. Probab.* **22** (1994) 566–625), and in particular, cover the entire L^1-phase or subcritical regime. These results allow us to obtain a nondegenerate limit for the appropriately rescaled size of thick points, thereby considerably refining estimates of (*Acta Math.* **186** (2001) 239–270).

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BOUNDING THE NUMBER OF SELF-AVOIDING WALKS: HAMMERSLEY–WELSH WITH POLYGON INSERTION

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Let $c_n = c_n(d)$ denote the number of self-avoiding walks of length n starting at the origin in the Euclidean nearest-neighbour lattice $\mathbb{Z}^d$. Let $\mu = \lim_n c_n^{1/n}$ denote the connective constant of $\mathbb{Z}^d$. In 1962, Hammersley and Welsh (*Quart. J. Math. Oxford Ser. (2)* **13** (1962) 108–110) proved that, for each $d \geq 2$, there exists a constant $C > 0$ such that $c_n \leq \exp(Cn^{1/2})\mu^n$ for all $n \in \mathbb{N}$. While it is anticipated that $c_n \mu^{-n}$ has a power-law growth in n, the best-known upper bound in dimension two has remained of the form $n^{1/2}$ inside the exponential.

The natural first improvement to demand for a given planar lattice is a bound of the form $c_n \leq \exp(Cn^{1/2-\varepsilon})\mu^n$, where μ denotes the connective constant of the lattice in question. We derive a bound of this form for two such lattices, for an explicit choice of $\varepsilon > 0$ in each case. For the hexagonal lattice $\mathbb{H}$, the bound is proved for all $n \in \mathbb{N}$; while for the Euclidean lattice $\mathbb{Z}^2$, it is proved for a set of $n \in \mathbb{N}$ of limit supremum density equal to one.

A power-law upper bound on $c_n \mu^{-n}$ for $\mathbb{H}$ is also proved, contingent on a nonquantitative assertion concerning this lattice's connective constant.

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Key words and phrases. Self-avoiding walk, Hammersley–Welsh bound.

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ELLIPTIC STOCHASTIC QUANTIZATION

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We prove an explicit formula for the law in zero of the solution of a class of elliptic SPDE in $\mathbb{R}^2$. This formula is the simplest instance of *dimensional reduction*, discovered in the physics literature by Parisi and Sourlas (*Phys. Rev. Lett.* **43** (1979) 744–745), which links the law of an elliptic SPDE in $d+2$ dimension with a Gibbs measure in d dimensions. This phenomenon is similar to the relation between a $\mathbb{R}^{d+1}$ dimensional parabolic SPDE and its $\mathbb{R}^d$ dimensional invariant measure. As such, dimensional reduction of elliptic SPDEs can be considered a sort of *elliptic stochastic quantisation* procedure in the sense of Nelson (*Phys. Rev.* **150** (1966) 1079–1085) and Parisi and Wu (*Sci. Sin.* **24** (1981) 483–496). Our proof uses in a fundamental way the representation of the law of the SPDE as a supersymmetric quantum field theory. Dimensional reduction for the supersymmetric theory was already established by Klein et al. (*Comm. Math. Phys.* **94** (1984) 459–482). We fix a subtle gap in their proof and also complete the dimensional reduction picture by providing the link between the elliptic SPDE and the supersymmetric model. Even in our $d=0$ context the arguments are nontrivial and a non-supersymmetric, elementary proof seems only to be available in the Gaussian case.

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GROWTH-FRAGMENTATION PROCESSES IN BROWNIAN MOTION INDEXED BY THE BROWNIAN TREE

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We consider the model of Brownian motion indexed by the Brownian tree. For every $r \geq 0$ and every connected component of the set of points where Brownian motion is greater than r, we define the boundary size of this component, and we then show that the collection of these boundary sizes evolves when r varies like a well-identified growth-fragmentation process. We then prove that the same growth-fragmentation process appears when slicing a Brownian disk at height r and considering the perimeters of the resulting connected components.

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POINTS OF INFINITE MULTIPLICITY OF PLANAR BROWNIAN MOTION: MEASURES AND LOCAL TIMES

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It is well known (see Dvoretzky, Erdős and Kakutani (*Bull. Res. Council Israel Sect. F* **7F** (1958) 175–180) and Le Gall (*J. Funct. Anal.* **71** (1987) 246–262)) that a planar Brownian motion $(B_t)_{t\geq 0}$ has points of infinite multiplicity, and these points form a dense set on the range. Our main result is the construction of a family of random measures, denoted by $\{\mathcal{M}_\infty^\alpha\}_{0<\alpha<2}$, that are supported by the set of the points of infinite multiplicity. We prove that for any $\alpha \in (0, 2)$, almost surely the Hausdorff dimension of $\mathcal{M}_\infty^\alpha$ equals $2 - \alpha$, and $\mathcal{M}_\infty^\alpha$ is supported by the set of thick points defined in Bass, Burdzy and Khoshnevisan (*Ann. Probab.* **22** (1994) 566–625) as well as by that defined in Dembo, Peres, Rosen and Zeitouni (*Acta Math.* **186** (2001) 239–270).

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AVERAGING DYNAMICS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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We consider slow/fast systems where the slow system is driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We show that unlike in the case $H = \frac{1}{2}$, convergence to the averaged solution takes place in probability and the limiting process solves the 'naïvely' averaged equation. Our proof strongly relies on the recently obtained stochastic sewing lemma.

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CONFLUENCE OF GEODESICS IN LIOUVILLE QUANTUM GRAVITY FOR $\gamma \in (0, 2)$

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We prove that for any metric, which one can associate with a Liouville quantum gravity (LQG) surface for $\gamma \in (0, 2)$ satisfying certain natural axioms, its geodesics exhibit the following confluence property. For any fixed point z, a.s. any two γ-LQG geodesics started from distinct points other than z must merge into each other and subsequently coincide until they reach z. This is analogous to the confluence of geodesics property for the Brownian map proven by Le Gall (*Acta Math.* **205** (2010) 287–360). Our results apply for the subsequential limits of Liouville first passage percolation and are an important input in the proof of the existence and uniqueness of the LQG metric for all $\gamma \in (0, 2)$.

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FROM NONLINEAR FOKKER–PLANCK EQUATIONS TO SOLUTIONS OF DISTRIBUTION DEPENDENT SDE

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We construct weak solutions to the McKean–Vlasov SDE

$$dX(t) = b\left(X(t), \frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)dt + \sigma\left(X(t), \frac{d\mathcal{L}_{X(t)}}{dt}(X(t))\right)dW(t)$$

on $\mathbb{R}^d$ for possibly degenerate diffusion matrices σ with $X(0)$ having a given law, which has a density with respect to Lebesgue measure, dx. Here, $\mathcal{L}_{X(t)}$ denotes the law of $X(t)$. Our approach is to first solve the corresponding nonlinear Fokker–Planck equations and then use the well-known superposition principle to obtain weak solutions of the above SDE.

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PERIODIC PÓLYA URNS, THE DENSITY METHOD AND ASYMPTOTICS OF YOUNG TABLEAUX

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Pólya urns are urns where at each unit of time a ball is drawn and replaced with some other balls according to its colour. We introduce a more general model: the replacement rule depends on the colour of the drawn ball *and* the value of the time (mod p). We extend the work of Flajolet et al. on Pólya urns: the generating function encoding the evolution of the urn is studied by methods of analytic combinatorics. We show that the initial *partial* differential equations lead to *ordinary* linear differential equations which are related to hypergeometric functions (giving the exact state of the urns at time n). When the time goes to infinity, we prove that these *periodic Pólya urns* have asymptotic fluctuations which are described by a product of generalized gamma distributions. With the additional help of what we call the *density method* (a method which offers access to enumeration and random generation of poset structures), we prove that the law of the southeast corner of a triangular Young tableau follows asymptotically a product of generalized gamma distributions. This allows us to tackle some questions related to the continuous limit of random Young tableaux and links with random surfaces.

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FINITARY ISOMORPHISMS OF BROWNIAN MOTIONS

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Ornstein and Shields (*Advances in Math.* **10** (1973) 143–146) proved that Brownian motion reflected on a bounded region is an infinite entropy Bernoulli flow, and, thus, Ornstein theory yielded the existence of a measure-preserving isomorphism between any two such Brownian motions. For fixed $h > 0$, we construct by elementary methods, isomorphisms with almost surely finite coding windows between Brownian motions reflected on the intervals $[0, qh]$ for all positive rationals q.

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NEAR-CRITICAL SPANNING FORESTS AND RENORMALIZATION

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We study random two-dimensional spanning forests in the plane that can be viewed both in the discrete case and in their appropriately taken scaling limits as a uniformly chosen spanning tree with some Poissonian deletion of edges or points. We show how to relate these scaling limits to a stationary distribution of a natural coalescent-type Markov process on a state space of abstract graphs with real-valued edge weights. This Markov process can be interpreted as a renormalization flow.

This provides a model for which one can rigorously implement the formalism proposed by the third author in order to relate the law of the scaling limit of a critical model to a stationary distribution of such a renormalization/Markov process. When starting from any two-dimensional lattice with constant edge weights, the Markov process does indeed converge in law to this stationary distribution that corresponds to a scaling limit of UST with Poissonian deletions.

The results of this paper heavily build on the convergence in distribution of branches of the UST to SLE_2 (a result by Lawler, Schramm and Werner) as well as on the convergence of the suitably renormalized length of the loop-erased random walk to the "natural parametrization" of the SLE_2 (a recent result by Lawler and Viklund).

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RANDOM WALKS ON DYNAMICAL RANDOM ENVIRONMENTS WITH NONUNIFORM MIXING

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In this paper, we study random walks on dynamical random environments in $1+1$ dimensions. Assuming that the environment is invariant under space-time shifts and fulfills a mild mixing hypothesis, we establish a law of large numbers and a concentration inequality around the asymptotic speed. The mixing hypothesis imposes a polynomial decay rate of covariances on the environment with sufficiently high exponent but does not impose uniform mixing. Examples of environments for which our methods apply include the contact process and Markovian environments with a positive spectral gap, such as the East model. For the East model, we also obtain that the distinguished zero satisfies a law of large numbers with strictly positive speed.

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ALGORITHMIC THRESHOLDS FOR TENSOR PCA

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We study the algorithmic thresholds for principal component analysis of Gaussian k-tensors with a planted rank-one spike, via Langevin dynamics and gradient descent. In order to efficiently recover the spike from natural initializations, the signal-to-noise ratio must diverge in the dimension. Our proof shows that the mechanism for the success/failure of recovery is the strength of the "curvature" of the spike on the maximum entropy region of the initial data. To demonstrate this, we study the dynamics on a generalized family of high-dimensional landscapes with planted signals, containing the spiked tensor models as specific instances. We identify thresholds of signal-to-noise ratios above which order 1 time recovery succeeds; in the case of the spiked tensor model, these match the thresholds conjectured for algorithms such as approximate message passing. Below these thresholds, where the curvature of the signal on the maximal entropy region is weak, we show that recovery from certain natural initializations takes at least stretched exponential time. Our approach combines global regularity estimates for spin glasses with pointwise estimates to study the recovery problem by a perturbative approach.

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FINITELY DEPENDENT PROCESSES ARE FINITARY

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We show that any finitely dependent invariant process on a transitive amenable graph is a finitary factor of an i.i.d. process. With an additional assumption on the geometry of the graph, namely that no two balls with different centers are identical, we further show that the i.i.d. process may be taken to have entropy arbitrarily close to that of the finitely dependent process. As an application, we give an affirmative answer to a question of Holroyd (*Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 753–765).

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