ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 49, No. 2—March 2021

THE ANNALS OF PROBABILITY Vol. 49, No. 2, pp. 527–1066 March 2021

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

IMS OFFICERS

President: Regina Y. Liu, Department of Statistics, Rutgers University, Piscataway, New Jersey 08854-8019, USA

President-Elect: Krzysztof Burdzy, Department of Mathematics, University of Washington, Seattle, Washington 98195-4350, USA

Past President: Susan Murphy, Department of Statistics, Harvard University, Cambridge, Massachusetts 02138-2901, USA

Executive Secretary: Edsel Peña, Department of Statistics, University of South Carolina, Columbia, South Carolina 29208-001, USA

Treasurer: Zhengjun Zhang, Department of Statistics, University of Wisconsin, Madison, Wisconsin 53706-1510, USA

Program Secretary: Ming Yuan, Department of Statistics, Columbia University, New York, NY 10027-5927, USA

IMS EDITORS

The Annals of Statistics. *Editors*: Richard J. Samworth, Statistical Laboratory, Centre for Mathematical Sciences, University of Cambridge, Cambridge, CB3 0WB, UK. Ming Yuan, Department of Statistics, Columbia University, New York, NY 10027, USA

The Annals of Applied Statistics. *Editor-in-Chief*: Karen Kafadar, Department of Statistics, University of Virginia, Heidelberg Institute for Theoretical Studies, Charlottesville, VA 22904-4135, USA

The Annals of Probability. *Editors*: Alice Guionnet, Unité de Mathématiques Pures et Appliquées, ENS de Lyon, Lyon, France. Christophe Garban, Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne, France

The Annals of Applied Probability. *Editors*: François Delarue, Laboratoire J. A. Dieudonné, Université de Nice Sophia-Antipolis, France-06108 Nice Cedex 2. Peter Friz, Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany and Weierstrass-Institut für Angewandte Analysis und Stochastik, 10117 Berlin, Germany

Statistical Science. *Editor*: Sonia Petrone, Department of Decision Sciences, Università Bocconi, 20100 Milano MI, Italy

The IMS Bulletin. *Editor*: Vlada Limic, UMR 7501 de l'Université de Strasbourg et du CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 49, Number 2, March 2021. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

Copyright © 2021 by the Institute of Mathematical Statistics
Printed in the United States of America

The Annals of Probability
2021, Vol. 49, No. 2, 527–555
https://doi.org/10.1214/20-AOP1454
© Institute of Mathematical Statistics, 2021

WEAK EXISTENCE AND UNIQUENESS FOR MCKEAN–VLASOV SDES WITH COMMON NOISE

BY WILLIAM R. P. HAMMERSLEY[*], DAVID ŠIŠKA[†] AND ŁUKASZ SZPRUCH[‡]

School of Mathematics, University of Edinburgh, [*]*w.r.p.hammersley@sms.ed.ac.uk;* [†]*d.siska@ed.ac.uk;* [‡]*l.szpruch@ed.ac.uk*

This paper concerns the McKean–Vlasov stochastic differential equation (SDE) with common noise. An appropriate definition of a weak solution to such an equation is developed. The importance of the notion of compatibility in this definition is highlighted by a demonstration of its role in connecting weak solutions to McKean–Vlasov SDEs with common noise and solutions to corresponding stochastic partial differential equations (SPDEs). By keeping track of the dependence structure between all components in a sequence of approximating processes, a compactness argument is employed to prove the existence of a weak solution assuming boundedness and joint continuity of the coefficients (allowing for degenerate diffusions). Weak uniqueness is established when the private (idiosyncratic) noise's diffusion coefficient is nondegenerate and the drift is regular in the total variation distance. This seems sharp when one considers using finite-dimensional noise to regularise an infinite dimensional problem. The proof relies on a suitably tailored cost function in the Monge–Kantorovich problem and representation of weak solutions via Girsanov transformations.

REFERENCES

[1] AHUJA, S. (2016). Wellposedness of mean field games with common noise under a weak monotonicity condition. *SIAM J. Control Optim.* **54** 30–48. MR3439756 https://doi.org/10.1137/140974730

[2] BARBU, V. and RÖCKNER, M. (2020). From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE. *Ann. Probab.* **48** 1902–1920. MR4124528 https://doi.org/10.1214/19-AOP1410

[3] BARBU, V., RÖCKNER, M. and RUSSO, F. (2017). Doubly probabilistic representation for the stochastic porous media type equation. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 2043–2073. MR3729647 https://doi.org/10.1214/16-AIHP783

[4] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[5] BLACKWELL, D. and DUBINS, L. E. (1983). An extension of Skorohod's almost sure representation theorem. *Proc. Amer. Math. Soc.* **89** 691–692. MR0718998 https://doi.org/10.2307/2044607

[6] BOGACHEV, V. I. (2007). *Measure Theory. Vol. I*, *II*. Springer, Berlin. MR2267655 https://doi.org/10.1007/978-3-540-34514-5

[7] BOSSY, M. and JABIR, J.-F. (2019). On the wellposedness of some McKean models with moderated or singular diffusion coefficient. In *Frontiers in Stochastic Analysis—BSDEs*, *SPDEs and Their Applications*. *Springer Proc. Math. Stat.* **289** 43–87. Springer, Cham. MR4008340 https://doi.org/10.1007/978-3-030-22285-7_2

[8] BRIAND, P., CARDALIAGUET, P., DE RAYNAL, P.-E. C. and HU, Y. (2019). Forward and backward stochastic differential equations with normal constraint in law. Available at arXiv:1903.01114.

[9] CAMPI, L. and FISCHER, M. (2018). N-player games and mean-field games with absorption. *Ann. Appl. Probab.* **28** 2188–2242. MR3843827 https://doi.org/10.1214/17-AAP1354

[10] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062 https://doi.org/10.2307/j.ctvckq7qf

[11] CARMONA, R. and DELARUE, F. (2017). *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, Berlin.

MSC2020 subject classifications. Primary 60H10; secondary 60H15.

Key words and phrases. Stochastic McKean–Vlasov equations, mean-field equations, Girsanov transformations.

[12] CARMONA, R., DELARUE, F. and LACKER, D. (2016). Mean field games with common noise. *Ann. Probab.* **44** 3740–3803. MR3572323 https://doi.org/10.1214/15-AOP1060

[13] COGHI, M. and GESS, B. (2019). Stochastic nonlinear Fokker–Planck equations. *Nonlinear Anal.* **187** 259–278. MR3954095 https://doi.org/10.1016/j.na.2019.05.003

[14] CRISAN, D., JANJIGIAN, C. and KURTZ, T. G. (2018). Particle representations for stochastic partial differential equations with boundary conditions. *Electron. J. Probab.* **23** 65. MR3835471 https://doi.org/10.1214/18-EJP186

[15] DAWSON, D. and VAILLANCOURT, J. (1995). Stochastic McKean–Vlasov equations. *NoDEA Nonlinear Differential Equations Appl.* **2** 199–229. MR1328577 https://doi.org/10.1007/BF01295311

[16] DE RAYNAL, P.-E. C. and FRIKHA, N. (2018). Well-posedness for some non-linear diffusion processes and related PDE on the Wasserstein space. Available at arXiv:1811.06904.

[17] DELARUE, F. (2019). Restoring uniqueness to mean-field games by randomizing the equilibria. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** 598–678. MR4022284 https://doi.org/10.1007/s40072-019-00135-9

[18] DELARUE, F. and FOGUEN TCHUENDOM, R. (2020). Selection of equilibria in a linear quadratic mean-field game. *Stochastic Process. Appl.* **130** 1000–1040. MR4046528 https://doi.org/10.1016/j.spa.2019.04.005

[19] HAMBLY, B. and SØJMARK, A. (2019). An SPDE model for systemic risk with endogenous contagion. *Finance Stoch.* **23** 535–594. MR3968278 https://doi.org/10.1007/s00780-019-00396-1

[20] HAMMERSLEY, W. R. P., ŠIŠKA, D. and SZPRUCH, Ł. (2018). McKean-Vlasov SDEs under measure dependent Lyapunov conditions. Available at arXiv:1802.03974.

[21] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927

[22] HUANG, X. (2019). Path-distribution dependent SDEs with singular coefficients. Available at arXiv:1902.08953.

[23] HUANG, X. and WANG, F.-Y. (2019). Distribution dependent SDEs with singular coefficients. *Stochastic Process. Appl.* **129** 4747–4770. MR4013879 https://doi.org/10.1016/j.spa.2018.12.012

[24] JABIN, P.-E. and WANG, Z. (2018). Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.* **214** 523–591. MR3858403 https://doi.org/10.1007/s00222-018-0808-y

[25] JABIR, J. F. (2019). Rate of propagation of chaos for diffusive stochastic particle systems via Girsanov transformation. Available at arXiv:1907.09096.

[26] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley and Los Angeles. MR0084985

[27] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

[28] KARATZAS, I. and SHREVE, S. (2012). *Brownian Motion and Stochastic Calculus*. Springer, Berlin.

[29] KOLOKOLTSOV, V. N. and TROEVA, M. (2017). Regularity and sensitivity for McKean–Vlasov SPDEs. *AIP Conf. Proc.* **1907** 030046.

[30] KOLOKOLTSOV, V. N. and TROEVA, M. (2019). On mean field games with common noise and McKean–Vlasov SPDEs. *Stoch. Anal. Appl.* **37** 522–549. MR3954570 https://doi.org/10.1080/07362994.2019.1592690

[31] KOLOKOLTSOV, V. N. and TROEVA, M. (2019). On mean field games with common noise based on stable-like processes. *Stat. Optim. Inf. Comput.* **7** 264–276. MR3955960 https://doi.org/10.19139/soic.v7i2.637

[32] KOLOKOLTSOV, V. N. and TROEVA, M. S. (2018). Regularity and sensitivity for McKean–Vlasov type SPDEs generated by stable-like processes. *Probl. Anal. Issues Anal.* **7** 69–81. MR3896524 https://doi.org/10.15393/j3.art.2018.5250

[33] KRYLOV, N. V. (1980). *Controlled Diffusion Processes*. *Applications of Mathematics* **14**. Springer, New York-Berlin. Translated from the Russian by A. B. Aries. MR0601776

[34] KURTZ, T. G. (2014). Weak and strong solutions of general stochastic models. *Electron. Commun. Probab.* **19** 58. MR3254737 https://doi.org/10.1214/ECP.v19-2833

[35] KURTZ, T. G. and XIONG, J. (1999). Particle representations for a class of nonlinear SPDEs. *Stochastic Process. Appl.* **83** 103–126. MR1705602 https://doi.org/10.1016/S0304-4149(99)00024-1

[36] LACKER, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014 https://doi.org/10.1007/s00440-015-0641-9

[37] LACKER, D. (2018). Dense sets of joint distributions appearing in filtration enlargements, stochastic control, and causal optimal transport. Available at arXiv:1805.03185.

[38] LACKER, D. (2018). On a strong form of propagation of chaos for McKean–Vlasov equations. *Electron. Commun. Probab.* **23** 45. MR3841406 https://doi.org/10.1214/18-ECP150

[39] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8

[40] LEDGER, S. J. and SØJMARK, A. (2018). At the mercy of the common noise: Blow-ups in a conditional McKean–Vlasov problem. Available at arXiv:1807.05126.

[41] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595 https://doi.org/10.1073/pnas.56.6.1907

[42] MEHRI, S. and STANNAT, W. (2019). Weak solutions to Vlasov–McKean equations under Lyapunov-type conditions. *Stoch. Dyn.* **19** 1950042. MR4033001 https://doi.org/10.1142/S0219493719500424

[43] MISHURA, Y. S. and VERETENNIKOV, A. Y. (2016). Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. Available at arXiv:1603.02212.

[44] PROTTER, P. (1990). *Stochastic Integration and Differential Equations*. *Applications of Mathematics* (*New York*) **21**. Springer, Berlin. MR1037262 https://doi.org/10.1007/978-3-662-02619-9

[45] RÖCKNER, M. and ZHANG, X. (2018). Well-posedness of distribution dependent SDEs with singular drifts. Available at arXiv:1809.02216.

[46] SKOROKHOD, A. V. (1965). *Studies in the Theory of Random Processes*. *Translated from the Russian by Scripta Technica*, *Inc*. Addison-Wesley, Reading, MA. MR0185620

[47] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[48] TARALDSEN, G. (2018). Optimal learning from the Doob–Dynkin lemma. Available at arXiv:1801.00974.

[49] TCHUENDOM, R. F. (2018). Uniqueness for linear-quadratic mean field games with common noise. *Dyn. Games Appl*. **8** 199–210. MR3764697 https://doi.org/10.1007/s13235-016-0200-8

[50] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. Springer, Berlin.

The Annals of Probability
2021, Vol. 49, No. 2, 556–636
https://doi.org/10.1214/20-AOP1456
© Institute of Mathematical Statistics, 2021

QUANTITATIVE HOMOGENIZATION OF THE PARABOLIC AND ELLIPTIC GREEN'S FUNCTIONS ON PERCOLATION CLUSTERS

BY PAUL DARIO[1] AND CHENLIN GU[2]

[1]*School of Mathematical Sciences, Tel Aviv University, pauldario@mail.tau.ac.il*
[2]*DMA, École Normale Supérieure, chenlin.gu@ens.fr*

We study the heat kernel and the Green's function on the infinite supercritical percolation cluster in dimension $d \geq 2$ and prove a quantitative homogenization theorem for these functions with an almost optimal rate of convergence. These results are a quantitative version of the local central limit theorem proved by Barlow and Hambly in (*Electron. J. Probab.* **14** (2009) 1–27). The proof relies on a structure of renormalization for the infinite percolation cluster introduced in (*Comm. Pure Appl. Math.* **71** (2018) 1717–1849), Gaussian bounds on the heat kernel established by Barlow in (*Ann. Probab.* **32** (2004) 3024–3084) and tools of the theory of quantitative stochastic homogenization. An important step in the proof is to establish a $C^{0,1}$-large-scale regularity theory for caloric functions on the infinite cluster and is of independent interest.

REFERENCES

[1] AIZENMAN, M., DELYON, F. and SOUILLARD, B. (1980). Lower bounds on the cluster size distribution. *J. Stat. Phys.* **23** 267–280. MR0594824 https://doi.org/10.1007/BF01011369

[2] ALVES, C. and SAPOZHNIKOV, A. (2019). Decoupling inequalities and supercritical percolation for the vacant set of random walk loop soup. *Electron. J. Probab.* **24** 110. MR4017128 https://doi.org/10.1214/19-ejp360

[3] ANDRES, S., BARLOW, M. T., DEUSCHEL, J.-D. and HAMBLY, B. M. (2013). Invariance principle for the random conductance model. *Probab. Theory Related Fields* **156** 535–580. MR3078279 https://doi.org/10.1007/s00440-012-0435-2

[4] ANDRES, S., CHIARINI, A., DEUSCHEL, J.-D. and SLOWIK, M. (2018). Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *Ann. Probab.* **46** 302–336. MR3758732 https://doi.org/10.1214/17-AOP1186

[5] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2015). Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43** 1866–1891. MR3353817 https://doi.org/10.1214/14-AOP921

[6] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2016). Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probab. Theory Related Fields* **164** 931–977. MR3477784 https://doi.org/10.1007/s00440-015-0623-y

[7] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2016). Heat kernel estimates for random walks with degenerate weights. *Electron. J. Probab.* **21** 33. MR3492937 https://doi.org/10.1214/16-EJP4382

[8] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2020). Green kernel asymptotics for two-dimensional random walks under random conductances. *Electron. Commun. Probab.* **25** 58.

[9] ANDRES, S., DEUSCHEL, J.-D. and SLOWIK, M. (2019). Heat kernel estimates and intrinsic metric for random walks with general speed measure under degenerate conductances. *Electron. Commun. Probab.* **24** 5. MR3916337 https://doi.org/10.1214/18-ECP207

[10] ANTAL, P. and PISZTORA, A. (1996). On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* **24** 1036–1048. MR1404543 https://doi.org/10.1214/aop/1039639377

[11] ARMSTRONG, S., BORDAS, A. and MOURRAT, J.-C. (2018). Quantitative stochastic homogenization and regularity theory of parabolic equations. *Anal. PDE* **11** 1945–2014. MR3812862 https://doi.org/10.2140/apde.2018.11.1945

MSC2020 subject classifications. 35B27, 60K37, 60K35.
Key words and phrases. Stochastic homogenization, local central limit theorem, parabolic equation, large-scale regularity, supercritical percolation.

[12] Armstrong, S. and Dario, P. (2018). Elliptic regularity and quantitative homogenization on percolation clusters. *Comm. Pure Appl. Math.* **71** 1717–1849. MR3847767 https://doi.org/10.1002/cpa.21726

[13] Armstrong, S., Kuusi, T. and Mourrat, J.-C. (2016). Mesoscopic higher regularity and subadditivity in elliptic homogenization. *Comm. Math. Phys.* **347** 315–361. MR3545509 https://doi.org/10.1007/s00220-016-2663-2

[14] Armstrong, S., Kuusi, T. and Mourrat, J.-C. (2017). The additive structure of elliptic homogenization. *Invent. Math.* **208** 999–1154. MR3648977 https://doi.org/10.1007/s00222-016-0702-4

[15] Armstrong, S., Kuusi, T. and Mourrat, J.-C. (2019). *Quantitative Stochastic Homogenization and Large-Scale Regularity*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **352**. Springer, Cham. MR3932093 https://doi.org/10.1007/978-3-030-15545-2

[16] Armstrong, S. and Lin, J. (2017). Optimal quantitative estimates in stochastic homogenization for elliptic equations in nondivergence form. *Arch. Ration. Mech. Anal.* **225** 937–991. MR3665674 https://doi.org/10.1007/s00205-017-1118-z

[17] Armstrong, S. N. and Smart, C. K. (2016). Quantitative stochastic homogenization of convex integral functionals. *Ann. Sci. Éc. Norm. Supér.* (4) **49** 423–481. MR3481355 https://doi.org/10.24033/asens.2287

[18] Aronson, D. G. (1967). Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73** 890–896. MR0217444 https://doi.org/10.1090/S0002-9904-1967-11830-5

[19] Avellaneda, M. and Lin, F.-H. (1987). Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.* **40** 803–847. MR0910954 https://doi.org/10.1002/cpa.3160400607

[20] Avellaneda, M. and Lin, F.-H. (1991). L^p bounds on singular integrals in homogenization. *Comm. Pure Appl. Math.* **44** 897–910. MR1127038 https://doi.org/10.1002/cpa.3160440805

[21] Barlow, M. T. (2004). Random walks on supercritical percolation clusters. *Ann. Probab.* **32** 3024–3084. MR2094438 https://doi.org/10.1214/009117904000000748

[22] Barlow, M. T. and Deuschel, J.-D. (2010). Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.* **38** 234–276. MR2599199 https://doi.org/10.1214/09-AOP481

[23] Barlow, M. T. and Hambly, B. M. (2009). Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electron. J. Probab.* **14** 1–27. MR2471657 https://doi.org/10.1214/ejp.v14-587

[24] Bella, P., Fehrman, B. and Otto, F. (2018). A Liouville theorem for elliptic systems with degenerate ergodic coefficients. *Ann. Appl. Probab.* **28** 1379–1422. MR3809467 https://doi.org/10.1214/17-AAP1332

[25] Bella, P. and Schäffner, M. (2020). Quenched invariance principle for random walks among random degenerate conductances. *Ann. Probab.* **48** 296–316. MR4079437 https://doi.org/10.1214/19-AOP1361

[26] Benjamini, I., Duminil-Copin, H., Kozma, G. and Yadin, A. (2015). Disorder, entropy and harmonic functions. *Ann. Probab.* **43** 2332–2373. MR3395463 https://doi.org/10.1214/14-AOP934

[27] Benjamini, I. and Mossel, E. (2003). On the mixing time of a simple random walk on the super critical percolation cluster. *Probab. Theory Related Fields* **125** 408–420. MR1967022 https://doi.org/10.1007/s00440-002-0246-y

[28] Berger, N. and Biskup, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453 https://doi.org/10.1007/s00440-006-0498-z

[29] Berger, N., Biskup, M., Hoffman, C. E. and Kozma, G. (2008). Anomalous heat-kernel decay for random walk among bounded random conductances. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 374–392. MR2446329 https://doi.org/10.1214/07-AIHP126

[30] Biskup, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373. MR2861133 https://doi.org/10.1214/11-PS190

[31] Biskup, M. and Boukhadra, O. (2012). Subdiffusive heat-kernel decay in four-dimensional i.i.d. random conductance models. *J. Lond. Math. Soc.* (2) **86** 455–481. MR2980920 https://doi.org/10.1112/jlms/jds012

[32] Biskup, M. and Prescott, T. M. (2007). Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** 1323–1348. MR2354160 https://doi.org/10.1214/EJP.v12-456

[33] Boukhadra, O. (2010). Heat-kernel estimates for random walk among random conductances with heavy tail. *Stochastic Process. Appl.* **120** 182–194. MR2576886 https://doi.org/10.1016/j.spa.2009.11.001

[34] Boukhadra, O. (2010). Standard spectral dimension for the polynomial lower tail random conductances model. *Electron. J. Probab.* **15** 2069–2086. MR2745726 https://doi.org/10.1214/EJP.v15-839

[35] Burton, R. M. and Keane, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505. MR0990777

[36] CHANG, Y. (2017). Supercritical loop percolation on $\mathbb{Z}^d$ for $d \geq 3$. *Stochastic Process. Appl.* **127** 3159–3186. MR3692311 https://doi.org/10.1016/j.spa.2017.02.003

[37] CHIARINI, A. and DEUSCHEL, J.-D. (2016). Invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1535–1563. MR3573286 https://doi.org/10.1214/15-AIHP688

[38] DAL MASO, G. and MODICA, L. (1986). Nonlinear stochastic homogenization. *Ann. Mat. Pura Appl.* (4) **144** 347–389. MR0870884 https://doi.org/10.1007/BF01760826

[39] DAL MASO, G. and MODICA, L. (1986). Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.* **368** 28–42. MR0850613

[40] DARIO, P. (2018). Optimal corrector estimates on percolation clusters. arXiv preprint. Available at arXiv:1805.00902.

[41] DAVIES, E. B. (1993). Large deviations for heat kernels on graphs. *J. Lond. Math. Soc.* (2) **47** 65–72. MR1200978 https://doi.org/10.1112/jlms/s2-47.1.65

[42] DE MASI, A., FERRARI, P. A., GOLDSTEIN, S. and WICK, W. D. (1989). An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.* **55** 787–855. MR1003538 https://doi.org/10.1007/BF01041608

[43] DELMOTTE, T. (1999). Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoam.* **15** 181–232. MR1681641 https://doi.org/10.4171/RMI/254

[44] DEUSCHEL, J.-D., NGUYEN, T. A. and SLOWIK, M. (2018). Quenched invariance principles for the random conductance model on a random graph with degenerate ergodic weights. *Probab. Theory Related Fields* **170** 363–386. MR3748327 https://doi.org/10.1007/s00440-017-0759-z

[45] DURRETT, R. (1984). *Brownian Motion and Martingales in Analysis*. *Wadsworth Mathematics Series*. Wadsworth International Group, Belmont, CA. MR0750829

[46] EGLOFFE, A.-C., GLORIA, A., MOURRAT, J.-C. and NGUYEN, T. N. (2015). Random walk in random environment, corrector equation and homogenized coefficients: From theory to numerics, back and forth. *IMA J. Numer. Anal.* **35** 499–545. MR3335214 https://doi.org/10.1093/imanum/dru010

[47] EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 https://doi.org/10.1090/gsm/019

[48] FLEGEL, F., HEIDA, M. and SLOWIK, M. (2019). Homogenization theory for the random conductance model with degenerate ergodic weights and unbounded-range jumps. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1226–1257. MR4010934 https://doi.org/10.1214/18-aihp917

[49] GIAQUINTA, M. and GIUSTI, E. (1982). On the regularity of the minima of variational integrals. *Acta Math.* **148** 31–46. MR0666107 https://doi.org/10.1007/BF02392725

[50] GIAQUINTA, M. and MODICA, G. (1979). Regularity results for some classes of higher order nonlinear elliptic systems. *J. Reine Angew. Math.* **311/312** 145–169. MR0549962

[51] GIUNTI, A., HÖFER, R. and VELÁZQUEZ, J. J. L. (2018). Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations* **43** 1377–1412. MR3915491 https://doi.org/10.1080/03605302.2018.1531425

[52] GIUSTI, E. (2003). *Direct Methods in the Calculus of Variations*. World Scientific Co., River Edge, NJ. MR1962933 https://doi.org/10.1142/9789812795557

[53] GLORIA, A., NEUKAMM, S. and OTTO, F. (2015). Quantification of ergodicity in stochastic homogenization: Optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.* **199** 455–515. MR3302119 https://doi.org/10.1007/s00222-014-0518-z

[54] GLORIA, A., NEUKAMM, S. and OTTO, F. (2020). A regularity theory for random elliptic operators. *Milan J. Math.* **88** 99–170. MR4103433 https://doi.org/10.1007/s00032-020-00309-4

[55] GLORIA, A. and OTTO, F. (2011). An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* **39** 779–856. MR2789576 https://doi.org/10.1214/10-AOP571

[56] GLORIA, A. and OTTO, F. (2012). An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.* **22** 1–28. MR2932541 https://doi.org/10.1214/10-AAP745

[57] GLORIA, A. and OTTO, F. (2015). The corrector in stochastic homogenization: Optimal rates, stochastic integrability, and fluctuations. arXiv preprint. Available at arXiv:1510.08290.

[58] GLORIA, A. and OTTO, F. (2017). Quantitative results on the corrector equation in stochastic homogenization. *J. Eur. Math. Soc.* (*JEMS*) **19** 3489–3548. MR3713047 https://doi.org/10.4171/JEMS/745

[59] GRAFAKOS, L. (2004). *Classical and Modern Fourier Analysis*. Pearson Education, Inc., Upper Saddle River, NJ. MR2449250

[60] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[61] GU, C. (2019). An efficient algorithm for solving elliptic problems on percolation clusters. arXiv preprint. Available at arXiv:1907.13571.

[62] KESTEN, H. and ZHANG, Y. (1990). The probability of a large finite cluster in supercritical Bernoulli percolation. *Ann. Probab.* **18** 537–555. MR1055419

[63] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrsch. Verw. Gebiete* **32** 111–131. MR0375412 https://doi.org/10.1007/BF00533093

[64] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 33–58. MR0402883 https://doi.org/10.1007/BF00532688

[65] KOZLOV, S. M. (1978). Averaging of differential operators with almost periodic rapidly oscillating coefficients. *Mat. Sb.* (*N.S.*) **107** 199–217, 317. MR0512007

[66] LAMACZ, A., NEUKAMM, S. and OTTO, F. (2015). Moment bounds for the corrector in stochastic homogenization of a percolation model. *Electron. J. Probab.* **20** 106. MR3418538 https://doi.org/10.1214/EJP.v20-3618

[67] MATHIEU, P. (2008). Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.* **130** 1025–1046. MR2384074 https://doi.org/10.1007/s10955-007-9465-z

[68] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 2287–2307. MR2345229 https://doi.org/10.1098/rspa.2007.1876

[69] MATHIEU, P. and REMY, E. (2004). Isoperimetry and heat kernel decay on percolation clusters. *Ann. Probab.* **32** 100–128. MR2040777 https://doi.org/10.1214/aop/1078415830

[70] MOURRAT, J.-C. (2011). Variance decay for functionals of the environment viewed by the particle. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 294–327. MR2779406 https://doi.org/10.1214/10-AIHP375

[71] NADDAF, A. and SPENCER, T. (1998). Estimates on the variance of some homogenization problems. Unpublished manuscript.

[72] NEUKAMM, S., SCHÄFFNER, M. and SCHLÖMERKEMPER, A. (2017). Stochastic homogenization of nonconvex discrete energies with degenerate growth. *SIAM J. Math. Anal.* **49** 1761–1809. MR3650427 https://doi.org/10.1137/16M1097705

[73] OSADA, H. (1983). Homogenization of diffusion processes with random stationary coefficients. In *Probability Theory and Mathematical Statistics* (*Tbilisi*, 1982). *Lecture Notes in Math.* **1021** 507–517. Springer, Berlin. MR0736016 https://doi.org/10.1007/BFb0072946

[74] PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1981). Boundary value problems with rapidly oscillating random coefficients. In *Random Fields*, *Vol. I*, *II* (*Esztergom*, 1979). *Colloquia Mathematica Societatis János Bolyai* **27** 835–873. North-Holland, Amsterdam. MR0712714

[75] PENROSE, M. D. and PISZTORA, A. (1996). Large deviations for discrete and continuous percolation. *Adv. in Appl. Probab.* **28** 29–52. MR1372330 https://doi.org/10.2307/1427912

[76] PISZTORA, A. (1996). Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Related Fields* **104** 427–466. MR1384040 https://doi.org/10.1007/BF01198161

[77] PROCACCIA, E. B., ROSENTHAL, R. and SAPOZHNIKOV, A. (2016). Quenched invariance principle for simple random walk on clusters in correlated percolation models. *Probab. Theory Related Fields* **166** 619–657. MR3568036 https://doi.org/10.1007/s00440-015-0668-y

[78] SAPOZHNIKOV, A. (2017). Random walks on infinite percolation clusters in models with long-range correlations. *Ann. Probab.* **45** 1842–1898. MR3650417 https://doi.org/10.1214/16-AOP1103

[79] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields* **129** 219–244. MR2063376 https://doi.org/10.1007/s00440-004-0336-0

[80] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. *Princeton Mathematical Series*. **30**. Princeton Univ. Press, Princeton, NJ. MR0290095

[81] YURINSKIĬ, V. V. (1986). Averaging of symmetric diffusion in a random medium. *Sibirsk. Mat. Zh.* **27** 167–180, 215. MR0867870

The Annals of Probability
2021, Vol. 49, No. 2, 637–670
https://doi.org/10.1214/20-AOP1457
© Institute of Mathematical Statistics, 2021

THE DERRIDA–RETAUX CONJECTURE ON RECURSIVE MODELS

BY XINXING CHEN[1], VICTOR DAGARD[2], BERNARD DERRIDA[3], YUEYUN HU[4], MIKHAIL LIFSHITS[5] AND ZHAN SHI[6]

[1]*School of Mathematical Sciences, Shanghai Jiaotong University, chenxinx@sjtu.edu.cn*
[2]*Laboratoire de Physique, École Normale Supérieure, victor.dagard@lps.ens.fr*
[3]*PSL, Collège de France, bernard.derrida@phys.ens.fr*
[4]*LAGA, Université Sorbonne Paris Nord, yueyun@math.univ-paris13.fr*
[5]*Department of Mathematics and Computer Sciences, St. Petersburg State University, mikhail@lifshits.org*
[6]*LPSM, Sorbonne Université Paris VI, zhan.shi@sorbonne-universite.fr*

We are interested in the nearly supercritical regime in a family of max-type recursive models studied by Collet, Eckman, Glaser and Martin (*Comm. Math. Phys.* **94** (1984) 353–370) and by Derrida and Retaux (*J. Stat. Phys.* **156** (2014) 268–290) and prove that, under a suitable integrability assumption on the initial distribution, the free energy vanishes at the transition with an essential singularity with exponent $\frac{1}{2}$. This gives a weaker answer to a conjecture of Derrida and Retaux (*J. Stat. Phys.* **156** (2014) 268–290). Other behaviours are obtained when the integrability condition is not satisfied.

REFERENCES

[1] ALDOUS, D. J. and BANDYOPADHYAY, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15** 1047–1110. MR2134098 https://doi.org/10.1214/105051605000000142

[2] ALEXANDER, K. S. (2008). The effect of disorder on polymer depinning transitions. *Comm. Math. Phys.* **279** 117–146. MR2377630 https://doi.org/10.1007/s00220-008-0425-5

[3] BERGER, Q., GIACOMIN, G. and LACOIN, H. (2019). Disorder and critical phenomena: The $\alpha = 0$ copolymer model. *Probab. Theory Related Fields* **174** 787–819. MR3980305 https://doi.org/10.1007/s00440-018-0870-9

[4] CHEN, X., DERRIDA, B., HU, Y., LIFSHITS, M. and SHI, Z. (2019). A max-type recursive model: Some properties and open questions. In *Sojourns in Probability Theory and Statistical Physics-III* 166–186. Springer, Singapore.

[5] COLLET, P., ECKMANN, J.-P., GLASER, V. and MARTIN, A. (1984). A spin glass with random couplings. *J. Stat. Phys.* **36** 89–106. MR0762033 https://doi.org/10.1007/BF01015728

[6] COLLET, P., ECKMANN, J.-P., GLASER, V. and MARTIN, A. (1984). Study of the iterations of a mapping associated to a spin glass model. *Comm. Math. Phys.* **94** 353–370. MR0763384

[7] DERRIDA, B., HAKIM, V. and VANNIMENUS, J. (1992). Effect of disorder on two-dimensional wetting. *J. Stat. Phys.* **66** 1189–1213. MR1156401 https://doi.org/10.1007/BF01054419

[8] DERRIDA, B. and RETAUX, M. (2014). The depinning transition in presence of disorder: A toy model. *J. Stat. Phys.* **156** 268–290.

[9] GIACOMIN, G. (2007). *Random Polymer Models*. Imperial College Press, London. MR2380992 https://doi.org/10.1142/9781860948299

[10] GIACOMIN, G. (2011). *Disorder and Critical Phenomena Through Basic Probability Models*: *École d'Été de Probabilités de Saint-Flour*. *Lecture Notes in Math.* **2025**. Springer, Heidelberg. MR2816225 https://doi.org/10.1007/978-3-642-21156-0

[11] GIACOMIN, G., LACOIN, H. and TONINELLI, F. (2010). Marginal relevance of disorder for pinning models. *Comm. Pure Appl. Math.* **63** 233–265. MR2588461 https://doi.org/10.1002/cpa.20301

[12] GIACOMIN, G. and TONINELLI, F. L. (2006). Smoothing effect of quenched disorder on polymer depinning transitions. *Comm. Math. Phys.* **266** 1–16. MR2231963 https://doi.org/10.1007/s00220-006-0008-2

[13] GOLDSCHMIDT, C. and PRZYKUCKI, M. (2019). Parking on a random tree. *Combin. Probab. Comput.* **28** 23–45. MR3917904 https://doi.org/10.1017/S0963548318000457

MSC2020 subject classifications. Primary 60J80; secondary 82B27.
Key words and phrases. Max-type recursive model, free energy, critical behavior, depinning transition.

[14] HU, Y., MALLEIN, B. and PAIN, M. (2020). An exactly solvable continuous-time Derrida–Retaux model. *Comm*. *Math*. *Phys*. **375** 605–651. MR4082171 https://doi.org/10.1007/s00220-019-03465-w

[15] HU, Y. and SHI, Z. (2018). The free energy in the Derrida–Retaux recursive model. *J*. *Stat*. *Phys*. **172** 718–741. MR3827300 https://doi.org/10.1007/s10955-018-2066-1

[16] JORDAN, J. (2002). Almost sure convergence for iterated functions of independent random variables. *Ann*. *Appl*. *Probab*. **12** 985–1000. MR1925449 https://doi.org/10.1214/aoap/1031863178

[17] LI, D. and ROGERS, T. D. (1999). Asymptotic behavior for iterated functions of random variables. *Ann*. *Appl*. *Probab*. **9** 1175–1201. MR1728559 https://doi.org/10.1214/aoap/1029962869

[18] MONTHUS, C. (2017). Strong disorder renewal approach to DNA denaturation and wetting: Typical and large deviation properties of the free energy. *J*. *Stat*. *Mech*. *Theory Exp*. **2017** 013301. MR3623903 https://doi.org/10.1088/1742-5468/aa53f8

[19] TANG, L. H. and CHATÉ, H. (2001). Rare-event induced binding transition of heteropolymers. *Phys*. *Rev*. *Lett*. **86** 830.

The Annals of Probability
2021, Vol. 49, No. 2, 671–731
https://doi.org/10.1214/20-AOP1458
© Institute of Mathematical Statistics, 2021

CUT-OFF FOR SANDPILES ON TILING GRAPHS

BY ROBERT HOUGH[1] AND HYOJEONG SON[2]

[1]*Department of Mathematics, Stony Brook University, robert.hough@stonybrook.edu*
[2]*Department of Mathematics, Washington University at St. Louis, hyojeong.son@wustl.edu*

Sandpile dynamics are considered on graphs constructed from periodic plane and space tilings by assigning a growing piece of the tiling, either torus or open boundary conditions. A general method of obtaining the Green's function of the tiling is given, and a total variation cut-off phenomenon is demonstrated under general conditions. It is shown that the boundary condition does not affect the mixing time for planar tilings. In a companion paper, computational methods are used to demonstrate that an open boundary condition alters the mixing time for the D4 lattice in dimension 4, while an asymptotic evaluation shows that it does not change the asymptotic mixing time for the cubic lattice $\mathbb{Z}^d$ for all sufficiently large d.

REFERENCES

[1] AZIMI-TAFRESHI, N., DASHTI-NASERABADI, H., MOGHIMI-ARAGHI, S. and RUELLE, P. (2010). The Abelian sandpile model on the honeycomb lattice. *J. Stat. Mech. Theory Exp.* **2010** P02004.

[2] BAK, P., TANG, C. and WIESENFELD, K. (1988). Self-organized criticality. *Phys. Rev. A* (3) **38** 364–374. MR0949160 https://doi.org/10.1103/PhysRevA.38.364

[3] BRANKOV, J. G., IVASHKEVICH, E. V. and PRIEZZHEV, V. B. (1993). Boundary effects in a two-dimensional Abelian sandpile. *J. Phys., I* **3** 1729–1740.

[4] CHATTERJEE, S. (2014). Invariant measures and the soliton resolution conjecture. *Comm. Pure Appl. Math.* **67** 1737–1842. MR3263670 https://doi.org/10.1002/cpa.21501

[5] CHU, Y. and HOUGH, R. (2019). Solution of the 15-puzzle problem. Preprint. Available at arXiv:1908.07106.

[6] DARTOIS, A., FIORENZI, F. and FRANCINI, P. (2003). Sandpile group on the graph D_n of the dihedral group. *European J. Combin.* **24** 815–824. MR2009395 https://doi.org/10.1016/S0195-6698(03)00104-5

[7] DEMBO, A., DING, J., MILLER, J. and PERES, Y. (2019). Cut-off for lamplighter chains on tori: Dimension interpolation and phase transition. *Probab. Theory Related Fields* **173** 605–650. MR3916115 https://doi.org/10.1007/s00440-018-0883-4

[8] DHAR, D. (1990). Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.* **64** 1613–1616. MR1044086 https://doi.org/10.1103/PhysRevLett.64.1613

[9] DHAR, D. and MAJUMDAR, S. N. (1992). Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model. *Phys. A, Stat. Mech. Appl.* **185** 129–145.

[10] DIACONIS, P. (1988). *Group Representations in Probability and Statistics. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **11**. IMS, Hayward, CA. MR0964069

[11] DIACONIS, P. and SHAHSHAHANI, M. (1987). Time to reach stationarity in the Bernoulli–Laplace diffusion model. *SIAM J. Math. Anal.* **18** 208–218. MR0871832 https://doi.org/10.1137/0518016

[12] FEY, A., LEVINE, L. and WILSON, D. B. (2010). Driving sandpiles to criticality and beyond. *Phys. Rev. Lett.* **104** 145703.

[13] GAMLIN, S. (2016). Boundary conditions in Abelian sandpiles. Dissertation, Univ. Bath.

[14] HOUGH, R. (2017). Mixing and cut-off in cycle walks. *Electron. J. Probab.* **22** Paper No. 90, 49. MR3718718 https://doi.org/10.1214/17-EJP108

[15] HOUGH, R. and SON, H. (2021). The spectrum of the abelian sandpile model. *Math. Comp.* **90** 441–469. MR4166468 https://doi.org/10.1090/mcom/3565

[16] HOUGH, R. D., JERISON, D. C. and LEVINE, L. (2019). Sandpiles on the square lattice. *Comm. Math. Phys.* **367** 33–87. MR3933404 https://doi.org/10.1007/s00220-019-03408-5

MSC2020 subject classifications. Primary 82C20; secondary 60B15, 60J10.
Key words and phrases. Abelian sandpile model, random walk on a group, spectral gap, cut-off phenomenon.

[17] IVASHKEVICH, E. V. (1994). Boundary height correlations in a two-dimensional Abelian sandpile. *J. Phys. A: Math. Gen.* **27** 3643–3653.

[18] JENG, M., PIROUX, G. and RUELLE, P. (2006). Height variables in the Abelian sandpile model: Scaling fields and correlations. *J. Stat. Mech. Theory Exp.* **2006**.

[19] JERISON, D. C., LEVINE, L. and PIKE, J. (2019). Mixing time and eigenvalues of the Abelian sandpile Markov chain. *Trans. Amer. Math. Soc.* **372** 8307–8345. MR4029698 https://doi.org/10.1090/tran/7585

[20] KASSEL, A. and WILSON, D. B. (2016). The looping rate and sandpile density of planar graphs. *Amer. Math. Monthly* **123** 19–39. MR3453533 https://doi.org/10.4169/amer.math.monthly.123.1.19

[21] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/CBO9780511750854

[22] LEVINE, L. and PROPP, J. (2010). What is ... a sandpile? *Notices Amer. Math. Soc.* **57** 976–979. MR2667495

[23] LIN, C.-Y. and HU, C.-K. (2002). Renormalization-group approach to an Abelian sandpile model on planar lattices. *Phys. Rev. E* **66** 021307.

[24] NASSOURI, E., OUARO, S. and TRAORÉ, U. (2017). Growing sandpile problem with Dirichlet and Fourier boundary conditions. *Electron. J. Differential Equations* **2017** Paper No. 300, 19. MR3748018

[25] OTTER, R. (1948). The number of trees. *Ann. of Math.* (2) **49** 583–599. MR0025715 https://doi.org/10.2307/1969046

[26] PAPOYAN, V. V. and SHCHERBAKOV, R. R. (1995). Abelian sandpile model on the Husimi lattice of square plaquettes. *J. Phys. A* **28** 6099–6107. MR1364785

[27] PEGDEN, W. and SMART, C. K. (2013). Convergence of the Abelian sandpile. *Duke Math. J.* **162** 627–642. MR3039676 https://doi.org/10.1215/00127094-2079677

[28] PIROUX, G. and RUELLE, P. (2005). Boundary height fields in the Abelian sandpile model. *J. Phys. A* **38** 1451–1472. MR2117262 https://doi.org/10.1088/0305-4470/38/7/004

[29] PRIEZZHEV, V. B. (1994). Structure of two-dimensional sandpile. I. Height probabilities. *J. Stat. Phys.* **74** 955–979.

[30] SADHU, T. and DHAR, D. (2010). Pattern formation in growing sandpiles with multiple sources or sinks. *J. Stat. Phys.* **138** 815–837. MR2594922 https://doi.org/10.1007/s10955-009-9901-3

[31] SCHMIDT, K. and VERBITSKIY, E. (2009). Abelian sandpiles and the harmonic model. *Comm. Math. Phys.* **292** 721–759. MR2551792 https://doi.org/10.1007/s00220-009-0884-3

[32] SOKOLOV, A. et al. (2015). Memory on multiple time-scales in an Abelian sandpile. *Phys. A, Stat. Mech. Appl.* **428** 295–301.

[33] SPITZER, F. (2013). *Principles of Random Walk*, 2nd ed. *Graduate Texts in Mathematics* **34**. Springer, New York. MR0388547

[34] STEELE, J. M. (2004). *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities. MAA Problem Books Series*. Cambridge Univ. Press, Cambridge. MR2062704 https://doi.org/10.1017/CBO9780511817106

[35] TAO, T. and VU, V. (2006). *Additive Combinatorics. Cambridge Studies in Advanced Mathematics* **105**. Cambridge Univ. Press, Cambridge. MR2289012 https://doi.org/10.1017/CBO9780511755149

The Annals of Probability
2021, Vol. 49, No. 2, 732–792
https://doi.org/10.1214/20-AOP1459
© Institute of Mathematical Statistics, 2021

TIGHTNESS AND TAILS OF THE MAXIMUM IN 3D ISING INTERFACES

BY REZA GHEISSARI* AND EYAL LUBETZKY†

Courant Institute of Mathematical Sciences, New York University, *reza@cims.nyu.edu; †eyal@courant.nyu.edu*

Consider the 3D Ising model on a box of side length n with minus boundary conditions above the xy-plane and plus boundary conditions below it. At low temperatures, Dobrushin (1972) showed that the interface separating the predominantly plus and predominantly minus regions is localized: its height above a fixed point has exponential tails. Recently, the authors proved a law of large numbers for the maximum height M_n of this interface: for every β large, $M_n/\log n \to c_\beta$ in probability as $n \to \infty$.

Here, we show that the laws of the centered maxima $(M_n - \mathbb{E}[M_n])_{n\geq 1}$ are uniformly tight. Moreover, even though this sequence does not converge, we prove that it has uniform upper and lower Gumbel tails (exponential right tails and doubly exponential left tails). Key to the proof is a sharp (up to $O(1)$ precision) understanding of the surface large deviations. This includes, in particular, the shape of a pillar that reaches near-maximum height, even at its base, where the interactions with neighboring pillars are dominant.

REFERENCES

[1] AIZENMAN, M. (1980). Translation invariance and instability of phase coexistence in the two-dimensional Ising system. *Comm. Math. Phys.* **73** 83–94. MR0573615

[2] BOLTHAUSEN, E., DEUSCHEL, J.-D. and GIACOMIN, G. (2001). Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.* **29** 1670–1692. MR1880237 https://doi.org/10.1214/aop/1015345767

[3] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69** 62–123. MR3433630 https://doi.org/10.1002/cpa.21621

[4] BRAMSON, M. and ZEITOUNI, O. (2012). Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **65** 1–20. MR2846636 https://doi.org/10.1002/cpa.20390

[5] BRICMONT, J., EL MELLOUKI, A. and FRÖHLICH, J. (1986). Random surfaces in statistical mechanics: Roughening, rounding, wetting, ... *J. Stat. Phys.* **42** 743–798. MR0833220 https://doi.org/10.1007/BF01010444

[6] BRICMONT, J., LEBOWITZ, J. L. and PFISTER, C. E. (1979). Nontranslation-invariant Gibbs states with coexisting phases. II. Cluster properties and surface tension. *Comm. Math. Phys.* **66** 21–36. MR0530914

[7] CAPUTO, P., LUBETZKY, E., MARTINELLI, F., SLY, A. and TONINELLI, F. L. (2014). Dynamics of $(2+1)$-dimensional SOS surfaces above a wall: Slow mixing induced by entropic repulsion. *Ann. Probab.* **42** 1516–1589. MR3262485 https://doi.org/10.1214/13-AOP836

[8] CAPUTO, P., LUBETZKY, E., MARTINELLI, F., SLY, A. and TONINELLI, F. L. (2016). Scaling limit and cube-root fluctuations in SOS surfaces above a wall. *J. Eur. Math. Soc.* (*JEMS*) **18** 931–995. MR3500829 https://doi.org/10.4171/JEMS/606

[9] CERF, R. and KENYON, R. (2001). The low-temperature expansion of the Wulff crystal in the 3D Ising model. *Comm. Math. Phys.* **222** 147–179. MR1853867 https://doi.org/10.1007/s002200100505

[10] DING, J. (2013). Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields* **157** 285–299. MR3101848 https://doi.org/10.1007/s00440-012-0457-9

[11] DOBRUSHIN, R. and HRYNIV, O. (1997). Fluctuations of the phase boundary in the 2D Ising ferromagnet. *Comm. Math. Phys.* **189** 395–445. MR1480026 https://doi.org/10.1007/s002200050209

MSC2020 subject classifications. 60K35, 82B20, 82B24.
Key words and phrases. 3D Ising model, low temperature interface, maximum of random surface.

[12] DOBRUSHIN, R., KOTECKÝ, R. and SHLOSMAN, S. (1992). *Wulff Construction*: *A Global Shape from Local Interaction*. *Translations of Mathematical Monographs* **104**. Amer. Math. Soc., Providence, RI. MR1181197 https://doi.org/10.1090/mmono/104

[13] DOBRUSHIN, L. R. (1973). An investigation of Gibbs states for three-dimensional lattice systems. *Teor*. *Veroyatn*. *Primen*. **18** 261–279. MR0325076

[14] DOBRUSHIN, R. L. (1972). The Gibbs state that describes the coexistence of phases for a three-dimensional Ising model. *Teor*. *Veroyatn*. *Primen*. **17** 619–639. MR0421546

[15] DOBRUSHIN, R. L. (1972). Asymptotical behavior of Gibbsian distributions for lattice systems and their dependence on the form of the container. *Teoret*. *Mat*. *Fiz*. **12** 115–134. MR0479233

[16] GHEISSARI, R. and LUBETZKY, E. Maximum and shape of interfaces in 3D Ising crystals. *Comm*. *Pure Appl*. *Math*. To appear.

[17] HIGUCHI, Y. (1981). On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model. In *Random Fields*, *Vol*. *I*, *II* (*Esztergom*, 1979). *Colloquia Mathematica Societatis János Bolyai* **27** 517–534. North-Holland, Amsterdam. MR0712693

[18] HRYNIV, O. (1998). On local behaviour of the phase separation line in the 2D Ising model. *Probab*. *Theory Related Fields* **110** 91–107. MR1602044 https://doi.org/10.1007/s004400050146

[19] IOFFE, D. (1994). Large deviations for the 2D Ising model: A lower bound without cluster expansions. *J*. *Stat*. *Phys*. **74** 411–432. MR1257822 https://doi.org/10.1007/BF02186818

[20] IOFFE, D. (1995). Exact large deviation bounds up to T_c for the Ising model in two dimensions. *Probab*. *Theory Related Fields* **102** 313–330. MR1339736 https://doi.org/10.1007/BF01192464

[21] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. *Springer Series in Statistics*. Springer, New York–Berlin. MR0691492

[22] LUBETZKY, E., MARTINELLI, F. and SLY, A. (2016). Harmonic pinnacles in the discrete Gaussian model. *Comm*. *Math*. *Phys*. **344** 673–717. MR3508158 https://doi.org/10.1007/s00220-016-2628-5

[23] PFISTER, C.-E. and VELENIK, Y. (1997). Large deviations and continuum limit in the 2D Ising model. *Probab*. *Theory Related Fields* **109** 435–506. MR1483597 https://doi.org/10.1007/s004400050139

[24] PFISTER, C.-E. and VELENIK, Y. (1999). Interface, surface tension and reentrant pinning transition in the 2D Ising model. *Comm*. *Math*. *Phys*. **204** 269–312. MR1704276 https://doi.org/10.1007/s002200050646

[25] ZEITOUNI, O. (2016). Branching random walks and Gaussian fields. In *Probability and Statistical Physics in St*. *Petersburg*. *Proc*. *Sympos*. *Pure Math*. **91** 437–471. Amer. Math. Soc., Providence, RI. MR3526836

The Annals of Probability
2021, Vol. 49, No. 2, 793–831
https://doi.org/10.1214/20-AOP1460
© Institute of Mathematical Statistics, 2021

DIFFUSIONS ON A SPACE OF INTERVAL PARTITIONS: POISSON–DIRICHLET STATIONARY DISTRIBUTIONS

BY NOAH FORMAN[1], SOUMIK PAL[2], DOUGLAS RIZZOLO[3] AND MATTHIAS WINKEL[4]

[1]*Department of Mathematics and Statistics, McMaster University, noah.forman@gmail.com*

[2]*Department of Mathematics, University of Washington, soumikpal@gmail.com*

[3]*Department of Mathematical Sciences, University of Delaware, drizzolo@udel.edu*

[4]*Department of Statistics, University of Oxford, winkel@stats.ox.ac.uk*

We introduce diffusions on a space of interval partitions of the unit interval that are stationary with the Poisson–Dirichlet laws with parameters $(\alpha, 0)$ and (α, α). The construction has two steps. The first is a general construction of interval partition processes obtained previously by decorating the jumps of a Lévy process with independent excursions. Here, we focus on the second step which requires explicit transition kernels and, what we call, pseudo-stationarity. This allows us to study processes obtained from the original construction via scaling and time-change. In a sequel paper we establish connections to diffusions on decreasing sequences introduced by Ethier and Kurtz (*Adv. in Appl. Probab.* **13** (1981) 429–452) and Petrov (*Funktsional. Anal. i Prilozhen.* **43** (2009) 45–66). The latter diffusions are continuum limits of up-down Markov chains on Chinese restaurant processes. Our construction is also a step toward resolving longstanding conjectures by Feng and Sun on measure-valued Poisson–Dirichlet diffusions and by Aldous on a continuum-tree-valued diffusion.

REFERENCES

[1] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326
[2] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226
[3] ALDOUS, D. (1999). From random walks on discrete tree space to diffusions on continuous tree space. Seminar on Stochastic Processes, Fields Institute. Available at https://www.fields.utoronto.ca/programs/scientific/98-99/stochastic_processes/abstracts.html.
[4] ALDOUS, D. (1999). Wright–Fisher diffusions with negative mutation rate! Available at http://www.stat.berkeley.edu/~aldous/Research/OP/fw.html.
[5] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564
[6] BERTOIN, J. (1996). On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. Lond. Math. Soc.* **28** 514–520. MR1396154 https://doi.org/10.1112/blms/28.5.514
[7] BERTOIN, J., CHAUMONT, L. and PITMAN, J. (2003). Path transformations of first passage bridges. *Electron. Commun. Probab.* **8** 155–166. MR2042754 https://doi.org/10.1214/ECP.v8-1096
[8] BOYLAN, E. S. (1964). Local times for a class of Markoff processes. *Illinois J. Math.* **8** 19–39. MR0158434
[9] BRODERICK, T., JORDAN, M. I. and PITMAN, J. (2013). Cluster and feature modeling from combinatorial stochastic processes. *Statist. Sci.* **28** 289–312. MR3135534 https://doi.org/10.1214/13-STS434
[10] COSTANTINI, C., DE BLASI, P., ETHIER, S. N., RUGGIERO, M. and SPANÒ, D. (2017). Wright–Fisher construction of the two-parameter Poisson–Dirichlet diffusion. *Ann. Appl. Probab.* **27** 1923–1950. MR3678488 https://doi.org/10.1214/16-AAP1252
[11] EISENBAUM, N. and KASPI, H. (1993). A necessary and sufficient condition for the Markov property of the local time process. *Ann. Probab.* **21** 1591–1598. MR1235430
[12] ETHIER, S. N. and KURTZ, T. G. (1981). The infinitely-many-neutral-alleles diffusion model. *Adv. in Appl. Probab.* **13** 429–452. MR0615945 https://doi.org/10.2307/1426779
[13] ETHIER, S. N. and KURTZ, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim.* **31** 345–386. MR1205982 https://doi.org/10.1137/0331019

MSC2020 subject classifications. Primary 60J25, 60J60, 60J80; Secondary 60G18, 60G52, 60G55.
Key words and phrases. Interval partition, Aldous diffusion, Poisson–Dirichlet distribution, infinitely-many-neutral-alleles model, excursion theory.

[14] FENG, S. (2010). *The Poisson–Dirichlet Distribution and Related Topics*: *Models and asymptotic behaviors*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2663265 https://doi.org/10.1007/978-3-642-11194-5

[15] FENG, S. and SUN, W. (2010). Some diffusion processes associated with two parameter Poisson–Dirichlet distribution and Dirichlet process. *Probab. Theory Related Fields* **148** 501–525. MR2678897 https://doi.org/10.1007/s00440-009-0238-2

[16] FENG, S. and SUN, W. (2019). A dynamic model for the two-parameter Dirichlet process. *Potential Anal.* **51** 147–164. MR3983502 https://doi.org/10.1007/s11118-018-9706-6

[17] FENG, S., SUN, W., WANG, F.-Y. and XU, F. (2011). Functional inequalities for the two-parameter extension of the infinitely-many-neutral-alleles diffusion. *J. Funct. Anal.* **260** 399–413. MR2737405 https://doi.org/10.1016/j.jfa.2010.10.005

[18] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2018). Uniform control of local times of spectrally positive stable processes. *Ann. Appl. Probab.* **28** 2592–2634. MR3843837 https://doi.org/10.1214/17-AAP1370

[19] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2018). Interval partition evolutions with emigration related to the Aldous diffusion. Available at arXiv:1804.01205 [math.PR].

[20] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2018). Aldous diffusion I: A projective system of continuum k-tree evolutions. Available at arXiv:1809.07756.

[21] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2020). Diffusions on a space of interval partitions: construction from marked Lévy processes. *Electron. J. Probab.* **25** Paper No. 133, 46. https://doi.org/10.1214/20-EJP521

[22] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2020). Interval partition diffusions: Connection with Petrov's Poisson–Dirichlet diffusions. Work in progress.

[23] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2020). Projections of the Aldous chain on binary trees: Intertwining and consistency. *Random Structures Algorithms* **57** 745–769. MR4144083 https://doi.org/10.1002/rsa.20930

[24] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2020). Aldous diffusion II: Properties of the continuum-tree-valued process. Work in progress.

[25] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2020). Metrics on sets of interval partitions with diversity. *Electron. Commun. Probab.* **25** Paper No. 38, 16. MR4112769 https://doi.org/10.1214/20-ecp317

[26] FORMAN, N., RIZZOLO, D., SHI, Q. and WINKEL, M. (2020). Diffusions on a space of interval partitions: The two-parameter model. Available at arXiv:2008.02823 [math.PR].

[27] FORMAN, N., RIZZOLO, D., SHI, Q. and WINKEL, M. (2020). A two-parameter family of measure-valued diffusions with Poisson–Dirichlet stationary distributions. Available at arXiv:2007.05250 [math.PR].

[28] GNEDIN, A. and PITMAN, J. (2005). Regenerative composition structures. *Ann. Probab.* **33** 445–479. MR2122798 https://doi.org/10.1214/009117904000000801

[29] GÖING-JAESCHKE, A. and YOR, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli* **9** 313–349. MR1997032 https://doi.org/10.3150/bj/1068128980

[30] HELLAND, I. S. (1978). Continuity of a class of random time transformations. *Stochastic Process. Appl.* **7** 79–99. MR0488203 https://doi.org/10.1016/0304-4149(78)90040-6

[31] ISHWARAN, H. and JAMES, L. F. (2001). Gibbs sampling methods for stick-breaking priors. *J. Amer. Statist. Assoc.* **96** 161–173. MR1952729 https://doi.org/10.1198/016214501750332758

[32] ITÔ, K. (1972). Poisson point processes attached to Markov processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971), *Vol. III*: *Probability Theory* 225–239. MR0402949

[33] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

[34] KAWAZU, K. and WATANABE, S. (1971). Branching processes with immigration and related limit theorems. *Teor. Veroyatn. Primen.* **16** 34–51. MR0290475

[35] KINGMAN, J. F. C.(1975). Random discrete distribution. *J. Roy. Statist. Soc. Ser. B* **37** 1–22. MR0368264

[36] LAMPERTI, J. (1967). Continuous state branching processes. *Bull. Amer. Math. Soc.* **73** 382–386. MR0208685 https://doi.org/10.1090/S0002-9904-1967-11762-2

[37] LAMPERTI, J. (1972). Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Gebiete* **22** 205–225. MR0307358 https://doi.org/10.1007/BF00536091

[38] LE GALL, J.-F. and MIERMONT, G. (2012). Scaling limits of random trees and planar maps. In *Probability and Statistical Physics in Two and More Dimensions*. *Clay Math. Proc.* **15** 155–211. Amer. Math. Soc., Providence, RI. MR3025391

[39] LI, Z. (2011). *Measure-Valued Branching Markov Processes*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2760602 https://doi.org/10.1007/978-3-642-15004-3

[40] LÖHR, W., MYTNIK, L. and WINTER, A. (2020). The Aldous chain on cladograms in the diffusion limit. *Ann. Probab.* **48** 2565–2590. MR4152651 https://doi.org/10.1214/20-AOP1431

[41] MILLER, J. and SHEFFIELD, S. (2019). Liouville quantum gravity spheres as matings of finite-diameter trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1712–1750. MR4010949 https://doi.org/10.1214/18-aihp932

[42] PAL, S. (2011). Analysis of market weights under volatility-stabilized market models. *Ann. Appl. Probab.* **21** 1180–1213. MR2830616 https://doi.org/10.1214/10-AAP725

[43] PAL, S. (2013). Wright–Fisher diffusion with negative mutation rates. *Ann. Probab.* **41** 503–526. MR3077518 https://doi.org/10.1214/11-AOP704

[44] PERMAN, M., PITMAN, J. and YOR, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields* **92** 21–39. MR1156448 https://doi.org/10.1007/BF01205234

[45] PETROV, L. A. (2009). A two-parameter family of infinite-dimensional diffusions on the Kingman simplex. *Funktsional. Anal. i Prilozhen.* **43** 45–66. MR2596654 https://doi.org/10.1007/s10688-009-0036-8

[46] PITMAN, J. (1997). Partition structures derived from Brownian motion and stable subordinators. *Bernoulli* **3** 79–96. MR1466546 https://doi.org/10.2307/3318653

[47] PITMAN, J. (2006). *Combinatorial Stochastic Processes*. *Lecture Notes in Math.* **1875**. Springer, Berlin. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard. MR2245368

[48] PITMAN, J. and WINKEL, M. (2009). Regenerative tree growth: Binary self-similar continuum random trees and Poisson–Dirichlet compositions. *Ann. Probab.* **37** 1999–2041. MR2561439 https://doi.org/10.1214/08-AOP445

[49] PITMAN, J. and WINKEL, M. (2018). Squared Bessel processes of positive and negative dimension embedded in Brownian local times. *Electron. Commun. Probab.* **23** Paper No. 74, 13. MR3866047 https://doi.org/10.1214/18-ECP174

[50] PITMAN, J. and YOR, M. (1982). A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete* **59** 425–457. MR0656509 https://doi.org/10.1007/BF00532802

[51] PITMAN, J. and YOR, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* **25** 855–900. MR1434129 https://doi.org/10.1214/aop/1024404422

[52] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[53] ROGERS, L. C. G. and WILLIAMS, D. (1994). *Diffusions, Markov Processes, and Martingales. Vol.* 1, 2nd ed. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, Chichester. Foundations. MR1331599

[54] RUGGIERO, M. (2014). Species dynamics in the two-parameter Poisson–Dirichlet diffusion model. *J. Appl. Probab.* **51** 174–190. MR3189450 https://doi.org/10.1239/jap/1395771422

[55] RUGGIERO, M. and WALKER, S. G. (2009). Countable representation for infinite dimensional diffusions derived from the two-parameter Poisson–Dirichlet process. *Electron. Commun. Probab.* **14** 501–517. MR2564485 https://doi.org/10.1214/ECP.v14-1508

[56] RUGGIERO, M., WALKER, S. G. and FAVARO, S. (2013). Alpha-diversity processes and normalized inverse-Gaussian diffusions. *Ann. Appl. Probab.* **23** 386–425. MR3059239 https://doi.org/10.1214/12-AAP846

[57] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author. MR1739520

[58] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[59] SHIGA, T. (1990). A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes. *J. Math. Kyoto Univ.* **30** 245–279. MR1068791 https://doi.org/10.1215/kjm/1250520071

[60] SHIGA, T. and WATANABE, S. (1973). Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrsch. Verw. Gebiete* **27** 37–46. MR0368192 https://doi.org/10.1007/BF00736006

[61] TAKSAR, M. I. (1980). Regenerative sets on real line. In *Seminar on Probability*, *XIV* (*Paris*, 1978/1979) (*French*). *Lecture Notes in Math.* **784** 437–474. Springer, Berlin. MR0580147

[62] WARREN, J. and YOR, M. (1998). The Brownian burglar: Conditioning Brownian motion by its local time process. In *Séminaire de Probabilités*, *XXXII*. *Lecture Notes in Math.* **1686** 328–342. Springer, Berlin. MR1655303 https://doi.org/10.1007/BFb0101767

The Annals of Probability
2021, Vol. 49, No. 2, 832–876
https://doi.org/10.1214/20-AOP1461
© Institute of Mathematical Statistics, 2021

KPZ EQUATION CORRELATIONS IN TIME

BY IVAN CORWIN[1], PROMIT GHOSAL[2] AND ALAN HAMMOND[3]

[1]*Department of Mathematics, Columbia University, ic2354@columbia.edu*
[2]*Department of Mathematics, Massachusetts Institute of Technology, promit@mit.edu*
[3]*Departments of Mathematics and Statistics, University of California, Berkeley, alanmh@berkeley.edu*

We consider the narrow wedge solution to the Kardar–Parisi–Zhang stochastic PDE under the characteristic $3 : 2 : 1$ scaling of time, space and fluctuations. We study the correlation of fluctuations at two different times. We show that, when the times are close to each other, the correlation approaches one at a power-law rate with exponent $2/3$, while, when the two times are remote from each other, the correlation tends to zero at a power-law rate with exponent $-1/3$. We also prove exponential-type tail bounds for differences of the solution at two space-time points.

Three main tools are pivotal to proving these results: (1) a representation for the two-time distribution in terms of two independent narrow wedge solutions, (2) the Brownian Gibbs property of the KPZ line ensemble and (3) recently proved one-point tail bounds on the narrow wedge solution.

REFERENCES

[1] ALBERTS, T., KHANIN, K. and QUASTEL, J. (2014). The intermediate disorder regime for directed polymers in dimension $1+1$. *Ann. Probab.* **42** 1212–1256. MR3189070 https://doi.org/10.1214/13-AOP858

[2] AMIR, G., CORWIN, I. and QUASTEL, J. (2011). Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions. *Comm. Pure Appl. Math.* **64** 466–537. MR2796514 https://doi.org/10.1002/cpa.20347

[3] BAIK, J. and LIU, Z. (2019). Multipoint distribution of periodic TASEP. *J. Amer. Math. Soc.* **32** 609–674. MR3981984 https://doi.org/10.1090/jams/915

[4] BALÁZS, M., QUASTEL, J. and SEPPÄLÄINEN, T. (2011). Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.* **24** 683–708. MR2784327 https://doi.org/10.1090/S0894-0347-2011-00692-9

[5] BASU, R. and GANGULY, S. (2018). Time correlation exponents in last passage percolation. Available at arXiv:1807.09260.

[6] BASU, R., GANGULY, S. and HAMMOND, A. (2019). Fractal geometry of airy$_2$ processes coupled via the airy sheet. Available at arXiv:1904.01717.

[7] BASU, R., GANGULY, S. and ZHANG, L. (2019). Temporal correlation in last passage percolation with flat initial condition via Brownian comparison. Available at arXiv:1912.04891.

[8] BEN-AROUS, G. (2003). Aging and spin-glass dynamics. Available at arXiv:math/0304364.

[9] BERTINI, L. and GIACOMIN, G. (1997). Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183** 571–607. MR1462228 https://doi.org/10.1007/s002200050044

[10] BORODIN, A., BUFETOV, A. and WHEELER, M. (2016). Between the stochastic six vertex model and Hall–Littlewood processes. arXiv E-prints.

[11] CAPUTO, P., IOFFE, D. and WACHTEL, V. (2019). Confinement of Brownian polymers under geometric area tilts. *Electron. J. Probab.* **24** 37. MR3940767 https://doi.org/10.1214/19-EJP283

[12] CAPUTO, P., IOFFE, D. and WACHTEL, V. (2019). Tightness and line ensembles for Brownian polymers under geometric area tilts. In *Statistical Mechanics of Classical and Disordered Systems*. *Springer Proc. Math. Stat.* **293** 241–266. Springer, Cham. MR4015014 https://doi.org/10.1007/978-3-030-29077-1_10

[13] CORWIN, I. (2012). The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl.* **1** 1130001. MR2930377 https://doi.org/10.1142/S2010326311300014

MSC2020 subject classifications. Primary 60H15; secondary 35R60, 82B44, 82B26.
Key words and phrases. KPZ equation, narrow wedge solution, time correlation, aging.

[14] CORWIN, I. (2018). Exactly solving the KPZ equation. In *Random Growth Models*. *Proc. Sympos. Appl. Math.* **75** 203–254. Amer. Math. Soc., Providence, RI. MR3838899
[15] CORWIN, I. and DIMITROV, E. (2018). Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall–Littlewood Gibbsian line ensembles. *Comm. Math. Phys.* **363** 435–501. MR3851820 https://doi.org/10.1007/s00220-018-3139-3
[16] CORWIN, I., FERRARI, P. L. and PÉCHÉ, S. (2012). Universality of slow decorrelation in KPZ growth. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 134–150. MR2919201 https://doi.org/10.1214/11-AIHP440
[17] CORWIN, I. and GHOSAL, P. (2020). Lower tail of the KPZ equation. *Duke Math. J.* **169** 1329–1395. MR4094738 https://doi.org/10.1215/00127094-2019-0079
[18] CORWIN, I. and GHOSAL, P. (2020). KPZ equation tails for general initial data. *Electron. J. Probab.* **25** 66. MR4115735 https://doi.org/10.1214/20-ejp467
[19] CORWIN, I. and HAMMOND, A. (2014). Brownian Gibbs property for Airy line ensembles. *Invent. Math.* **195** 441–508. MR3152753 https://doi.org/10.1007/s00222-013-0462-3
[20] CORWIN, I. and HAMMOND, A. (2016). KPZ line ensemble. *Probab. Theory Related Fields* **166** 67–185. MR3547737 https://doi.org/10.1007/s00440-015-0651-7
[21] CORWIN, I., O'CONNELL, N., SEPPÄLÄINEN, T. and ZYGOURAS, N. (2014). Tropical combinatorics and Whittaker functions. *Duke Math. J.* **163** 513–563. MR3165422 https://doi.org/10.1215/00127094-2410289
[22] CORWIN, I. and QUASTEL, J. (2013). Crossover distributions at the edge of the rarefaction fan. *Ann. Probab.* **41** 1243–1314. MR3098678 https://doi.org/10.1214/11-AOP725
[23] CORWIN, I., QUASTEL, J. and REMENIK, D. (2015). Renormalization fixed point of the KPZ universality class. *J. Stat. Phys.* **160** 815–834. MR3373642 https://doi.org/10.1007/s10955-015-1243-8
[24] CORWIN, I. and SHEN, H. (2020). Some recent progress in singular stochastic partial differential equations. *Bull. Amer. Math. Soc.* (*N.S.*) **57** 409–454. MR4108091 https://doi.org/10.1090/bull/1670
[25] DAUVERGNE, D., ORTMANN, J. and VIRÁG, B. (2018). The directed landscape. Available at arXiv:1812.00309.
[26] DAUVERGNE, D. and VIRÁG, B. (2018). Basic properties of the Airy line ensemble. Available at arXiv:1812.00311.
[27] DE NARDIS, J. and LE DOUSSAL, P. (2017). Tail of the two-time height distribution for KPZ growth in one dimension. *J. Stat. Mech. Theory Exp.* **5** 053212. MR3664392 https://doi.org/10.1088/1742-5468/aa6bce
[28] DE NARDIS, J. and LE DOUSSAL, P. (2018). Two-time height distribution for 1D KPZ growth: The recent exact result and its tail via replica. *J. Stat. Mech. Theory Exp.* **9** 093203. MR3869061 https://doi.org/10.1088/1742-5468/aad6b3
[29] DE NARDIS, J., LE DOUSSAL, P. and TAKEUCHI, K. A. (2017). Memory and universality in interface growth. *Phys. Rev. Lett.* **118** 125701. MR3682581 https://doi.org/10.1103/PhysRevLett.118.125701
[30] DEMBO, A. and DEUSCHEL, J.-D. (2007). Aging for interacting diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **43** 461–480. MR2329512 https://doi.org/10.1016/j.anihpb.2006.07.001
[31] DOTSENKO, V. (2013). Two-time free energy distribution function in $(1+1)$ directed polymers. *J. Stat. Mech. Theory Exp.* **6** P06017. MR3090739 https://doi.org/10.1088/1742-5468/2013/06/p06017
[32] DOTSENKO, V. (2015). Two-time free energy distribution function in the KPZ problem. Available at arXiv:1507.06135.
[33] DOTSENKO, V. (2016). On two-time distribution functions in $(1+1)$ random directed polymers. *J. Phys. A* **49** 27LT01. MR3512099 https://doi.org/10.1088/1751-8113/49/27/27LT01
[34] DOUSSAL, P. L. (2017). Maximum of an Airy process plus Brownian motion and memory in Kardar–Parisi–Zhang growth. *Phys Rev E* **96** 060101. https://doi.org/10.1103/PhysRevE.96.060101
[35] FERRARI, P. L. (2008). Slow decorrelations in Kardar–Parisi–Zhang growth. *J. Stat. Mech.* **07** P07022.
[36] FERRARI, P. L. and OCCELLI, A. (2019). Time-time covariance for last passage percolation with generic initial profile. *Math. Phys. Anal. Geom.* **22** 1. MR3895778 https://doi.org/10.1007/s11040-018-9300-6
[37] FERRARI, P. L. and SPOHN, H. (2016). On time correlations for KPZ growth in one dimension. *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** 074. MR3529743 https://doi.org/10.3842/SIGMA.2016.074
[38] GONÇALVES, P. and JARA, M. (2014). Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.* **212** 597–644. MR3176353 https://doi.org/10.1007/s00205-013-0693-x
[39] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6. MR3406823 https://doi.org/10.1017/fmp.2015.2
[40] GUBINELLI, M. and PERKOWSKI, N. (2017). KPZ reloaded. *Comm. Math. Phys.* **349** 165–269. MR3592748 https://doi.org/10.1007/s00220-016-2788-3

[41] GUBINELLI, M. and PERKOWSKI, N. (2018). Energy solutions of KPZ are unique. *J. Amer. Math. Soc.* **31** 427–471. MR3758149 https://doi.org/10.1090/jams/889

[42] HAIRER, M. (2013). Solving the KPZ equation. *Ann. of Math.* (2) **178** 559–664. MR3071506 https://doi.org/10.4007/annals.2013.178.2.4

[43] HAMMOND, A. (2016). Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation. *Mem. Amer. Math. Soc.* To appear. Available at arXiv:1609.02971.

[44] HAMMOND, A. (2019). Modulus of continuity of polymer weight profiles in Brownian last passage percolation. *Ann. Probab.* **47** 3911–3962. MR4038045 https://doi.org/10.1214/19-aop1350

[45] HAMMOND, A. (2019). A patchwork quilt sewn from Brownian fabric: Regularity of polymer weight profiles in Brownian last passage percolation. *Forum Math. Pi* **7** e2. MR3987302 https://doi.org/10.1017/fmp.2019.2

[46] HAMMOND, A. (2020). Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *Proc. Lond. Math. Soc.* (3) **120** 370–433. MR4008374 https://doi.org/10.1112/plms.12292

[47] HAMMOND, A. and SARKAR, S. (2020). Modulus of continuity for polymer fluctuations and weight profiles in Poissonian last passage percolation. *Electron. J. Probab.* **25** 29. MR4073690 https://doi.org/10.1214/20-ejp430

[48] JOHANSSON, K. (2017). Two time distribution in Brownian directed percolation. *Comm. Math. Phys.* **351** 441–492. MR3613511 https://doi.org/10.1007/s00220-016-2660-5

[49] JOHANSSON, K. (2019). The two-time distribution in geometric last-passage percolation. *Probab. Theory Related Fields* **175** 849–895. MR4026607 https://doi.org/10.1007/s00440-019-00901-9

[50] JOHANSSON, K. (2019). The long ans short time asymptotics of the two-time distribution in local random growth. Available at arXiv:1904.08195.

[51] JOHANSSON, K. and RAHMAN, M. (2019). Multi-time distribution in discrete polynuclear growth. Available at arXiv:1906.01053.

[52] KARDAR, M., PARISI, G. and ZHANG, Y. C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892.

[53] KUPIAINEN, A. (2016). Renormalization group and stochastic PDEs. *Ann. Henri Poincaré* **17** 497–535. MR3459120 https://doi.org/10.1007/s00023-015-0408-y

[54] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346 https://doi.org/10.1007/978-3-662-03990-8

[55] LIGGETT, T. M. (2005). *Interacting Particle Systems. Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374

[56] LIU, Z. (2019). Multi-time distribution of TASEP.

[57] MATETSKI, K., QUASTEL, J. and REMENIK, D. (2017). The KPZ fixed point. Available at arXiv:1701.00018.

[58] O'CONNELL, N. (2012). Directed polymers and the quantum Toda lattice. *Ann. Probab.* **40** 437–458. MR2952082 https://doi.org/10.1214/10-AOP632

[59] PIMENTEL, L. P. R. (2018). Local behaviour of airy processes. *J. Stat. Phys.* **173** 1614–1638. MR3880997 https://doi.org/10.1007/s10955-018-2147-1

[60] PRÄHOFER, M. and SPOHN, H. (2002). Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* **108** 1071–1106. MR1933446 https://doi.org/10.1023/A:1019791415147

[61] TAKEUCHI, K. and SANO, M. (2012). Evidence for geometry-dependent universal fluctuations of the Kardar–Parisi–Zhang interfaces in liquid–crystal turbulence. *J. Stat. Phys.* **147** 853–890.

[62] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[63] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

The Annals of Probability
2021, Vol. 49, No. 2, 877–907
https://doi.org/10.1214/20-AOP1462
© Institute of Mathematical Statistics, 2021

A SOLUTION TO THE MONGE TRANSPORT PROBLEM FOR BROWNIAN MARTINGALES

BY NASSIF GHOUSSOUB*, YOUNG-HEON KIM† AND AARON ZEFF PALMER‡

*Department of Mathematics, University of British Columbia, *nassif@math.ubc.ca; †yhkim@math.ubc.ca; ‡azp@math.ubc.ca*

We provide a solution to the problem of optimal transport by Brownian martingales in general dimensions whenever the transport cost satisfies certain subharmonic properties in the target variable as well as a stochastic version of the standard "twist condition" frequently used in deterministic Monge transport theory. This setting includes, in particular, the case of the distance cost $c(x, y) = |x - y|$. We prove existence and uniqueness of the solution and characterize it as the first time Brownian motion hits a barrier that is determined by solutions to a corresponding dual problem.

REFERENCES

[1] AMBROSIO, L. (2003). Lecture notes on optimal transport problems. In *Mathematical Aspects of Evolving Interfaces* (*Funchal*, 2000). *Lecture Notes in Math*. **1812** 1–52. Springer, Berlin. MR2011032 https://doi.org/10.1007/978-3-540-39189-0_1

[2] BAXTER, J. R. and CHACON, R. V. (1974). Potentials of stopped distributions. *Illinois J. Math*. **18** 649–656. MR0358960

[3] BEIGLBÖCK, M., COX, A. M. G. and HUESMANN, M. (2017). Optimal transport and Skorokhod embedding. *Invent. Math*. **208** 327–400. MR3639595 https://doi.org/10.1007/s00222-016-0692-2

[4] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and PENKNER, F. (2013). Model-independent bounds for option prices—a mass transport approach. *Finance Stoch*. **17** 477–501. MR3066985 https://doi.org/10.1007/s00780-013-0205-8

[5] BEIGLBÖCK, M. and JUILLET, N. (2016). On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.* **44** 42–106. MR3456332 https://doi.org/10.1214/14-AOP966

[6] BEIGLBÖCK, M., LIM, T. and OBŁÓJ, J. (2019). Dual attainment for the martingale transport problem. *Bernoulli* **25** 1640–1658. MR3961225 https://doi.org/10.3150/17-BEJ1015

[7] BEIGLBÖCK, M., LIM, T. and OBŁÓJ, J. (2019). Dual attainment for the martingale transport problem. *Bernoulli* **25** 1640–1658. MR3961225 https://doi.org/10.3150/17-BEJ1015

[8] BEIGLBÖCK, M., NUTZ, M. and TOUZI, N. (2017). Complete duality for martingale optimal transport on the line. *Ann. Probab.* **45** 3038–3074. MR3706738 https://doi.org/10.1214/16-AOP1131

[9] BENAMOU, J.-D. and BRENIER, Y. (2000). A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. *Numer. Math*. **84** 375–393. MR1738163 https://doi.org/10.1007/s002110050002

[10] BERNARD, P. and BUFFONI, B. (2007). Optimal mass transportation and Mather theory. *J. Eur. Math. Soc*. (*JEMS*) **9** 85–121. MR2283105 https://doi.org/10.4171/JEMS/74

[11] BRENIER, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math*. **44** 375–417. MR1100809 https://doi.org/10.1002/cpa.3160440402

[12] CAFFARELLI, L. A. (1996). Allocation maps with general cost functions. In *Partial Differential Equations and Applications*. *Lecture Notes in Pure and Applied Mathematics* **177** 29–35. Dekker, New York. MR1371577

[13] CAFFARELLI, L. A., FELDMAN, M. and MCCANN, R. J. (2002). Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs. *J. Amer. Math. Soc*. **15** 1–26. MR1862796 https://doi.org/10.1090/S0894-0347-01-00376-9

[14] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc*. (*N.S.*) **27** 1–67. MR1118699 https://doi.org/10.1090/S0273-0979-1992-00266-5

[15] DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential*. *North-Holland Mathematics Studies* **29**. North-Holland, Amsterdam. MR0521810

MSC2020 subject classifications. Primary 49, 60; secondary 52.
Key words and phrases. Optimal transport, Skorokhod embedding, variational inequality.

[16] DOLINSKY, Y. and SONER, H. M. (2014). Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields* **160** 391–427. MR3256817 https://doi.org/10.1007/s00440-013-0531-y

[17] DUBINS, L. E. (1968). On a theorem of Skorohod. *Ann. Math. Stat.* **39** 2094–2097. MR0234520 https://doi.org/10.1214/aoms/1177698036

[18] EDGAR, G. A., MILLET, A. and SUCHESTON, L. (1982). On compactness and optimality of stopping times. In *Martingale Theory in Harmonic Analysis and Banach Spaces* (*Cleveland*, *Ohio*, 1981). *Lecture Notes in Math.* **939** 36–61. Springer, Berlin. MR0668536

[19] EVANS, L. C. and GANGBO, W. (1999). *Differential Equations Methods for the Monge–Kantorovich Mass Transfer Problem* **653**. Am. Math. Soc., Providence.

[20] FALKNER, N. (1980). On Skorohod embedding in n-dimensional Brownian motion by means of natural stopping times. In *Seminar on Probability*, *XIV* (*Paris*, 1978/1979) (*French*). *Lecture Notes in Math.* **784** 357–391. Springer, Berlin. MR0580142

[21] FUKUDA, I., ISHII, H. and TSUTSUMI, M. (1993). Uniqueness of solutions to the Cauchy problem for $u_t - u\Delta u + \gamma|\nabla u|^2 = 0$. *Differential Integral Equations* **6** 1231–1252. MR1235190

[22] GANGBO, W. (1995). Habilitation thesis, Universite de Metz. Available at http://people.math.gatech.edu/gangbo/publications/habilitation.pdf.

[23] GANGBO, W. and MCCANN, R. J. (1996). The geometry of optimal transportation. *Acta Math.* **177** 113–161. MR1440931 https://doi.org/10.1007/BF02392620

[24] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2017). Optimal Brownian Stopping between radially symmetric marginals in general dimensions. ArXiv e-prints.

[25] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2019). Structure of optimal martingale transport plans in general dimensions. *Ann. Probab.* **47** 109–164. MR3909967 https://doi.org/10.1214/18-AOP1258

[26] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2018). Optimal transport with controlled dynamics and free end times. *SIAM J. Control Optim.* **56** 3239–3259. MR3853603 https://doi.org/10.1137/18M1178037

[27] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2019). PDE methods for optimal Skorokhod embeddings. *Calc. Var. Partial Differential Equations* **58** 113. MR3959935 https://doi.org/10.1007/s00526-019-1563-7

[28] GRIGOR'YAN, A. (1999). Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. *Bull. Amer. Math. Soc.* (*N.S.*) **36** 135–249. MR1659871 https://doi.org/10.1090/S0273-0979-99-00776-4

[29] GUO, G., TAN, X. and TOUZI, N. (2016). On the monotonicity principle of optimal Skorokhod embedding problem. *SIAM J. Control Optim.* **54** 2478–2489. MR3549873 https://doi.org/10.1137/15M1025268

[30] HENRY-LABORDÈRE, P. (2017). *Model-Free Hedging*: *A Martingale Optimal Transport Viewpoint*. *Chapman & Hall/CRC Financial Mathematics Series*. CRC Press, Boca Raton, FL. MR3699668

[31] HENRY-LABORDÈRE, P. and TOUZI, N. (2016). An explicit martingale version of the one-dimensional Brenier theorem. *Finance Stoch.* **20** 635–668. MR3519164 https://doi.org/10.1007/s00780-016-0299-x

[32] HOBSON, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris–Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math.* **2003** 267–318. Springer, Berlin. MR2762363 https://doi.org/10.1007/978-3-642-14660-2_4

[33] HOBSON, D. and KLIMMEK, M. (2012). Model-independent hedging strategies for variance swaps. *Finance Stoch.* **16** 611–649. MR2972236 https://doi.org/10.1007/s00780-012-0190-3

[34] HOBSON, D. and KLIMMEK, M. (2015). Robust price bounds for the forward starting straddle. *Finance Stoch.* **19** 189–214. MR3292129 https://doi.org/10.1007/s00780-014-0249-4

[35] KANTOROVITCH, L. (1942). On the translocation of masses. *C. R.* (*Dokl.*) *Acad. Sci. URSS* **37** 199–201. MR0009619

[36] KINDERLEHRER, D. and STAMPACCHIA, G. (2000). *An Introduction to Variational Inequalities and Their Applications*. *Classics in Applied Mathematics* **31**. SIAM, Philadelphia, PA. MR1786735 https://doi.org/10.1137/1.9780898719451

[37] LEVIN, V. (1999). Abstract cyclical monotonicity and Monge solutions for the general Monge–Kantorovich problem. *Set-Valued Anal.* **7** 7–32. MR1699061 https://doi.org/10.1023/A:1008753021652

[38] LIONS, P.-L. (1983). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* **8** 1229–1276. MR0709162 https://doi.org/10.1080/03605308308820301

[39] MA, X.-N., TRUDINGER, N. S. and WANG, X.-J. (2005). Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal.* **177** 151–183. MR2188047 https://doi.org/10.1007/s00205-005-0362-9

[40] MONGE, G. (1781). Mémoire sur la théorie des déblais et des remblais. *Histoire de L'Académie Royale des Sciences de Paris*.

[41] OBŁÓJ, J. (2004). The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1** 321–392.
[42] PRATELLI, A. (2007). On the equality between Monge's infimum and Kantorovich's minimum in optimal mass transportation. *Ann. Inst. Henri Poincaré Probab. Stat.* **43** 1–13. MR2288266 https://doi.org/10.1016/j.anihpb.2005.12.001
[43] ROOT, D. H. (1969). The existence of certain stopping times on Brownian motion. *Ann. Math. Stat.* **40** 715–718. MR0238394 https://doi.org/10.1214/aoms/1177697749
[44] ROST, H. (1971). The stopping distributions of a Markov Process. *Invent. Math.* **14** 1–16. MR0346920 https://doi.org/10.1007/BF01418740
[45] SKOROKHOD, A. V. (1965). *Studies in the Theory of Random Processes. Translated from the Russian by Scripta Technica, Inc.* Addison-Wesley, Reading, MA. MR0185620
[46] STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Stat.* **36** 423–439. MR0177430 https://doi.org/10.1214/aoms/1177700153
[47] SYLVESTRE, L. (2015). Viscosity solutions of elliptic equations. *Lect. Notes Second Chicago Summer Sch. Anal.* Available at http://math.uchicago.edu/~luis/preprints/viscosity-solutions.pdf.
[48] TRUDINGER, N. S. and WANG, X.-J. (2001). On the Monge mass transfer problem. *Calc. Var. Partial Differential Equations* **13** 19–31. MR1854255 https://doi.org/10.1007/PL00009922

The Annals of Probability
2021, Vol. 49, No. 2, 908–943
https://doi.org/10.1214/20-AOP1464
© Institute of Mathematical Statistics, 2021

A ROUGH SUPER-BROWNIAN MOTION

BY NICOLAS PERKOWSKI* AND TOMMASO ROSATI†

*Institut für Mathematik, Freie Universität Berlin, *perkowski@math.fu-berlin.de; †rosati.tom@fu-berlin.de*

We study the scaling limit of a branching random walk in static random environment in dimension $d = 1, 2$ and show that it is given by a super-Brownian motion in a white noise potential. In dimension 1 we characterize the limit as the unique weak solution to the stochastic PDE

$$\partial_t \mu = (\Delta + \xi)\mu + \sqrt{2\nu\mu}\tilde{\xi}$$

for independent space white noise ξ and space-time white noise $\tilde{\xi}$. In dimension 2 the study requires paracontrolled theory and the limit process is described via a martingale problem. In both dimensions we prove persistence of this rough version of the super-Brownian motion.

REFERENCES

[1] ALBEVERIO, S., BOGACHEV, L. V., MOLCHANOV, S. A. and YAROVAYA, E. B. (2000). Annealed moment Lyapunov exponents for a branching random walk in a homogeneous random branching environment. *Markov Process*. *Related Fields* **6** 473–516. MR1805091

[2] ALLEZ, R. and CHOUK, K. (2015). The continuous Anderson hamiltonian in dimension two. arXiv preprint. Available at arXiv:1511.02718.

[3] BARTSCH, C., GANTERT, N. and KOCHLER, M. (2009). Survival and growth of a branching random walk in random environment. *Markov Process*. *Related Fields* **15** 525–548. MR2598127

[4] CHAKRABORTY, P. and TINDEL, S. (2019). Rough differential equations with power type nonlinearities. *Stochastic Process*. *Appl*. **129** 1533–1555. MR3944775 https://doi.org/10.1016/j.spa.2018.05.010

[5] CHEN, X. (2014). Quenched asymptotics for Brownian motion in generalized Gaussian potential. *Ann*. *Probab*. **42** 576–622. MR3178468 https://doi.org/10.1214/12-AOP830

[6] CHENTSOV, N. (1956). Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov–Smirnov tests. *Theory Probab*. *Appl*. **1** 140–144.

[7] CHOUK, K. and VAN ZUIJLEN, W. (2019). Asymptotics of the eigenvalues of the Anderson Hamiltonian with white noise potential in two dimensions. arXiv preprint. Available at arXiv:1907.01352.

[8] CORWIN, I. and TSAI, L.-C. (2018). SPDE limit of weakly inhomogeneous ASEP. arXiv preprint. Available at arXiv:1806.09682.

[9] CRISAN, D. (2004). Superprocesses in a Brownian environment. *Proc*. *R*. *Soc*. *Lond*. *Ser*. *A Math*. *Phys*. *Eng*. *Sci*. **460** 243–270. Stochastic analysis with applications to mathematical finance.

[10] DAWSON, D. A., MAISONNEUVE, B. and SPENCER, J. (1993). *École D'Été de Probabilités de Saint-Flour XXI*—1991. *Lecture Notes in Math*. **1541**. Springer, Berlin. Papers from the school held in Saint-Flour, August 18–September 4, 1991, Edited by P. L. Hennequin. MR1242574 https://doi.org/10.1007/BFb0084189

[11] ETHERIDGE, A. M. (2000). *An Introduction to Superprocesses*. *University Lecture Series* **20**. Amer. Math. Soc., Providence, RI. MR1779100 https://doi.org/10.1090/ulect/020

[12] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[13] FUKUSHIMA, M. and NAKAO, S. (1976/77). On spectra of the Schrödinger operator with a white Gaussian noise potential. *Z*. *Wahrsch*. *Verw*. *Gebiete* **37** 267–274. MR0438481 https://doi.org/10.1007/BF00537493

[14] GANTERT, N., MÜLLER, S., POPOV, S. and VACHKOVSKAIA, M. (2010). Survival of branching random walks in random environment. *J*. *Theoret*. *Probab*. **23** 1002–1014. MR2735734 https://doi.org/10.1007/s10959-009-0227-5

MSC2020 subject classifications. Primary 60H17; secondary 60H25, 60L40.
Key words and phrases. Super-Brownian motion, singular, stochastic PDEs, parabolic Anderson model.

[15] GÄRTNER, J. and MOLCHANOV, S. A. (1990). Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.* **132** 613–655. MR1069840

[16] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6. MR3406823 https://doi.org/10.1017/fmp.2015.2

[17] GUBINELLI, M. and PERKOWSKI, N. (2017). KPZ reloaded. *Comm. Math. Phys.* **349** 165–269. MR3592748 https://doi.org/10.1007/s00220-016-2788-3

[18] GUBINELLI, M., UGURCAN, B. and ZACHHUBER, I. (2020). Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** 82–149. MR4058957 https://doi.org/10.1007/s40072-019-00143-9

[19] GÜN, O., KÖNIG, W. and SEKULOVIĆ, O. (2013). Moment asymptotics for branching random walks in random environment. *Electron. J. Probab.* **18** 63. MR3078022 https://doi.org/10.1214/ejp.v18-2212

[20] HAIRER, M. (2014). A theory of regularity structures. *Invent. Math.* **198** 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4

[21] HAIRER, M. and LABBÉ, C. (2015). A simple construction of the continuum parabolic Anderson model on $\mathbf{R}^2$. *Electron. Commun. Probab.* **20** 43. MR3358965 https://doi.org/10.1214/ECP.v20-4038

[22] HAIRER, M. and LABBÉ, C. (2018). Multiplicative stochastic heat equations on the whole space. *J. Eur. Math. Soc.* (*JEMS*) **20** 1005–1054. MR3779690 https://doi.org/10.4171/JEMS/781

[23] HARRIS, T. E. (1951). Some mathematical models for branching processes. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 305–328. Univ. California Press, Berkeley and Los Angeles. MR0045331

[24] HARRIS, T. E. (2002). *The Theory of Branching Processes*. *Dover Phoenix Editions*. Dover, Mineola, NY. Corrected reprint of the 1963 original [Springer, Berlin; MR0163361 (29 #664)]. MR1991122

[25] KÖNIG, W. (2016). *The Parabolic Anderson Model*. *Pathways in Mathematics*. Birkhäuser/Springer, [Cham], Basel. Random walk in random potential. MR3526112 https://doi.org/10.1007/978-3-319-33596-4

[26] KONNO, N. and SHIGA, T. (1988). Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* **79** 201–225. MR0958288 https://doi.org/10.1007/BF00320919

[27] LABBÉ, C. (2019). The continuous Anderson Hamiltonian in $d \leq 3$. *J. Funct. Anal.* **277** 3187–3235. MR3997634 https://doi.org/10.1016/j.jfa.2019.05.027

[28] MARTIN, J. and PERKOWSKI, N. (2019). Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 2058–2110. MR4029148 https://doi.org/10.1214/18-AIHP942

[29] MYTNIK, L. (1996). Superprocesses in random environments. *Ann. Probab.* **24** 1953–1978. MR1415235 https://doi.org/10.1214/aop/1041903212

[30] MYTNIK, L. and XIONG, J. (2007). Local extinction for superprocesses in random environments. *Electron. J. Probab.* **12** 1349–1378. MR2354161 https://doi.org/10.1214/EJP.v12-457

[31] PETROV, V. V. (1995). *Limit Theorems of Probability Theory*. *Oxford Studies in Probability* **4**. The Clarendon Press, Oxford University Press, New York. Sequences of independent random variables, Oxford Science Publications. MR1353441

[32] REIMERS, M. (1989). One-dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields* **81** 319–340. MR0983088 https://doi.org/10.1007/BF00340057

[33] ROSATI, T. C. (2020). Killed rough super-Brownian motion. *Electron. Commun. Probab.* **25** 44. MR4116392 https://doi.org/10.1214/20-ecp319

[34] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV—1984*. *Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

[35] ZEL'DOVICH, Y. B., MOLCHANOV, S. A., RUZMAĬKIN, A. A. and SOKOLOV, D. D. (1987). Intermittency in random media. *Usp. Fiz. Nauk* **152** 3–32. MR0921018 https://doi.org/10.1070/PU1987v030n05ABEH002867

The Annals of Probability
2021, Vol. 49, No. 2, 944–996
https://doi.org/10.1214/20-AOP1465
© Institute of Mathematical Statistics, 2021

PROPAGATION OF CHAOS FOR MEAN FIELD ROUGH DIFFERENTIAL EQUATIONS

BY ISMAËL BAILLEUL[1], RÉMI CATELLIER[2,*] AND FRANÇOIS DELARUE[2,†]

[1] *Université Rennes 1, CNRS, IRMAR, ismael.bailleul@univ-rennes1.fr*
[2] *Université Côte d'Azur, CNRS, LJAD, [*]remi.catellier@unice.fr; [†]francois.delarue@unice.fr*

We address propagation of chaos for large systems of rough differential equations associated with random rough differential equations of mean field type

$$dX_t = \mathrm{V}(X_t, \mathcal{L}(X_t))\,dt + \mathrm{F}(X_t, \mathcal{L}(X_t))\,dW_t,$$

where W is a random rough path and $\mathcal{L}(X_t)$ is the law of X_t. We prove propagation of chaos, and provide also an explicit optimal convergence rate. The analysis is based upon the tools we developed in our companion paper (*Electron. J. Probab.* **25** (2020) 21) for solving mean field rough differential equations and in particular upon a corresponding version of the Itô-Lyons continuity theorem. The rate of convergence is obtained by a coupling argument developed first by Sznitman for particle systems with Brownian inputs.

REFERENCES

[1] ALBEVERIO, S., KONDRATIEV, Y. G. and RÖCKNER, M. (1996). Differential geometry of Poisson spaces. *C. R. Acad. Sci. Paris Sér. I Math.* **323** 1129–1134. MR1423438

[2] ALBEVERIO, S., KONDRATIEV, YU. G. and RÖCKNER, M. (1998). Analysis and geometry on configuration spaces. *J. Funct. Anal.* **154** 444–500. MR1612725 https://doi.org/10.1006/jfan.1997.3183

[3] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2005). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2129498

[4] BAILLEUL, I. (2015). Flows driven by rough paths. *Rev. Mat. Iberoam.* **31** 901–934. MR3420480 https://doi.org/10.4171/RMI/858

[5] BAILLEUL, I., CATELLIER, R. and DELARUE, F. (2020). Solving mean field rough differential equations. *Electron. J. Probab.* **25** Paper No. 21. MR4073682 https://doi.org/10.1214/19-ejp409

[6] BAILLEUL, I., CATELLIER, R. and DELARUE, F. (2020). Solving mean field rough differential equations. *Electron. J. Probab.* **25** Paper No. 21, 51. MR4073682 https://doi.org/10.1214/19-ejp409

[7] BAILLEUL, I. and RIEDEL, S. (2019). Rough flows. *J. Math. Soc. Japan* **71** 915–978. MR3984248 https://doi.org/10.2969/jmsj/80108010

[8] BARBU, V. and RÖCKNER, M. (2020). From nonlinear Fokker–Planck equations to solutions of distribution dependent SDE. *Ann. Probab.* **48** 1902–1920. MR4124528 https://doi.org/10.1214/19-AOP1410

[9] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[10] BUDHIRAJA, A., DUPUIS, P. and FISCHER, M. (2012). Large deviation properties of weakly interacting processes via weak convergence methods. *Ann. Probab.* **40** 74–102. MR2917767 https://doi.org/10.1214/10-AOP616

[11] BUDHIRAJA, A. and WU, R. (2017). Moderate deviation principles for weakly interacting particle systems. *Probab. Theory Related Fields* **168** 721–771. MR3663630 https://doi.org/10.1007/s00440-016-0723-3

[12] CARDALIAGUET, P. (2013). Notes on mean field games. https://www.ceremade.dauphine.fr/cardaliaguet/MFG20130420.pdf.

[13] CARMONA, R. and DELARUE, F. (2015). Forward-backward stochastic differential equations and controlled McKean–Vlasov dynamics. *Ann. Probab.* **43** 2647–2700. MR3395471 https://doi.org/10.1214/14-AOP946

MSC2020 subject classifications. 60H10, 60G99.
Key words and phrases. Random rough differential equations, particle system, mean field interaction, propagation of chaos, convergence rate.

[14] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. I: Mean Field FBSDEs, Control, and Games. Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669

[15] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. II: Mean Field Games with Common Noise and Master Equations. Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660

[16] CASS, T., DOS REIS, G. and SLAKELD, W. (2019). The support of McKean Vlasov equations driven by Brownian motion. Available at arXiv:1911.01992.

[17] CASS, T., LITTERER, C. and LYONS, T. (2013). Integrability and tail estimates for Gaussian rough differential equations. *Ann. Probab.* **41** 3026–3050. MR3112937 https://doi.org/10.1214/12-AOP821

[18] CASS, T. and LYONS, T. (2015). Evolving communities with individual preferences. *Proc. Lond. Math. Soc.* (3) **110** 83–107. MR3299600 https://doi.org/10.1112/plms/pdu040

[19] COGHI, M., DEUSCHEL, J.-D., FRIZ, P. and MAURELLI, M. (2018). Pathwise McKean–Vlasov theory. Available at arXiv:1812.11773.

[20] COGHI, M. and FLANDOLI, F. (2016). Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26** 1407–1442. MR3513594 https://doi.org/10.1214/15-AAP1120

[21] COGHI, M. and GESS, B. (2019). Stochastic nonlinear Fokker–Planck equations. *Nonlinear Anal.* **187** 259–278. MR3954095 https://doi.org/10.1016/j.na.2019.05.003

[22] COGHI, M. and NILSSEN, T. (2019). Rough nonlocal diffusions. Available at arXiv:1905.07270.

[23] DAWSON, D. and VAILLANCOURT, J. (1995). Stochastic McKean–Vlasov equations. *NoDEA Nonlinear Differential Equations Appl.* **2** 199–229. MR1328577 https://doi.org/10.1007/BF01295311

[24] DAWSON, D. A. and GÄRTNER, J. (1987). Large deviations from the McKean–Vlasov limit for weakly interacting diffusions. *Stochastics* **20** 247–308. MR0885876 https://doi.org/10.1080/17442508708833446

[25] DEUSCHEL, J.-D., FRIZ, P. K., MAURELLI, M. and SLOWIK, M. (2018). The enhanced Sanov theorem and propagation of chaos. *Stochastic Process. Appl.* **128** 2228–2269. MR3804792 https://doi.org/10.1016/j.spa.2017.09.010

[26] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[27] FRIZ, P. and VICTOIR, N. (2010). Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 369–413. MR2667703 https://doi.org/10.1214/09-AIHP202

[28] FRIZ, P. K. and HAIRER, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures. Universitext*. Springer, Cham. MR3289027 https://doi.org/10.1007/978-3-319-08332-2

[29] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669 https://doi.org/10.1017/CBO9780511845079

[30] GANGBO, W. and TUDORASCU, A. (2019). On differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations. *J. Math. Pures Appl.* (9) **125** 119–174. MR3944201 https://doi.org/10.1016/j.matpur.2018.09.003

[31] GÄRTNER, J. (1988). On the McKean-Vlasov limit for interacting diffusions. *Math. Nachr.* **137** 197–248. MR0968996 https://doi.org/10.1002/mana.19881370116

[32] GUBINELLI, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. MR2091358 https://doi.org/10.1016/j.jfa.2004.01.002

[33] JABIN, P.-E. and WANG, Z. (2018). Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.* **214** 523–591. MR3858403 https://doi.org/10.1007/s00222-018-0808-y

[34] JORDAN, R., KINDERLEHRER, D. and OTTO, F. (1998). The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** 1–17. MR1617171 https://doi.org/10.1137/S0036141096303359

[35] JOURDAIN, B. and MÉLÉARD, S. (1998). Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. Henri Poincaré Probab. Stat.* **34** 727–766. MR1653393 https://doi.org/10.1016/S0246-0203(99)80002-8

[36] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley and Los Angeles. MR0084985

[37] KOLOKOLTSOV, V. N. and TROEVA, M. (2019). On mean field games with common noise and McKean-Vlasov SPDEs. *Stoch. Anal. Appl.* **37** 522–549. MR3954570 https://doi.org/10.1080/07362994.2019.1592690

[38] LIONS, P.-L. Théorie des jeux à champs moyen et applications. Lectures at the Collège de France, 2007–2008. Available at http://www.college-de-france.fr/default/EN/all/equ_der/cours_et_seminaires.htm.

[39] LOTT, J. (2008). Some geometric calculations on Wasserstein space. *Comm. Math. Phys.* **277** 423–437. MR2358290 https://doi.org/10.1007/s00220-007-0367-3

[40] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527 https://doi.org/10.4171/RMI/240

[41] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595 https://doi.org/10.1073/pnas.56.6.1907

[42] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299 https://doi.org/10.1007/BFb0093177

[43] OTTO, F. (2001). The geometry of dissipative evolution equations: The porous medium equation. *Comm. Partial Differential Equations* **26** 101–174. MR1842429 https://doi.org/10.1081/PDE-100002243

[44] OVERBECK, L., RÖCKNER, M. and SCHMULAND, B. (1995). An analytic approach to Fleming–Viot processes with interactive selection. *Ann. Probab.* **23** 1–36. MR1330758

[45] REN, P. and WANG, F.-Y. (2019). Derivative formulas in measure on Riemannian manifold. Available at arXiv:1908.03711.

[46] ROSENTHAL, H. P. (1972). On the span in L^p of sequences of independent random variables. II. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California, Berkeley, Calif.*, 1970/1971), *Vol. II*: *Probability Theory* 149–167. MR0440354

[47] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[48] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete* **46** 67–105. MR0512334 https://doi.org/10.1007/BF00535689

[49] TANAKA, H. (1984). Limit theorems for certain diffusion processes with interaction. In *Stochastic Analysis* (*Katata/Kyoto*, 1982). *North-Holland Math. Library* **32** 469–488. North-Holland, Amsterdam. MR0780770 https://doi.org/10.1016/S0924-6509(08)70405-7

[50] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058

[51] WU, C. and ZHANG, J. (2017). An elementary proof for the structure of derivatives in probability measures. Available at arXiv:1705.08046.

The Annals of Probability
2021, Vol. 49, No. 2, 997–1032
https://doi.org/10.1214/20-AOP1468
© Institute of Mathematical Statistics, 2021

SMALL GAPS OF CIRCULAR β-ENSEMBLE

BY RENJIE FENG[1] AND DONGYI WEI[2]

[1]*Department of Mathematics, University of Science and Technology of China, renjiefeng.math@gmail.com*
[2]*Department of Mathematics, Peking University, jnwdyi@pku.edu.cn*

In this article, we study the smallest gaps of the circular β-ensemble (CβE) on the unit circle, where β is any positive integer. The main result is that the smallest gaps, after being normalized by $n^{\frac{\beta+2}{\beta+1}}$, will converge in distribution to a Poisson point process with some explicit intensity. And thus one can derive the limiting density of the kth smallest gap, which is proportional to $x^{k(\beta+1)-1}e^{-x^{\beta+1}}$. In particular, the results apply to the classical COE, CUE and CSE in random matrix theory. The essential part of the proof is to derive several identities and inequalities regarding the Selberg integral, which should have their own interest.

REFERENCES

[1] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[2] AOMOTO, K. (1987). On the complex Selberg integral. *Quart*. *J*. *Math*. *Oxford Ser*. (2) **38** 385–399. MR0916224 https://doi.org/10.1093/qmath/38.4.385

[3] AOMOTO, K. (1987). Jacobi polynomials associated with Selberg integrals. *SIAM J*. *Math*. *Anal*. **18** 545–549. MR0876291 https://doi.org/10.1137/0518042

[4] BEN AROUS, G. and BOURGADE, P. (2013). Extreme gaps between eigenvalues of random matrices. *Ann*. *Probab*. **41** 2648–2681. MR3112927 https://doi.org/10.1214/11-AOP710

[5] DYSON, F. J. (1962). The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics. *J*. *Math*. *Phys*. **3** 1199–1215. MR0177643 https://doi.org/10.1063/1.1703863

[6] FENG, R., TIAN, G. and WEI, D. (2019). Small gaps of GOE. *Geom*. *Funct*. *Anal*. **29** 1794–1827. MR4034920 https://doi.org/10.1007/s00039-019-00520-5

[7] FENG, R. and WEI, D. Large gaps of CUE and GUE. Available at arXiv:1807.02149.

[8] FIGALLI, A. and GUIONNET, A. (2016). Universality in several-matrix models via approximate transport maps. *Acta Math*. **217** 81–176. MR3646880 https://doi.org/10.1007/s11511-016-0142-4

[9] FORRESTER, P. J. (2010). *Log-Gases and Random Matrices*. *London Mathematical Society Monographs Series* **34**. Princeton Univ. Press, Princeton, NJ. MR2641363 https://doi.org/10.1515/9781400835416

[10] LANDON, B., LOPATTO, P. and MARCINEK, J. Comparison theorem for some extremal eigenvalue statistics. Available at arXiv:1812.10022.

[11] SELBERG, A. (1944). Bemerkninger om et multipelt integral. *Norsk*. *Mat*. *Tidsskr*. **24** 71–78.

[12] SHI, D. and JIANG, Y. Smallest gaps between eigenvalues of random matrices with complex Ginibre, Wishart and universal unitary ensembles. Available at arXiv:1207.4240.

[13] SMACZYNSKI, M., TKOCZ, T., KUS, M. and ZYCZKOWSKI, K. (2013). Extremal spacings between eigenphases of random unitary matrices and their tensor products. *Phys*. *Rev*. *E* **88** 052902.

[14] SOSHNIKOV, A. (2000). Determinantal random point fields. *Russian Math*. *Surveys* **55** 923–975.

[15] SOSHNIKOV, A. (2005). Statistics of extreme spacing in determinantal random point processes. *Mosc*. *Math*. *J*. **5** 705–719. MR2241818 https://doi.org/10.17323/1609-4514-2005-5-3-705-719

[16] VINSON, J. (2001). Closest spacing of eigenvalues Ph.D. thesis, Princeton University.

MSC2020 subject classifications. 60B20.
Key words and phrases. Smallest gaps, circular β-ensemble, random matrices.

The Annals of Probability
2021, Vol. 49, No. 2, 1033–1066
https://doi.org/10.1214/20-AOP1469
© Institute of Mathematical Statistics, 2021

A GENERALISATION OF THE HONEYCOMB DIMER MODEL TO HIGHER DIMENSIONS

BY PIET LAMMERS

Statistical Laboratory, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge,
p.g.lammers@statslab.cam.ac.uk

Linde, Moore and Nordahl introduced a generalisation of the honeycomb dimer model to higher dimensions. The purpose of this article is to describe a number of structural properties of this generalised model. First, it is shown that the samples of the model are in one-to-one correspondence with the perfect matchings of a hypergraph. This leads to a generalised Kasteleyn theory: the partition function of the model equals the Cayley hyperdeterminant of the adjacency hypermatrix of the hypergraph. Second, we prove an identity which relates the covariance matrix of the random height function directly to the random geometrical structure of the model. This identity is known in the planar case but is new for higher dimensions. It relies on a more explicit formulation of Sheffield's *cluster swap* which is made possible by the structure of the honeycomb dimer model. Finally, we use the special properties of this explicit cluster swap to give a new and simplified proof of strict convexity of the surface tension in this case.

REFERENCES

[1] BAXTER, R. J. (1982). *Exactly Solved Models in Statistical Mechanics*. Academic Press, London. MR0690578

[2] BURTON, R. M. and KEANE, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505. MR0990777

[3] COHN, H., KENYON, R. and PROPP, J. (2001). A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** 297–346. MR1815214 https://doi.org/10.1090/S0894-0347-00-00355-6

[4] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications*. *Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. MR2571413 https://doi.org/10.1007/978-3-642-03311-7

[5] EDWARDS, R. G. and SOKAL, A. D. (1988). Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm. *Phys. Rev. D* (3) **38** 2009–2012. MR0965465 https://doi.org/10.1103/PhysRevD.38.2009

[6] GEORGII, H.-O. (2011). *Gibbs Measures and Phase Transitions*, 2nd ed. *De Gruyter Studies in Mathematics* **9**. de Gruyter, Berlin. MR2807681 https://doi.org/10.1515/9783110250329

[7] GRIMMETT, G. (2018). *Probability on Graphs*: *Random Processes on Graphs and Lattices*, 2nd ed. *Institute of Mathematical Statistics Textbooks* **8**. Cambridge Univ. Press, Cambridge. MR3751350 https://doi.org/10.1017/9781108528986

[8] KASTELEYN, P. W. (1961). The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice. *Physica* **27** 1209–1225.

[9] KASTELEYN, P. W. (1963). Dimer statistics and phase transitions. *J. Math. Phys.* **4** 287–293. MR0153427 https://doi.org/10.1063/1.1703953

[10] KASTELEYN, P. W. (1967). Graph theory and crystal physics. In *Graph Theory and Theoretical Physics* 43–110. Academic Press, London. MR0253689

[11] KENYON, R. (2014). Conformal invariance of loops in the double-dimer model. *Comm. Math. Phys.* **326** 477–497. MR3165463 https://doi.org/10.1007/s00220-013-1881-0

[12] KENYON, R. (2018). Limit shapes and their analytic parameterizations. *Proc. Int. Cong. Math.* **3** 3123–3138.

[13] KENYON, R. and OKOUNKOV, A. (2007). Limit shapes and the complex Burgers equation. *Acta Math.* **199** 263–302. MR2358053 https://doi.org/10.1007/s11511-007-0021-0

MSC2020 subject classifications. Primary 82B41, 60K35; secondary 60F10.

Key words and phrases. Dimer model, random tilings, height functions, variational principle, limit shapes, surface tension strict convexity.

[14] KENYON, R., OKOUNKOV, A. and SHEFFIELD, S. (2006). Dimers and amoebae. *Ann. of Math.* (2) **163** 1019–1056. MR2215138 https://doi.org/10.4007/annals.2006.163.1019

[15] KENYON, R. and PEMANTLE, R. (2016). Double-dimers, the Ising model and the hexahedron recurrence. *J. Combin. Theory Ser. A* **137** 27–63. MR3403514 https://doi.org/10.1016/j.jcta.2015.07.005

[16] KRIEGER, A., MENZ, G. and TASSY, M. (2019). Deducing a variational principle with minimal a priori assumptions. Preprint. Available at arXiv:1910.01042v2.

[17] LAMMERS, P. G. and TASSY, M. (2020). Variational principle for weakly dependent random fields. *J. Stat. Phys.* **179** 846–870. MR4102443 https://doi.org/10.1007/s10955-020-02538-8

[18] LIEB, E. H. (1967). Residual entropy of square ice. *Phys. Rev.* **162** 162–172.

[19] LINDE, J., MOORE, C. and NORDAHL, M. G. (2001). An n-dimensional generalization of the rhombus tiling. In *Discrete Models*: *Combinatorics*, *Computation*, *and Geometry* (*Paris*, 2001). *Discrete Math. Theor. Comput. Sci. Proc.*, *AA* 023–042. Maison Inform. Math. Discrèt. (MIMD), Paris. MR1888761

[20] ONSAGER, L. (1944). Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.* (2) **65** 117–149. MR0010315

[21] SHEFFIELD, S. (2005). Random surfaces. *Astérisque* **304**. MR2251117

[22] SUTHERLAND, B. (1967). Exact solution of a two-dimensional model for hydrogen-bonded crystals. *Phys. Rev. Lett.* **19** 103–104.

[23] SWENDSEN, R. H. and WANG, J.-S. (1987). Nonuniversal critical dynamics in Monte Carlo simulations. *Phys. Rev. Lett.* **58** 86–88.

[24] TEMPERLEY, H. N. V. and FISHER, M. E. (1961). Dimer problem in statistical mechanics—an exact result. *Philos. Mag.* (8) **6** 1061–1063. MR0136398

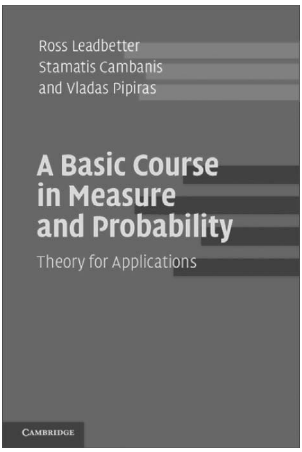
[25] YANG, C. N. (1952). The spontaneous magnetization of a two-dimensional Ising model. *Phys. Rev.* (2) **85** 808–816. MR0051740

[26] YANG, C. P. (1967). Exact solution of a model of two-dimensional ferroelectrics in an arbitrary external electric field. *Phys. Rev. Lett.* **19** 586–588.

The Annals of Probability

Vol. 49 **May 2021** **No. 3**

Articles



A Basic Course in Measure and Probability: *Theory for Applications*

Ross Leadbetter, Stamatis Cambanis, and Vladas Pipiras

Originating from the authors' own graduate course at the University of North Carolina, this material has been thoroughly tried and tested over many years, making the book perfect for a two-term course or for self-study. It provides a concise introduction that covers all of the measure theory and probability most useful for statisticians, including Lebesgue integration, limit theorems in probability, martingales, and some theory of stochastic processes. Readers can test their understanding of the material through the 300 exercises provided.

The book is especially useful for graduate students in statistics and related fields of application (biostatistics, econometrics, finance, meteorology, machine learning, and so on) who want to shore up their mathematical foundation. The authors establish common ground for students of varied interests which will serve as a firm 'take-off point' for them as they specialize in areas that exploit mathematical machinery.

Special price for IMS members

Claim your **40% discount:** use the code IMSSERIES2 at checkout

Hardback US$69 (was $115)
Paperback $30 (was $50)

www.cambridge.org/9781107652521