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IDENTIFICATION OF THE POLARON MEASURE IN STRONG COUPLING AND THE PEKAR VARIATIONAL FORMULA

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The path measure corresponding to the *Fröhlich polaron* appearing in quantum statistical mechanics is defined as the tilted measure

$$\mathrm{d}\widehat{\mathbb{P}}_{\varepsilon,T} = \frac{1}{Z(\varepsilon,T)} \exp\left\{\frac{1}{2}\int_{-T}^{T}\int_{-T}^{T} \frac{\varepsilon \mathrm{e}^{-\varepsilon|t-s|}}{|\omega(t)-\omega(s)|}\,\mathrm{d}s\,\mathrm{d}t\right\} \mathrm{d}\mathbb{P}.$$

Here, $\varepsilon > 0$ is a constant known as the *Kac parameter* or the *inverse-coupling parameter*, and $\mathbb{P}$ is the distribution of the increments of the three-dimensional Brownian motion. In (*Comm. Pure Appl. Math.* **73** (2020) 350–383) it was shown that, when $\varepsilon > 0$ is sufficiently small or sufficiently large, the (thermodynamic) limit $\lim_{T\to\infty}\widehat{\mathbb{P}}_{\varepsilon,T} = \widehat{\mathbb{P}}_\varepsilon$ exists as a process with stationary increments, and this limit was identified explicitly as a mixture of Gaussian processes. In the present article the *strong coupling limit* or the *vanishing Kac parameter* limit $\lim_{\varepsilon\to 0}\widehat{\mathbb{P}}_\varepsilon$ is investigated. It is shown that this limit exists and coincides with the *increments* of the so-called *Pekar process*, a stationary diffusion with generator $\frac{1}{2}\Delta + (\nabla\psi/\psi)\cdot\nabla$, where ψ is the unique (up to spatial translations) maximizer of the *Pekar variational problem*

$$g_0 = \sup_{\|\psi\|_2=1}\left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi^2(x)\psi^2(y)|x-y|^{-1}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\|\nabla\psi\|_2^2\right\}.$$

As the Pekar process was also earlier shown (*Ann. Probab.* **44** (2016) 3934–3964; *Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2017) 2214–2228; *Comm. Pure Appl. Math.* **70** (2017) 1598–1629) to be the limiting object of the *mean-field polaron measures*, the present identification of the strong coupling limit is a rigorous justification of the mean-field approximation of the polaron problem (on the level of path measures) conjectured by Spohn in (*Ann. Physics* **175** (1987) 278–318). Replacing the Coulomb potential by continuous function vanishing at infinity and assuming uniqueness (modulo translations) of the relevant variational problem, our proof also shows that path measures coming from a Kac interaction of the above form with translation invariance in space converge to the increments of the corresponding mean-field model.

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ON THE ABSOLUTE CONTINUITY OF RANDOM NODAL VOLUMES

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We study the absolute continuity with respect to the Lebesgue measure of the distribution of the nodal volume associated with a smooth, nondegenerate and stationary Gaussian field $(f(x), x \in \mathbb{R}^d)$. Under mild conditions, we prove that in dimension $d \geq 3$, the distribution of the nodal volume has an absolutely continuous component plus a possible singular part. This singular part is actually unavoidable bearing in mind that some Gaussian processes have a positive probability to keep a constant sign on some compact domain. Our strategy mainly consists in proving closed Kac–Rice type formulas allowing one to express the volume of the set $\{f = 0\}$ as integrals of explicit functionals of $(f, \nabla f, \text{Hess}(f))$ and next to deduce that the random nodal volume belongs to the domain of a suitable Malliavin gradient. The celebrated Bouleau–Hirsch criterion then gives conditions ensuring the absolute continuity.

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ON THE NUMBER OF MAXIMAL PATHS IN DIRECTED LAST-PASSAGE PERCOLATION

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We show that the number of maximal paths in directed last-passage percolation on the hypercubic lattice $\mathbb{Z}^d$ $(d \geq 2)$ in which weights take finitely many values is typically exponentially large.

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INVERTING THE MARKOVIAN PROJECTION, WITH AN APPLICATION TO LOCAL STOCHASTIC VOLATILITY MODELS

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We study two-dimensional stochastic differential equations (SDEs) of McKean–Vlasov type in which the conditional distribution of the second component of the solution given the first enters the equation for the first component of the solution. Such SDEs arise when one tries to invert the Markovian projection developed in (*Probab. Theory Related Fields* **71** (1986) 501–516), typically to produce an Itô process with the fixed-time marginal distributions of a given one-dimensional diffusion but richer dynamical features. We prove the strong existence of stationary solutions for these SDEs as well as their strong uniqueness in an important special case. Variants of the SDEs discussed in this paper enjoy frequent application in the calibration of local stochastic volatility models in finance, despite the very limited theoretical understanding.

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DETERMINISTIC WALKS IN RANDOM ENVIRONMENT

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Motivated by the random Lorentz gas, we study deterministic walks in random environment and show that (in simple, yet relevant cases) they can be reduced to a class of random walks in random environment where the jump probability depends (weakly) on the past. In addition, we prove few basic results (hopefully, the germ of a general theory) for the latter purely probabilistic model.

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MARTINGALE BENAMOU–BRENIER: A PROBABILISTIC PERSPECTIVE

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In classical optimal transport, the contributions of Benamou–Brenier and McCann regarding the time-dependent version of the problem are cornerstones of the field and form the basis for a variety of applications in other mathematical areas.

We suggest a Benamou–Brenier type formulation of the martingale transport problem for given d-dimensional distributions μ, ν in convex order. The unique solution $M^* = (M_t^*)_{t\in[0,1]}$ of this problem turns out to be a Markov-martingale which has several notable properties: In a specific sense it mimics the movement of a Brownian particle as closely as possible subject to the conditions $M_0^* \sim \mu$, $M_1^* \sim \nu$. Similar to McCann's displacement-interpolation, M^* provides a time-consistent interpolation between μ and ν. For particular choices of the initial and terminal law, M^* recovers archetypical martingales such as Brownian motion, geometric Brownian motion, and the Bass martingale. Furthermore, it yields a natural approximation to the local vol model and a new approach to Kellerer's theorem.

This article is parallel to the work of Huesmann–Trevisan, who consider a related class of problems from a PDE-oriented perspective.

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FRACTIONAL DIFFUSION LIMIT FOR A KINETIC EQUATION WITH AN INTERFACE

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We consider the limit of a linear kinetic equation with reflection-transmission-absorption at an interface and with a degenerate scattering kernel. The equation arises from a microscopic chain of oscillators in contact with a heat bath. In the absence of the interface, the solutions exhibit a superdiffusive behavior in the long time limit. With the interface, the long time limit is the unique solution of a version of the fractional in space heat equation with reflection-transmission-absorption at the interface. The limit problem corresponds to a certain stable process that is either absorbed, reflected or transmitted upon crossing the interface.

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LIMIT PROFILE FOR RANDOM TRANSPOSITIONS[1]

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We present an improved version of Diaconis–Shahshahani upper bound lemma, which is used to compute the limiting value of the distance to stationarity. We then apply it to the random transposition shuffle.

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SOLUTION OF THE KOLMOGOROV EQUATION FOR TASEP

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We provide a direct and elementary proof that the formula obtained in (Matetski, Quastel and Remenik (2016)) for the TASEP transition probabilities for general (one-sided) initial data solves the Kolmogorov backward equation. The same method yields the solution for the related PushASEP particle system.

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ANOMALOUS DIFFUSION FOR MULTI-DIMENSIONAL CRITICAL KINETIC FOKKER–PLANCK EQUATIONS

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We consider a particle moving in $d \geq 2$ dimensions, its velocity being a reversible diffusion process, with identity diffusion coefficient, of which the invariant measure behaves, roughly, like $(1 + |v|)^{-\beta}$ as $|v| \to \infty$, for some constant $\beta > 0$. We prove that for large times, after a suitable rescaling, the position process resembles a Brownian motion if $\beta \geq 4+d$, a stable process if $\beta \in [d, 4+d)$ and an integrated multi-dimensional generalization of a Bessel process if $\beta \in (d-2, d)$. The critical cases $\beta = d$, $\beta = 1+d$ and $\beta = 4+d$ require special rescalings.

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NONLINEAR LARGE DEVIATION BOUNDS WITH APPLICATIONS TO WIGNER MATRICES AND SPARSE ERDŐS–RÉNYI GRAPHS

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We prove general nonlinear large deviation estimates similar to Chatterjee–Dembo's original bounds, except that we do not require any second order smoothness. Our approach relies on convex analysis arguments and is valid for a broad class of distributions. Our results are then applied in three different setups. Our first application consists in the mean-field approximation of the partition function of the Ising model under an optimal assumption on the spectra of the adjacency matrices of the sequence of graphs. Next, we apply our general large deviation bound to investigate the large deviation of the traces of powers of Wigner matrices with sub-Gaussian entries and the upper tail of cycles counts in sparse Erdős–Rényi graphs down to the sparsity threshold $n^{-1/2}$.

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MIXING TIME OF THE ADJACENT WALK ON THE SIMPLEX

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By viewing the N-simplex as the set of positions of $N-1$ ordered particles on the unit interval, the adjacent walk is the continuous-time Markov chain obtained by updating independently at rate 1 the position of each particle with a sample from the uniform distribution over the interval given by the two particles adjacent to it. We determine its spectral gap and prove that both the total variation distance and the separation distance to the uniform distribution exhibit a cutoff phenomenon, with mixing times that differ by a factor 2. The results are extended to the family of log-concave distributions obtained by replacing the uniform sampling by a symmetric log-concave Beta distribution.

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THE CLT IN HIGH DIMENSIONS: QUANTITATIVE BOUNDS VIA MARTINGALE EMBEDDING

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We introduce a new method for obtaining quantitative convergence rates for the central limit theorem (CLT) in a high-dimensional setting. Using our method, we obtain several new bounds for convergence in transportation distance and entropy, and in particular: (a) We improve the best known bound, obtained by the third named author (*Probab. Theory Related Fields* **170** (2018) 821–845), for convergence in quadratic Wasserstein transportation distance for bounded random vectors; (b) we derive the first nonasymptotic convergence rate for the entropic CLT in arbitrary dimension, for general log-concave random vectors (this adds to (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 777–790), where a finite Fisher information is assumed); (c) we give an improved bound for convergence in transportation distance under a log-concavity assumption and improvements for both metrics under the assumption of strong log-concavity. Our method is based on martingale embeddings and specifically on the Skorokhod embedding constructed in (*Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2016) 1259–1280).

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LARGE DEVIATIONS AND LOCALIZATION OF THE MICROCANONICAL ENSEMBLES GIVEN BY MULTIPLE CONSTRAINTS

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We develop a unified theory to analyze the microcanonical ensembles with several constraints given by unbounded observables. Several interesting phenomena that do not occur in the single constraint case can happen under the multiple constraints case. We systematically analyze the detailed structures of such microcanonical ensembles in two orthogonal directions using the theory of large deviations. First of all, we establish the equivalence of ensembles result, which exhibits an interesting phase transition phenomenon. Secondly, we study the localization and delocalization phenomena by obtaining large deviation results for the joint law of empirical distributions and the maximum component. Some concrete examples for which the theory applies will be given as well.

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THE ALDOUS CHAIN ON CLADOGRAMS IN THE DIFFUSION LIMIT

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In (*Combin. Probab. Comput.* **9** (2000) 191–204), Aldous investigates a symmetric Markov chain on cladograms and gives bounds on its mixing and relaxation times. The latter bound was sharpened in (*Random Structures Algorithms* **20** (2002) 59–70). In the present paper, we encode cladograms as binary, algebraic measure trees and show that this Markov chain on cladograms with a fixed number of leaves converges in distribution as the number of leaves tends to infinity. We give a rigorous construction of the limit as the solution of a well-posed martingale problem. The existence of a continuum limit diffusion was conjectured by Aldous, and we therefore refer to it as Aldous diffusion. We show that the Aldous diffusion is a Feller process with continuous paths, and the algebraic measure Brownian CRT is its unique invariant distribution.

Furthermore, we consider the vector of the masses of the three subtrees connected to a sampled branch point. In the Brownian CRT, its annealed law is known to be the Dirichlet distribution. Here, we give an explicit expression for the infinitesimal evolution of its quenched law under the Aldous diffusion.

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ANISOTROPIC BOOTSTRAP PERCOLATION IN THREE DIMENSIONS

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Consider a p-random subset A of initially infected vertices in the discrete cube $[L]^3$, and assume that the neighborhood of each vertex consists of the a_i nearest neighbors in the $\pm e_i$-directions for each $i \in \{1, 2, 3\}$, where $a_1 \le a_2 \le a_3$. Suppose we infect any healthy vertex $x \in [L]^3$ already having $a_3 + 1$ infected neighbors, and that infected sites remain infected forever. In this paper, we determine the critical length for percolation up to a constant factor in the exponent, for all triples (a_1, a_2, a_3). To do so, we introduce a new algorithm called the *beams process* and prove an exponential decay property for a family of subcritical two-dimensional bootstrap processes.

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LIMITING ENTROPY OF DETERMINANTAL PROCESSES

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We extend Lyons's tree entropy theorem to general determinantal measures. As a byproduct we show that the sofic entropy of an invariant determinantal measure does not depend on the chosen sofic approximation.

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ERRATA: MEAN FIELD GAMES WITH COMMON NOISE

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This note corrects Lemma 3.7 in our paper (*Ann. Probab.* **44** (2016) 3740–3803). The main results of the paper remain correct as stated.

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