ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 49, No. 3—May 2021

THE ANNALS OF PROBABILITY Vol. 49, No. 3, pp. 1067–1606 May 2021

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 49, Number 3, May 2021. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Probability
2021, Vol. 49, No. 3, 1067–1096
https://doi.org/10.1214/20-AOP1470

PERSISTENCE OF GAUSSIAN STATIONARY PROCESSES: A SPECTRAL PERSPECTIVE

BY NAOMI FELDHEIM[1], OHAD FELDHEIM[2] AND SHAHAF NITZAN[3]

[1]*Department of Mathematics, Bar-Ilan University, naomi.feldheim@biu.ac.il*
[2]*Einstein Institute for Mathematics, Hebrew University, ohad.feldheim@mail.huji.ac.il*
[3]*School of Mathematics, Georgia Institute of Technology, shahaf.nitzan@math.gatech.edu*

We study the persistence probability of a centered stationary Gaussian process on $\mathbb{Z}$ or $\mathbb{R}$, that is, its probability to remain positive for a long time. We describe the delicate interplay between this probability and the behavior of the spectral measure of the process near zero and infinity.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[2] ANTEZANA, J., BUCKLEY, J., MARZO, J. and OLSEN, J.-F. (2012). Gap probabilities for the cardinal sine. *J. Math. Anal. Appl.* **396** 466–472. MR2961239 https://doi.org/10.1016/j.jmaa.2012.06.022

[3] AURZADA, F. and SIMON, T. (2015). Persistence probabilities and exponents. In *Lévy Matters. V. Functionals of Lévy Processes*. *Lecture Notes in Math.* **2149** 183–224. Springer, Cham. MR3468226 https://doi.org/10.1007/978-3-319-23138-9_3

[4] BERNSTEIN, S. N. (1937). *Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of a Single Real Variable*. State United Scientific and Technical Publishing House, Moscow (in Russian).

[5] BORICHEV, A., SODIN, M. and WEISS, B. (2018). Spectra of stationary processes on $\mathbb{Z}$. In 50 *Years with Hardy Spaces* (A. Baranov, S. Kisliakov and N. Nikolsk eds.). *Operator Theory: Advances and Applications* **261** 141–157. MR3792094

[6] BOURGAIN, J. and TZAFRIRI, L. (1987). Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis. *Israel J. Math.* **57** 137–224. MR0890420 https://doi.org/10.1007/BF02772174

[7] BRAY, A. J., MAJUMDAR, S. N. and SCHEHR, G. (2013). Persistence and first-passage properties in non-equilibrium systems. *Adv. Phys.* **62** 225–361.

[8] DE BOOR, C. (1986). B(asic)-spline basics. MRC Technical Summary Report # 2952. Available at http://ftp.cs.wisc.edu/Approx/bsplbasic.pdf.

[9] DEMBO, A. and MUKHERJEE, S. (2015). No zero-crossings for random polynomials and the heat equation. *Ann. Probab.* **43** 85–118. MR3298469 https://doi.org/10.1214/13-AOP852

[10] DEMBO, A. and MUKHERJEE, S. (2017). Persistence of Gaussian processes: Non-summable correlations. *Probab. Theory Related Fields* **169** 1007–1039. MR3719062 https://doi.org/10.1007/s00440-016-0746-9

[11] EREMENKO, A. and NOVIKOV, D. (2004). Oscillation of Fourier integrals with a spectral gap. *J. Math. Pures Appl.* (9) **83** 313–365. MR2038065 https://doi.org/10.1016/S0021-7824(03)00064-3

[12] FELDHEIM, N., FELDHEIM, O., JAYE, B., NAZAROV, F. and NITZAN, S. (2018). On the probability that a stationary Gaussian process with spectral gap remains non-negative on a long interval. *Int. Math. Res. Not.*, rny248.

[13] FELDHEIM, N. D. and FELDHEIM, O. N. (2015). Long gaps between sign-changes of Gaussian stationary processes. *Int. Math. Res. Not. IMRN* **2015** 3021–3034. MR3373043 https://doi.org/10.1093/imrn/rnu020

[14] GANZBURG, M. I. (2001). Polynomial inequalities on measurable sets and their applications. *Constr. Approx.* **17** 275–306. MR1814358 https://doi.org/10.1007/s003650010020

MSC2020 subject classifications. 60G15, 60G10, 42A38.
Key words and phrases. Gaussian process, stationary process, spectral measure, persistence, gap probability, one-sided barrier, Chebyshev polynomials.

[15] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRAG, B. (2009). *Zeroes of Gaussian Analytic Functions and Determinantal Processes*. *University Lecture Series* **51**. Amer. Math. Soc., Providence.

[16] KAHANE, J.-P. (1993). *Some Random Series of Functions*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **5**. Cambridge Univ. Press, Cambridge. MR0833073

[17] KATZNELSON, Y. (2004). *An Introduction to Harmonic Analysis*, 3rd ed. *Cambridge Mathematical Library*. Cambridge Univ. Press, Cambridge. MR2039503 https://doi.org/10.1017/CBO9781139165372

[18] KRISHNA, M. and KRISHNAPUR, M. (2016). Persistence probabilities in centered, stationary, Gaussian processes in discrete time. *Indian J. Pure Appl. Math.* **47** 183–194. MR3517621 https://doi.org/10.1007/s13226-016-0183-6

[19] LATALA, R. and MATLAK, D. (2017). Royen's proof of the Gaussian correlation inequality. In *Geometric Aspects of Functional Analysis*: *Israel Seminar* (*GAFA*) 2014–2016 (B. Klartag and E. Milman, eds.). *Lecture Notes in Math.* **2169**. Springer, Cham. MR3645127

[20] LI, W. V. and SHAO, Q.-M. (2001). Gaussian processes: Inequalities, small ball probabilities and applications. In *Stochastic Processes*: *Theory and Methods* (C. R. Rao and D. Shanbhag eds.). *Handbook of Statistics* **19** 533–598. Elsevier, New York. MR1861734 https://doi.org/10.1016/S0169-7161(01)19019-X

[21] LIFSHITS, M. A. (1995). *Gaussian Random Functions*. *Mathematics and Its Applications* **322**. Kluwer Academic, Dordrecht. MR1472736 https://doi.org/10.1007/978-94-015-8474-6

[22] LIN, Z. and BAI, Z. (2011). *Probability Inequalities*, 1st ed. Springer, Berlin.

[23] NEWELL, G. F. and ROSENBLATT, M. (1962). Zero crossing probabilities for Gaussian stationary processes. *Ann. Math. Stat.* **33** 1306–1313. MR0141153 https://doi.org/10.1214/aoms/1177704363

[24] ROYEN, T. (2014). A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions. *Far East J. Theor. Stat.* **48** 139–145. MR3289621

[25] SCHATZMANN, M. (2002). *Numerical Analysis*: *A Mathematical Introduction*. Clarendon Press, Oxford.

[26] SCHEHR, G. and MAJUMDAR, S. N. (2008). Real roots of random polynomials and zero crossing properties of diffusion equation. *J. Stat. Phys.* **132** 235–273. MR2415102 https://doi.org/10.1007/s10955-008-9574-3

[27] SLEPIAN, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell Syst. Tech. J.* **41** 463–501. MR0133183 https://doi.org/10.1002/j.1538-7305.1962.tb02419.x

[28] SODIN, M. Private communication.

[29] VERSHYNIN, R. (2000). Coordinate restrictions of linear operators in ℓ_2^n. Preprint. Available at arXiv:math/0011232.

The Annals of Probability
2021, Vol. 49, No. 3, 1097–1128
https://doi.org/10.1214/20-AOP1471

RANDOM WALK ON RANDOM PLANAR MAPS: SPECTRAL DIMENSION, RESISTANCE AND DISPLACEMENT

BY EWAIN GWYNNE[*] AND JASON MILLER[†]

Department of Mathematics, University of Cambridge, [*]eg558@cam.ac.uk; [†]jpmiller@statslab.cam.ac.uk

We study simple random walk on the class of random planar maps which can be encoded by a two-dimensional random walk with i.i.d.. increments or a two-dimensional Brownian motion via a "mating-of-trees" type bijection. This class includes the uniform infinite planar triangulation (UIPT), the infinite-volume limits of random planar maps weighted by the number of spanning trees, bipolar orientations, or Schnyder woods they admit, and the γ-mated-CRT map for $\gamma \in (0,2)$. For each of these maps, we obtain an upper bound for the Green's function on the diagonal, an upper bound for the effective resistance to the boundary of a metric ball, an upper bound for the return probability of the random walk to its starting point after n steps, and a lower bound for the graph-distance displacement of the random walk, all of which are sharp up to polylogarithmic factors.

When combined with work of Lee (2017), our bound for the return probability shows that the spectral dimension of each of these random planar maps is a.s. equal to 2, that is, the (quenched) probability that the simple random walk returns to its starting point after $2n$ steps is $n^{-1+o_n(1)}$. Our results also show that the amount of time that it takes a random walk to exit a metric ball is at least its volume (up to a polylogarithmic factor). In the special case of the UIPT, this implies that random walk typically travels at least $n^{1/4-o_n(1)}$ units of graph distance in n units of time. The matching upper bound for the displacement is proven by Gwynne and Hutchcroft (*Probab. Theory Related Fields* **178** (2020) 567–611). These two works together resolve a conjecture of Benjamini and Curien (*Geom. Funct. Anal.* **23** (2013) 501–531) in the UIPT case.

Our proofs are based on estimates for the mated-CRT map (which come from its relationship to SLE-decorated Liouville quantum gravity) and a strong coupling of the mated-CRT map with the other random planar map models.

REFERENCES

[1] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326

[2] ALDOUS, D. (1991). The continuum random tree. II. An overview. In *Stochastic Analysis* (*Durham*, 1990). *London Mathematical Society Lecture Note Series* **167** 23–70. Cambridge Univ. Press, Cambridge. MR1166406 https://doi.org/10.1017/CBO9780511662980.003

[3] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226

[4] AMBJØRN, J., ANAGNOSTOPOULOS, K. N., JENSEN, L., ICHIHARA, T. and WATABIKI, Y. (1998). Quantum geometry and diffusion. *J. High Energy Phys.* **11** Paper 22, 16. MR1667363 https://doi.org/10.1088/1126-6708/1998/11/022

[5] AMBJØRN, J., NIELSEN, J. L., ROLF, J., BOULATOV, D. and WATABIKI, Y. (1998). The spectral dimension of 2D quantum gravity. *J. High Energy Phys.* **2** Paper 10, 8. MR1613966 https://doi.org/10.1088/1126-6708/1998/02/010

[6] ANDRES, S. and KAJINO, N. (2016). Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probab. Theory Related Fields* **166** 713–752. MR3568038 https://doi.org/10.1007/s00440-015-0670-4

MSC2020 subject classifications. 60G50, 60J67, 60G60.

Key words and phrases. Random planar maps, uniform infinite planar triangulation, spectral dimension, random walk, return probability, Liouville quantum gravity, Schramm–Loewner evolution.

[7] ANGEL, O. (2003). Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.* **13** 935–974. MR2024412 https://doi.org/10.1007/s00039-003-0436-5

[8] ANGEL, O., BARLOW, M. T., GUREL-GUREVICH, O. and NACHMIAS, A. (2016). Boundaries of planar graphs, via circle packings. *Ann. Probab.* **44** 1956–1984. MR3502598 https://doi.org/10.1214/15-AOP1014

[9] ANGEL, O., HUTCHCROFT, T., NACHMIAS, A. and RAY, G. (2016). Unimodular hyperbolic triangulations: Circle packing and random walk. *Invent. Math.* **206** 229–268. MR3556528 https://doi.org/10.1007/s00222-016-0653-9

[10] ANGEL, O. and SCHRAMM, O. (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241** 191–213. MR2013797 https://doi.org/10.1007/978-1-4419-9675-6_16

[11] BENJAMINI, I. and CURIEN, N. (2013). Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity via pioneer points. *Geom. Funct. Anal.* **23** 501–531. MR3053754 https://doi.org/10.1007/s00039-013-0212-0

[12] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** no. 23, 13. MR1873300 https://doi.org/10.1214/EJP.v6-96

[13] BERESTYCKI, N. and GWYNNE, E. (2020). Random walks on mated-CRT planar maps and Liouville Brownian motion. ArXiv e-prints.

[14] BERNARDI, O. (2007). Bijective counting of Kreweras walks and loopless triangulations. *J. Combin. Theory Ser. A* **114** 931–956. MR2333142 https://doi.org/10.1016/j.jcta.2006.09.009

[15] BERNARDI, O. (2007). Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.* **14** Research Paper 9, 36. MR2285813

[16] BERNARDI, O., HOLDEN, N. and SUN, X. (2018). Percolation on triangulations: A bijective path to Liouville quantum gravity. ArXiv e-prints.

[17] BISKUP, M., DING, J. and GOSWAMI, S. (2020). Return probability and recurrence for the random walk driven by two-dimensional Gaussian free field. *Comm. Math. Phys.* **373** 45–106. MR4050092 https://doi.org/10.1007/s00220-019-03589-z

[18] CARMESIN, J. and GEORGAKOPOULOS, A. (2020). Every planar graph with the Liouville property is amenable. *Random Structures Algorithms* **57** 706–729. MR4144081 https://doi.org/10.1002/rsa.20936

[19] CHEN, L. (2017). Basic properties of the infinite critical-FK random map. *Ann. Inst. Henri Poincaré D* **4** 245–271. MR3713017 https://doi.org/10.4171/AIHPD/40

[20] CURIEN, N., HUTCHCROFT, T. and NACHMIAS, A. (2017). Geometric and spectral properties of causal maps. *Geom. Funct. Anal.* To appear.

[21] CURIEN, N. and MARZOUK, C. (2019). Infinite stable Boltzmann planar maps are subdiffusive. ArXiv e-prints.

[22] DING, J. and GOSWAMI, S. (2019). Upper bounds on Liouville first-passage percolation and Watabiki's prediction. *Comm. Pure Appl. Math.* **72** 2331–2384. MR4011862 https://doi.org/10.1002/cpa.21846

[23] DING, J. and GWYNNE, E. (2020). The fractal dimension of Liouville quantum gravity: Universality, monotonicity, and bounds. *Comm. Math. Phys.* **374** 1877–1934. MR4076090 https://doi.org/10.1007/s00220-019-03487-4

[24] DING, J., ZEITOUNI, O. and ZHANG, F. (2019). Heat kernel for Liouville Brownian motion and Liouville graph distance. *Comm. Math. Phys.* **371** 561–618. MR4019914 https://doi.org/10.1007/s00220-019-03467-8

[25] DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2014). Liouville quantum gravity as a mating of trees. *Astérisque*. To appear.

[26] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 https://doi.org/10.1007/s00222-010-0308-1

[27] GEORGAKOPOULOS, A. (2016). The boundary of a square tiling of a graph coincides with the Poisson boundary. *Invent. Math.* **203** 773–821. MR3461366 https://doi.org/10.1007/s00222-015-0601-0

[28] GILL, J. T. and ROHDE, S. (2013). On the Riemann surface type of random planar maps. *Rev. Mat. Iberoam.* **29** 1071–1090. MR3090146 https://doi.org/10.4171/RMI/749

[29] GUREL-GUREVICH, O. and NACHMIAS, A. (2013). Recurrence of planar graph limits. *Ann. of Math.* (2) **177** 761–781. MR3010812 https://doi.org/10.4007/annals.2013.177.2.10

[30] GWYNNE, E., HOLDEN, N. and MILLER, J. (2020). An almost sure KPZ relation for SLE and Brownian motion. *Ann. Probab.* **48** 527–573. MR4089487 https://doi.org/10.1214/19-AOP1385

[31] GWYNNE, E., HOLDEN, N. and SUN, X. (2019). A distance exponent for Liouville quantum gravity. *Probab. Theory Related Fields* **173** 931–997. MR3936149 https://doi.org/10.1007/s00440-018-0846-9

[32] GWYNNE, E., HOLDEN, N. and SUN, X. (2019). Mating of trees for random planar maps and Liouville quantum gravity: A survey. ArXiv e-prints.

[33] GWYNNE, E., HOLDEN, N. and SUN, X. (2020). A mating-of-trees approach for graph distances in random planar maps. *Probab. Theory Related Fields* **177** 1043–1102. MR4126936 https://doi.org/10.1007/s00440-020-00969-8

[34] GWYNNE, E. and HUTCHCROFT, T. (2020). Anomalous diffusion of random walk on random planar maps. *Probab. Theory Related Fields* **178** 567–611. MR4146545 https://doi.org/10.1007/s00440-020-00986-7

[35] GWYNNE, E., KASSEL, A., MILLER, J. and WILSON, D. B. (2018). Active spanning trees with bending energy on planar maps and SLE-decorated Liouville quantum gravity for $\kappa > 8$. *Comm. Math. Phys.* **358** 1065–1115. MR3778352 https://doi.org/10.1007/s00220-018-3104-1

[36] GWYNNE, E. and MILLER, J. (2019). Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Invent. Math.* To appear.

[37] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2017). The Tutte embedding of the mated-CRT map converges to Liouville quantum gravity. ArXiv e-prints.

[38] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2018). An invariance principle for ergodic scale-free random environments. ArXiv e-prints.

[39] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2019). Harmonic functions on mated-CRT maps. *Electron. J. Probab.* **24** Paper No. 58, 55. MR3978208 https://doi.org/10.1214/19-EJP325

[40] GWYNNE, E. and PFEFFER, J. (2019). KPZ formulas for the Liouville quantum gravity metric. *Trans. Amer. Math. Soc.* To appear.

[41] GWYNNE, E. and PFEFFER, J. (2019). Bounds for distances and geodesic dimension in Liouville first passage percolation. *Electron. Commun. Probab.* **24** Paper No. 56, 12. MR4003130 https://doi.org/10.1214/19-ecp248

[42] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[43] KENYON, R., MILLER, J., SHEFFIELD, S. and WILSON, D. B. (2019). Bipolar orientations on planar maps and SLE_{12}. *Ann. Probab.* **47** 1240–1269. MR3945746 https://doi.org/10.1214/18-AOP1282

[44] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 33–58. MR0402883 https://doi.org/10.1007/BF00532688

[45] KOZMA, G. and NACHMIAS, A. (2009). The Alexander–Orbach conjecture holds in high dimensions. *Invent. Math.* **178** 635–654. MR2551766 https://doi.org/10.1007/s00222-009-0208-4

[46] LEE, J. R. (2017). Conformal growth rates and spectral geometry on distributional limits of graphs. ArXiv e-prints.

[47] LEE, J. R. (2018). Discrete uniformizing metrics on distributional limits of sphere packings. *Geom. Funct. Anal.* **28** 1091–1130. MR3820440 https://doi.org/10.1007/s00039-018-0442-2

[48] LEE, J. R. (2020). Relations between scaling exponents in unimodular random graphs. ArXiv e-prints.

[49] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. With a chapter by James G. Propp and David B. Wilson. MR2466937 https://doi.org/10.1090/mbk/058

[50] LI, Y., SUN, X. and WATSON, S. S. (2017). Schnyder woods, SLE(16), and Liouville quantum gravity. ArXiv E-prints.

[51] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. *Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[52] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[53] MILLER, J. and SHEFFIELD, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2

[54] MULLIN, R. C. (1967). On the enumeration of tree-rooted maps. *Canad. J. Math.* **19** 174–183. MR0205882 https://doi.org/10.4153/CJM-1967-010-x

[55] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218

[56] RHODES, R. and VARGAS, V. (2014). Spectral dimension of Liouville quantum gravity. *Ann. Henri Poincaré* **15** 2281–2298. MR3272822 https://doi.org/10.1007/s00023-013-0308-y

[57] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084 https://doi.org/10.1007/BF02803524

[58] SCHRAMM, O. and SHEFFIELD, S. (2013). A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields* **157** 47–80. MR3101840 https://doi.org/10.1007/s00440-012-0449-9

[59] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 https://doi.org/10.1007/s00440-006-0050-1

[60] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[61] SHEFFIELD, S. (2016). Quantum gravity and inventory accumulation. *Ann. Probab.* **44** 3804–3848. MR3572324 https://doi.org/10.1214/15-AOP1061

[62] STEPHENSON, K. (2003). Circle packing: A mathematical tale. *Notices Amer. Math. Soc.* **50** 1376–1388. MR2011604

[63] TUTTE, W. T. (1968). On the enumeration of planar maps. *Bull. Amer. Math. Soc.* **74** 64–74. MR0218276 https://doi.org/10.1090/S0002-9904-1968-11877-4

[64] ZAITSEV, A. YU. (1998). Multidimensional version of the results of Komlós, Major and Tusnády for vectors with finite exponential moments. *ESAIM Probab. Stat.* **2** 41–108. MR1616527 https://doi.org/10.1051/ps:1998103

The Annals of Probability
2021, Vol. 49, No. 3, 1129–1163
https://doi.org/10.1214/20-AOP1473

RATES OF CONVERGENCE TO EQUILIBRIUM FOR POTLATCH AND SMOOTHING PROCESSES

BY SAYAN BANERJEE[1] AND KRZYSZTOF BURDZY[2]

[1]*Department of Statistics and Operations Research, University of North Carolina, sayan@email.unc.edu*
[2]*Department of Mathematics, University of Washington, burdzy@uw.edu*

We analyze the local and global smoothing rates of the smoothing process and obtain convergence rates to stationarity for the dual process known as the potlatch process. For general finite graphs we connect the smoothing and convergence rates to the spectral gap of the associated Markov chain. We perform a more detailed analysis of these processes on the torus. Polynomial corrections to the smoothing rates are obtained. They show that local smoothing happens faster than global smoothing. These polynomial rates translate to rates of convergence to stationarity in L^2-Wasserstein distance for the potlatch process on $\mathbb{Z}^d$.

REFERENCES

[1] ACEMOĞLU, D., COMO, G., FAGNANI, F. and OZDAGLAR, A. (2013). Opinion fluctuations and disagreement in social networks. *Math*. *Oper*. *Res*. **38** 1–27. MR3029476 https://doi.org/10.1287/moor.1120.0570

[2] ALDOUS, D. and LANOUE, D. (2012). A lecture on the averaging process. *Probab*. *Surv*. **9** 90–102. MR2908618 https://doi.org/10.1214/11-PS184

[3] BALÁZS, M., RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2006). The random average process and random walk in a space-time random environment in one dimension. *Comm*. *Math*. *Phys*. **266** 499–545. MR2238887 https://doi.org/10.1007/s00220-006-0036-y

[4] BILLEY, S., BURDZY, K., PAL, S. and SAGAN, B. E. (2015). On meteors, earthworms and WIMPs. *Ann*. *Appl*. *Probab*. **25** 1729–1779. MR3348994 https://doi.org/10.1214/14-AAP1035

[5] BURDZY, K. (2015). Meteor process on $\mathbb{Z}^d$. *Probab*. *Theory Related Fields* **163** 667–711. MR3418753 https://doi.org/10.1007/s00440-014-0602-8

[6] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J*. *Amer*. *Math*. *Soc*. **23** 831–851. MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4

[7] COX, J. T., KLENKE, A. and PERKINS, E. A. (2000). Convergence to equilibrium and linear systems duality. In *Stochastic Models* (*Ottawa*, *ON*, 1998). *CMS Conf*. *Proc*. **26** 41–66. Amer. Math. Soc., Providence, RI. MR1765002

[8] DURRETT, R. and LIGGETT, T. M. (1983). Fixed points of the smoothing transformation. *Z*. *Wahrsch*. *Verw*. *Gebiete* **64** 275–301. MR0716487 https://doi.org/10.1007/BF00532962

[9] FELLER, W. (1941). On the integral equation of renewal theory. *Ann*. *Math*. *Stat*. **12** 243–267. MR0005419 https://doi.org/10.1214/aoms/1177731708

[10] FERRARI, P. A. and FONTES, L. R. G. (1998). Fluctuations of a surface submitted to a random average process. *Electron*. *J*. *Probab*. **3** no. 6, 34. MR1624854 https://doi.org/10.1214/EJP.v3-28

[11] FERRARI, P. A., GALVES, A. and LANDIM, C. (2000). Rate of convergence to equilibrium of symmetric simple exclusion processes. *Markov Process*. *Related Fields* **6** 73–88. MR1758983

[12] FORSSTRÖM, M. P. and JONASSON, J. (2017). The spectrum and convergence rates of exclusion and interchange processes on the complete graph. *J*. *Theoret*. *Probab*. **30** 639–654. MR3647074 https://doi.org/10.1007/s10959-015-0660-6

[13] FRASCA, P., RAVAZZI, C., TEMPO, R. and ISHII, H. (2013). Gossips and prejudices: Ergodic randomized dynamics in social networks. *IFAC Proc*. *Vol*. **46** 212–219.

[14] GHADERI, J. and SRIKANT, R. (2014). Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate. *Automatica J*. *IFAC* **50** 3209–3215. MR3284157 https://doi.org/10.1016/j.automatica.2014.10.034

MSC2020 subject classifications. Primary 60K35, 82C22; secondary37A25, 60F25.
Key words and phrases. Potlatch process, meteor process, smoothing process, Wasserstein distance, convergence rate.

[15] HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378. MR0488377 https://doi.org/10.1214/aop/1176995523

[16] HOLLEY, R. and LIGGETT, T. M. (1981). Generalized potlatch and smoothing processes. *Z. Wahrsch. Verw. Gebiete* **55** 165–195. MR0608015 https://doi.org/10.1007/BF00535158

[17] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR0959133 https://doi.org/10.1007/978-3-662-02514-7

[18] JANVRESSE, E., LANDIM, C., QUASTEL, J. and YAU, H. T. (1999). Relaxation to equilibrium of conservative dynamics. I. Zero-range processes. *Ann. Probab.* **27** 325–360. MR1681098 https://doi.org/10.1214/aop/1022677265

[19] LAWLER, G. F. (2010). *Random Walk and the Heat Equation*. *Student Mathematical Library* **55**. Amer. Math. Soc., Providence, RI. MR2732325 https://doi.org/10.1090/stml/055

[20] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. With a chapter by James G. Propp and David B. Wilson. MR2466937 https://doi.org/10.1090/mbk/058

[21] LIGGETT, T. M. (1985). *Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[22] LIGGETT, T. M. and SPITZER, F. (1981). Ergodic theorems for coupled random walks and other systems with locally interacting components. *Z. Wahrsch. Verw. Gebiete* **56** 443–468. MR0621659 https://doi.org/10.1007/BF00531427

[23] NAGAHATA, Y. (2012). Lower bound estimate of the spectral gap for simple exclusion process with degenerate rates. *Electron. J. Probab.* **17** no. 92, 19. MR2994840 https://doi.org/10.1214/EJP.v17-1916

[24] NAGAHATA, Y. and YOSHIDA, N. (2009). Central limit theorem for a class of linear systems. *Electron. J. Probab.* **14** 960–977. MR2506122 https://doi.org/10.1214/EJP.v14-644

[25] NAGAHATA, Y. and YOSHIDA, N. (2010). Localization for a class of linear systems. *Electron. J. Probab.* **15** 636–653. MR2650776 https://doi.org/10.1214/EJP.v15-757

[26] NAGAHATA, Y. and YOSHIDA, N. (2010). A note on the diffusive scaling limit for a class of linear systems. *Electron. Commun. Probab.* **15** 68–78. MR2595684 https://doi.org/10.1214/ECP.v15-1530

[27] RAJESH, R. and MAJUMDAR, S. N. (2001). Exact tagged particle correlations in the random average process. *Phys. Rev. E* **64** 036103.

[28] ROUSSIGNOL, M. (1980). Un processus de saut sur **R** à une infinité de particules. *Ann. Inst. Henri Poincaré B, Calc. Probab. Stat.* **16** 101–108. MR0585902

[29] SALOFF-COSTE, L. (1997). Lectures on finite Markov chains. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1996). *Lecture Notes in Math.* **1665** 301–413. Springer, Berlin. MR1490046 https://doi.org/10.1007/BFb0092621

[30] SHAH, D. (2009). Gossip algorithms. *Found. Trends Netw.* **3** 1–125.

[31] SHI, G., PROUTIERE, A., JOHANSSON, M., BARAS, J. S. and JOHANSSON, K. H. (2016). The evolution of beliefs over signed social networks. *Oper. Res.* **64** 585–604. MR3515199 https://doi.org/10.1287/opre.2015.1448

[32] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[33] YILDIZ, E., ACEMOGLU, D., OZDAGLAR, A. E., SABERI, A. and SCAGLIONE, A. (2011). Discrete opinion dynamics with stubborn agents. Available at SSRN 1744113.

The Annals of Probability
2021, Vol. 49, No. 3, 1164–1204
https://doi.org/10.1214/20-AOP1474

ON THE DERIVATIVE MARTINGALE IN A BRANCHING RANDOM WALK

BY DARIUSZ BURACZEWSKI[1], ALEXANDER IKSANOV[2] AND BASTIEN MALLEIN[3]

[1]*Mathematical Institute, University of Wrocław, dbura@math.uni.wroc.pl*
[2]*Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, iksan@univ.kiev.ua*
[3]*Laboratoire Analyse, Géométrie et Applications UMR 7539, Université Sorbonne Paris Nord, mallein@math.univ-paris13.fr*

We work under the Aïdékon–Chen conditions which ensure that the derivative martingale in a supercritical branching random walk on the line converges almost surely to a nondegenerate nonnegative random variable that we denote by Z. It is shown that $\mathbb{E}Z\,\mathbb{1}_{\{Z\le x\}} = \log x + o(\log x)$ as $x \to \infty$. Also, we provide necessary and sufficient conditions under which $\mathbb{E}Z\,\mathbb{1}_{\{Z\le x\}} = \log x + \text{const} + o(1)$ as $x \to \infty$. This more precise asymptotics is a key tool for proving distributional limit theorems which quantify the rate of convergence of the derivative martingale to its limit Z. The methodological novelty of the present paper is a three terms representation of a subharmonic function of, at most, linear growth for a killed centered random walk of finite variance. This yields the aforementioned asymptotics and should also be applicable to other models.

REFERENCES

[1] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680 https://doi.org/10.1214/12-AOP750

[2] AÏDÉKON, E. and SHI, Z. (2014). The Seneta–Heyde scaling for the branching random walk. *Ann. Probab.* **42** 959–993. MR3189063 https://doi.org/10.1214/12-AOP809

[3] ALSMEYER, G. and MALLEIN, B. (2019). A simple method to find all solutions to the functional equation of the smoothing transform. Preprint. Available at arXiv:1907.04111.

[4] ALSMEYER, G. and MEINERS, M. (2013). Fixed points of the smoothing transform: Two-sided solutions. *Probab. Theory Related Fields* **155** 165–199. MR3010396 https://doi.org/10.1007/s00440-011-0395-y

[5] BERTOIN, J. and DONEY, R. A. (1994). On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22** 2152–2167. MR1331218

[6] BIGGINS, J. D. and KYPRIANOU, A. E. (1997). Seneta–Heyde norming in the branching random walk. *Ann. Probab.* **25** 337–360. MR1428512 https://doi.org/10.1214/aop/1024404291

[7] BIGGINS, J. D. and KYPRIANOU, A. E. (2004). Measure change in multitype branching. *Adv. in Appl. Probab.* **36** 544–581. MR2058149 https://doi.org/10.1239/aap/1086957585

[8] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1989). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR1015093

[9] BROFFERIO, S., BURACZEWSKI, D. and DAMEK, E. (2012). On the invariant measure of the random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 377–395. MR2954260 https://doi.org/10.1214/10-AIHP406

[10] BURACZEWSKI, D. (2007). On invariant measures of stochastic recursions in a critical case. *Ann. Appl. Probab.* **17** 1245–1272. MR2344306 https://doi.org/10.1214/105051607000000140

[11] BURACZEWSKI, D. (2009). On tails of fixed points of the smoothing transform in the boundary case. *Stochastic Process. Appl.* **119** 3955–3961. MR2552312 https://doi.org/10.1016/j.spa.2009.09.005

[12] CHEN, X. (2015). A necessary and sufficient condition for the nontrivial limit of the derivative martingale in a branching random walk. *Adv. in Appl. Probab.* **47** 741–760. MR3406606 https://doi.org/10.1239/aap/1444308880

[13] CORLESS, R. M., GONNET, G. H., HARE, D. E. G., JEFFREY, D. J. and KNUTH, D. E. (1996). On the Lambert W function. *Adv. Comput. Math.* **5** 329–359. MR1414285 https://doi.org/10.1007/BF02124750

MSC2020 subject classifications. Primary 60G50, 60J80; secondary 60F05, 60G42.
Key words and phrases. Branching random walk, derivative martingale, killed random walk, rate of convergence, subharmonic function, tail behavior.

[14] DENISOV, D. and WACHTEL, V. (2015). Random walks in cones. *Ann. Probab.* **43** 992–1044. MR3342657 https://doi.org/10.1214/13-AOP867

[15] DENISOV, D. and WACHTEL, V. (2016). Universality of local times of killed and reflected random walks. *Electron. Commun. Probab.* **21** Paper No. 1, 11. MR3485370 https://doi.org/10.1214/15-ECP3995

[16] DONEY, R. A. (1980). Moments of ladder heights in random walks. *J. Appl. Probab.* **17** 248–252. MR0557453 https://doi.org/10.2307/3212942

[17] DONEY, R. A. (1983). A note on conditioned random walk. *J. Appl. Probab.* **20** 409–412. MR0698545 https://doi.org/10.2307/3213815

[18] DONEY, R. A. (1998). The Martin boundary and ratio limit theorems for killed random walks. *J. Lond. Math. Soc.* (2) **58** 761–768. MR1678162 https://doi.org/10.1112/S0024610798006826

[19] DURRETT, R. and LIGGETT, T. M. (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete* **64** 275–301. MR0716487 https://doi.org/10.1007/BF00532962

[20] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, *Vol. II*, 2nd ed. Wiley, New York. MR0270403

[21] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley Co., Reading. MR0233400

[22] GRADSHTEYN, I. S. and RYZHIK, I. M. (2007). *Table of Integrals*, *Series*, *and Products*, 7th ed. Academic Press, San Diego.

[23] IKSANOV, A. (2016). *Renewal Theory for Perturbed Random Walks and Similar Processes*. *Probability and Its Applications*. Birkhäuser/Springer, Cham. MR3585464 https://doi.org/10.1007/978-3-319-49113-4

[24] IKSANOV, A., KOLESKO, K. and MEINERS, M. (2020). Fluctuations of Biggins' martingales at complex parameters. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2445–2479. MR4164844 https://doi.org/10.1214/20-AIHP1046

[25] IKSANOV, A. and MEINERS, M. (2015). Fixed points of multivariate smoothing transforms with scalar weights. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 69–114. MR3340373

[26] KAHANE, J.-P. and PEYRIÈRE, J. (1976). Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.* **22** 131–145. MR0431355 https://doi.org/10.1016/0001-8708(76)90151-1

[27] LIU, Q. (1998). Fixed points of a generalized smoothing transformation and applications to the branching random walk. *Adv. in Appl. Probab.* **30** 85–112. MR1618888 https://doi.org/10.1239/aap/1035227993

[28] LYONS, R. (1997). A simple path to Biggins' martingale convergence for branching random walk. In *Classical and Modern Branching Processes* (*Minneapolis*, *MN*, 1994). *IMA Vol. Math. Appl.* **84** 217–221. Springer, New York. MR1601749 https://doi.org/10.1007/978-1-4612-1862-3_17

[29] MADAULE, T. (2016). The tail distribution of the derivative martingale and the global minimum of the branching random walk. Preprint. Available at arXiv:1606.03211.

[30] MADAULE, T. (2017). Convergence in law for the branching random walk seen from its tip. *J. Theoret. Probab.* **30** 27–63. MR3615081 https://doi.org/10.1007/s10959-015-0636-6

[31] MAILLARD, P. (2012). Branching Brownian motion with selection. Ph.D. thesis. Available at arXiv:1210.3500.

[32] MAILLARD, P. and PAIN, M. (2019). 1-stable fluctuations in branching Brownian motion at critical temperature I: The derivative martingale. *Ann. Probab.* **47** 2953–3002. MR4021242 https://doi.org/10.1214/18-AOP1329

[33] MALLEIN, B. (2016). Asymptotic of the maximal displacement in a branching random walk. *Grad. J. Math.* **1** 92–104. MR3850767

[34] MALLEIN, B. (2018). Genealogy of the extremal process of the branching random walk. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 1065–1087. MR3852245 https://doi.org/10.30757/alea.v15-39

[35] MEINERS, M. and MENTEMEIER, S. (2017). Solutions to complex smoothing equations. *Probab. Theory Related Fields* **168** 199–268. MR3651052 https://doi.org/10.1007/s00440-016-0709-1

[36] PEYRIÈRE, J. (1974). Turbulence et dimension de Hausdorff. *C. R. Acad. Sci. Paris Sér. A* **278** 567–569. MR0431354

[37] PORT, S. C. and STONE, C. J. (1969). Potential theory of random walks on Abelian groups. *Acta Math.* **122** 19–114. MR0261706 https://doi.org/10.1007/BF02392007

[38] REN, Y.-X., SONG, R., SUN, Z. and ZHAO, J. (2019). Stable central limit theorems for super Ornstein–Uhlenbeck processes. *Electron. J. Probab.* **24** Paper No. 141, 42. MR4049077 https://doi.org/10.1214/19-ejp396

[39] RESNICK, S. (2005). *Adventures in Stochastic Processes*, 4th ed. Birkhäuser, Inc., Boston, MA. MR1181423

[40] SGIBNEV, M. S. (1982). Renewal theorem in the case of an infinite variance. *Sib. Math. J.* **22** 787–796.

[41] SHI, Z. (2015). *Branching Random Walks*. *Lecture Notes in Math.* **2151**. Springer, Cham. MR3444654 https://doi.org/10.1007/978-3-319-25372-5

[42] SPITZER, F. (2001). *Principles of Random Walk*, 2nd ed. *Graduate Texts in Mathematics* **34**. Springer, New York. MR0388547

[43] TANAKA, H. (1989). Time reversal of random walks in one-dimension. *Tokyo J. Math.* **12** 159–174. MR1001739 https://doi.org/10.3836/tjm/1270133555

[44] TSIRELSON, B. (2013). From uniform renewal theorem to uniform large and moderate deviations for renewal-reward processes. *Electron. Commun. Probab.* **18** no. 52, 13. MR3078015 https://doi.org/10.1214/ECP.v18-2719

The Annals of Probability
2021, Vol. 49, No. 3, 1205–1236
https://doi.org/10.1214/20-AOP1475

DIFFUSION APPROXIMATION FOR FULLY COUPLED STOCHASTIC DIFFERENTIAL EQUATIONS

BY MICHAEL RÖCKNER[1] AND LONGJIE XIE[2]

[1]*Fakultät für Mathematik, Universität Bielefeld, roeckner@math.uni-bielefeld.de*

[2]*School of Mathematics and Statistics and Research Institute of Mathematical Science, Jiangsu Normal University, longjiexie@jsnu.edu.cn*

We consider a Poisson equation in $\mathbb{R}^d$ for the elliptic operator corresponding to an ergodic diffusion process. Optimal regularity and smoothness with respect to the parameter are obtained under mild conditions on the coefficients. The result is then applied to establish a general diffusion approximation for fully coupled multitime scales stochastic differential equations with only Hölder continuous coefficients. Four different averaged equations as well as rates of convergence are obtained. Moreover, the convergence is shown to rely only on the regularities of the coefficients with respect to the slow variable and does not depend on their regularities with respect to the fast component.

REFERENCES

[1] ABDULLE, A., PAVLIOTIS, G. A. and VAES, U. (2017). Spectral methods for multiscale stochastic differential equations. *SIAM/ASA J. Uncertain. Quantificat.* **5** 720–761. MR3683698 https://doi.org/10.1137/16M1094117

[2] ABOURASHCHI, N. and VERETENNIKOV, A. YU. (2010). On stochastic averaging and mixing. *Theory Stoch. Process.* **16** 111–129. MR2779833

[3] BAHVALOV, N. S. (1975). Averaging of partial differential equations with rapidly oscillating coefficients. *Sov. Math., Dokl.* **16** 351–355. MR0380075

[4] BAKHTIN, V. and KIFER, Y. (2004). Diffusion approximation for slow motion in fully coupled averaging. *Probab. Theory Related Fields* **129** 157–181. MR2063374 https://doi.org/10.1007/s00440-003-0326-7

[5] BALL, K., KURTZ, T. G., POPOVIC, L. and REMPALA, G. (2006). Asymptotic analysis of multiscale approximations to reaction networks. *Ann. Appl. Probab.* **16** 1925–1961. MR2288709 https://doi.org/10.1214/105051606000000420

[6] BRÉHIER, C.-E. (2013). Analysis of an HMM time-discretization scheme for a system of stochastic PDEs. *SIAM J. Numer. Anal.* **51** 1185–1210. MR3040961 https://doi.org/10.1137/110853078

[7] BRÉHIER, C.-E. (2020). Orders of convergence in the averaging principle for SPDEs: The case of a stochastically forced slow component. *Stochastic Process. Appl.* **130** 3325–3368. MR4092407 https://doi.org/10.1016/j.spa.2019.09.015

[8] BRÉHIER, C.-E. and KOPEC, M. (2017). Approximation of the invariant law of SPDEs: Error analysis using a Poisson equation for a full-discretization scheme. *IMA J. Numer. Anal.* **37** 1375–1410. MR3671499 https://doi.org/10.1093/imanum/drw030

[9] CATTIAUX, P., CHAFAÏ, D. and GUILLIN, A. (2012). Central limit theorems for additive functionals of ergodic Markov diffusions processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 337–382. MR3069369

[10] CERRAI, S. (2009). A Khasminskii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.* **19** 899–948. MR2537194 https://doi.org/10.1214/08-AAP560

[11] CERRAI, S. and FREIDLIN, M. (2009). Averaging principle for a class of stochastic reaction-diffusion equations. *Probab. Theory Related Fields* **144** 137–177. MR2480788 https://doi.org/10.1007/s00440-008-0144-z

[12] CERRAI, S. and LUNARDI, A. (2017). Averaging principle for nonautonomous slow-fast systems of stochastic reaction-diffusion equations: The almost periodic case. *SIAM J. Math. Anal.* **49** 2843–2884. MR3679916 https://doi.org/10.1137/16M1063307

MSC2020 subject classifications. Primary 60H10, 60J60; secondary 35B30.

Key words and phrases. Poisson equation, multiscale system, averaging principle, diffusion approximation, homogenization.

[13] CHEVYREV, I., FRIZ, P. K., KOREPANOV, A. and MELBOURNE, I. (2020). Superdiffusive limits for deterministic fast-slow dynamical systems. *Probab. Theory Related Fields* **178** 735–770. MR4168387 https://doi.org/10.1007/s00440-020-00988-5

[14] DUPUIS, P. and SPILIOPOULOS, K. (2012). Large deviations for multiscale diffusion via weak convergence methods. *Stochastic Process. Appl.* **122** 1947–1987. MR2914778 https://doi.org/10.1016/j.spa.2011.12.006

[15] E, W., LIU, D. and VANDEN-EIJNDEN, E. (2005). Analysis of multiscale methods for stochastic differential equations. *Comm. Pure Appl. Math.* **58** 1544–1585. MR2165382 https://doi.org/10.1002/cpa.20088

[16] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[17] FREIDLIN, M. (1985). *Functional Integration and Partial Differential Equations. Annals of Mathematics Studies* **109**. Princeton Univ. Press, Princeton, NJ. MR0833742 https://doi.org/10.1515/9781400881598

[18] FREIDLIN, M. I. and WENTZELL, A. D. (2012). *Random Perturbations of Dynamical Systems*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. Springer, Heidelberg. Translated from the 1979 Russian original by Joseph Szücs. MR2953753 https://doi.org/10.1007/978-3-642-25847-3

[19] GAO, H. and DUAN, J. (2003). Dynamics of quasi-geostrophic fluid motion with rapidly oscillating Coriolis force. *Nonlinear Anal. Real World Appl.* **4** 127–138. MR1932145 https://doi.org/10.1016/S1468-1218(02)00018-4

[20] GIVON, D. (2007). Strong convergence rate for two-time-scale jump-diffusion stochastic differential systems. *Multiscale Model. Simul.* **6** 577–594. MR2338495 https://doi.org/10.1137/060673345

[21] GLYNN, P. W. and MEYN, S. P. (1996). A Liapounov bound for solutions of the Poisson equation. *Ann. Probab.* **24** 916–931. MR1404536 https://doi.org/10.1214/aop/1039639370

[22] GONZALES-GARGATE, I. I. and RUFFINO, P. R. (2016). An averaging principle for diffusions in foliated spaces. *Ann. Probab.* **44** 567–588. MR3456346 https://doi.org/10.1214/14-AOP982

[23] HAIRER, M. and PARDOUX, E. (2008). Homogenization of periodic linear degenerate PDEs. *J. Funct. Anal.* **255** 2462–2487. MR2473263 https://doi.org/10.1016/j.jfa.2008.04.014

[24] HASHEMI, S. N. and HEUNIS, A. J. (2005). On the Poisson equation for singular diffusions. *Stochastics* **77** 155–189. MR2151665 https://doi.org/10.1080/10451120500114045

[25] HONORÉ, I., MENOZZI, S. and PAGÈS, G. (2020). Non-asymptotic Gaussian estimates for the recursive approximation of the invariant distribution of a diffusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 1559–1605. MR4116702 https://doi.org/10.1214/19-AIHP985

[26] KELLY, D. and VANDEN-EIJNDEN, E. (2017). Fluctuations in the heterogeneous multiscale methods for fast-slow systems. *Res. Math. Sci.* **4** Paper No. 23, 26. MR3736718 https://doi.org/10.1186/s40687-017-0112-2

[27] KHASMINSKII, R. Z. and YIN, G. (2004). On averaging principles: An asymptotic expansion approach. *SIAM J. Math. Anal.* **35** 1534–1560. MR2083789 https://doi.org/10.1137/S0036141002403973

[28] KHASMINSKII, R. Z. and YIN, G. (2005). Limit behavior of two-time-scale diffusions revisited. *J. Differential Equations* **212** 85–113. MR2130548 https://doi.org/10.1016/j.jde.2004.08.013

[29] KIFER, Y. (1992). Averaging in dynamical systems and large deviations. *Invent. Math.* **110** 337–370. MR1185587 https://doi.org/10.1007/BF01231336

[30] KIFER, Y. (2001). Averaging and climate models. In *Stochastic Climate Models* (*Chorin*, 1999). *Progress in Probability* **49** 171–188. Birkhäuser, Basel. MR1948296

[31] KUEHN, C. (2015). *Multiple Time Scale Dynamics. Applied Mathematical Sciences* **191**. Springer, Cham. MR3309627 https://doi.org/10.1007/978-3-319-12316-5

[32] LADYŽENSKAJA, O. A., SOLONNIKOV, V. A. and URAL'CEVA, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type. Translations of Mathematical Monographs* **23**. Amer. Math. Soc., Providence, RI. Translated from the Russian by S. Smith. MR0241822

[33] LI, X.-M. (2008). An averaging principle for a completely integrable stochastic Hamiltonian system. *Nonlinearity* **21** 803–822. MR2399826 https://doi.org/10.1088/0951-7715/21/4/008

[34] LIU, D. (2010). Strong convergence of principle of averaging for multiscale stochastic dynamical systems. *Commun. Math. Sci.* **8** 999–1020. MR2744917

[35] MAJDA, A. J., TIMOFEYEV, I. and VANDEN EIJNDEN, E. (2001). A mathematical framework for stochastic climate models. *Comm. Pure Appl. Math.* **54** 891–974. MR1829529 https://doi.org/10.1002/cpa.1014

[36] MATTINGLY, J. C., STUART, A. M. and TRETYAKOV, M. V. (2010). Convergence of numerical time-averaging and stationary measures via Poisson equations. *SIAM J. Numer. Anal.* **48** 552–577. MR2669996 https://doi.org/10.1137/090770527

[37] MORSE, M. R. and SPILIOPOULOS, K. (2017). Moderate deviations for systems of slow-fast diffusions. *Asymptot. Anal.* **105** 97–135. MR3729828 https://doi.org/10.3233/asy-171434

[38] PAGÈS, G. and PANLOUP, F. (2012). Ergodic approximation of the distribution of a stationary diffusion: Rate of convergence. *Ann. Appl. Probab.* **22** 1059–1100. MR2977986 https://doi.org/10.1214/11-AAP779

[39] PAPANICOLAOU, G. C., STROOCK, D. and VARADHAN, S. R. S. (1977). Martingale approach to some limit theorems. In *Papers from the Duke Turbulence Conference* (*Duke Univ., Durham, N.C.*, 1976). *Duke Univ. Math. Ser.* **III**. Paper No. 6, ii+120 pp. MR0461684

[40] PARDOUX, É. (1999). Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: A probabilistic approach. *J. Funct. Anal.* **167** 498–520. MR1716206 https://doi.org/10.1006/jfan.1999.3441

[41] PARDOUX, E. and VERETENNIKOV, A. YU. (2001). On the Poisson equation and diffusion approximation. I. *Ann. Probab.* **29** 1061–1085. MR1872736 https://doi.org/10.1214/aop/1015345596

[42] PARDOUX, È. and VERETENNIKOV, A. YU. (2003). On Poisson equation and diffusion approximation. II. *Ann. Probab.* **31** 1166–1192. MR1988467 https://doi.org/10.1214/aop/1055425774

[43] PARDOUX, E. and VERETENNIKOV, A. YU. (2005). On the Poisson equation and diffusion approximation. III. *Ann. Probab.* **33** 1111–1133. MR2135314 https://doi.org/10.1214/009117905000000062

[44] PAVLIOTIS, G. A. and STUART, A. M. (2008). *Multiscale Methods: Averaging and Homogenization. Texts in Applied Mathematics* **53**. Springer, New York. MR2382139

[45] PUHALSKII, A. A. (2016). On large deviations of coupled diffusions with time scale separation. *Ann. Probab.* **44** 3111–3186. MR3531687 https://doi.org/10.1214/15-AOP1043

[46] RÖCKNER, M., SUN, X. and XIE, L. Strong and weak convergence in averaging principle for SDEs with Hölder coefficients. https://arxiv.org/pdf/1907.09256.pdf.

[47] RÖCKNER, M. and XIE, L. Averaging principle and normal deviations for multiscale stochastic systems. https://arxiv.org/pdf/2008.04822.pdf.

[48] SPILIOPOULOS, K. (2014). Fluctuation analysis and short time asymptotics for multiple scales diffusion processes. *Stoch. Dyn.* **14** 1350026, 22. MR3213185 https://doi.org/10.1142/S0219493713500263

[49] VANDEN-EIJNDEN, E. (2003). Numerical techniques for multi-scale dynamical systems with stochastic effects. *Commun. Math. Sci.* **1** 385–391. MR1980483

[50] VERETENNIKOV, A. YU. (1991). On the averaging principle for systems of stochastic differential equations. *Math. USSR, Sb.* **69** 271–284. MR1046602 https://doi.org/10.1070/SM1991v069n01ABEH001237

[51] VERETENNIKOV, A. YU. (1997). On polynomial mixing bounds for stochastic differential equations. *Stochastic Process. Appl.* **70** 115–127. MR1472961 https://doi.org/10.1016/S0304-4149(97)00056-2

[52] VERETENNIKOV, A. YU. (2011). On Sobolev solutions of Poisson equations in $\mathbb{R}^d$ with a parameter. *J. Math. Sci.* (*N. Y.*) **179** 48–79. 1, Problems in mathematical analysis. No. 61. MR3014098 https://doi.org/10.1007/s10958-011-0582-5

[53] VERETENNIKOV, A. YU. and KULIK, A. M. (2011). The extended Poisson equation for weakly ergodic Markov processes. *Teor. Ĭmovīr. Mat. Stat.* **85** 22–38. MR2933700

[54] WANG, W. and ROBERTS, A. J. (2012). Average and deviation for slow-fast stochastic partial differential equations. *J. Differential Equations* **253** 1265–1286. MR2927381 https://doi.org/10.1016/j.jde.2012.05.011

[55] ZHANG, B., FU, H., WAN, L. and LIU, J. (2018). Weak order in averaging principle for stochastic differential equations with jumps. *Adv. Difference Equ.* Paper No. 197, 20. MR3805681 https://doi.org/10.1186/s13662-018-1638-3

The Annals of Probability
2021, Vol. 49, No. 3, 1237–1256
https://doi.org/10.1214/20-AOP1476

A SHAPE THEOREM FOR THE ORTHANT MODEL

BY MARK HOLMES[1] AND THOMAS S. SALISBURY[2]

[1]*School of Mathematics and Statistics, University of Melbourne, holmes.m@unimelb.edu.au*
[2]*Deptartment of Mathematics and Statistics, York University, salt@yorku.ca*

We study a particular model of a random medium, called the orthant model, in general dimensions $d \geq 2$. Each site $x \in \mathbb{Z}^d$ independently has arrows pointing to its positive neighbours $x + e_i$, $i = 1, \dots, d$ with probability p and, otherwise, to its negative neighbours $x - e_i$, $i = 1, \dots, d$ (with probability $1 - p$). We prove a shape theorem for the set of sites reachable by following arrows, starting from the origin, when p is large. The argument uses subadditivity, as would be expected from the shape theorems arising in the study of first passage percolation. The main difficulty to overcome is that the primary objects of study are not stationary which is a key requirement of the subadditive ergodic theorem.

REFERENCES

[1] ANTAL, P. and PISZTORA, A. (1996). On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.* **24** 1036–1048. MR1404543 https://doi.org/10.1214/aop/1039639377

[2] AUFFINGER, A., DAMRON, M. and HANSON, J. (2017). 50 *Years of First-Passage Percolation*. *University Lecture Series* **68**. Amer. Math. Soc., Providence, RI. MR3729447 https://doi.org/10.1090/ulect/068

[3] CERF, R. and THÉRET, M. (2016). Weak shape theorem in first passage percolation with infinite passage times. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1351–1381. MR3531712 https://doi.org/10.1214/15-AIHP686

[4] COX, J. T. and DURRETT, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603. MR0624685

[5] DREWITZ, A., RÁTH, B. and SAPOZHNIKOV, A. (2014). On chemical distances and shape theorems in percolation models with long-range correlations. *J. Math. Phys.* **55** 083307. MR3390739 https://doi.org/10.1063/1.4886515

[6] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

[7] GARET, O. and MARCHAND, R. (2004). Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Stat.* **8** 169–199. MR2085613 https://doi.org/10.1051/ps:2004009

[8] GRIMMETT, G. R. and MARSTRAND, J. M. (1990). The supercritical phase of percolation is well behaved. *Proc. R. Soc. Lond. Ser. A* **430** 439–457. MR1068308 https://doi.org/10.1098/rspa.1990.0100

[9] HOLMES, M. and SALISBURY, T. S. (2014). Degenerate random environments. *Random Structures Algorithms* **45** 111–137. MR3231085 https://doi.org/10.1002/rsa.20473

[10] HOLMES, M. and SALISBURY, T. S. (2014). Random walks in degenerate random environments. *Canad. J. Math.* **66** 1050–1077. MR3251764 https://doi.org/10.4153/CJM-2013-017-3

[11] HOLMES, M. and SALISBURY, T. S. (2016). Forward clusters for degenerate random environments. *Combin. Probab. Comput.* **25** 744–765. MR3531440 https://doi.org/10.1017/S0963548316000158

[12] HOLMES, M. and SALISBURY, T. S. (2017). Conditions for ballisticity and invariance principle for random walk in non-elliptic random environment. *Electron. J. Probab.* **22** 81. MR3710801 https://doi.org/10.1214/17-EJP107

[13] HOLMES, M. and SALISBURY, T. S. (2019). Phase transitions for degenerate random environments. Preprint.

[14] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 125–264. Springer, Berlin. MR0876084 https://doi.org/10.1007/BFb0074919

[15] LIGGETT, T. M. (1985). An improved subadditive ergodic theorem. *Ann. Probab.* **13** 1279–1285. MR0806224

MSC2020 subject classifications. 60K35.
Key words and phrases. Orthant model, shape theorem, subadditivity, random environments.

[16] MOURRAT, J.-C. (2012). Lyapunov exponents, shape theorems and large deviations for the random walk in random potential. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 165–211. MR2923190

[17] ZEITOUNI, O. (2004). Random walks in random environment. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math.* **1837** 189–312. Springer, Berlin. MR2071631 https://doi.org/10.1007/978-3-540-39874-5_2

The Annals of Probability
2021, Vol. 49, No. 3, 1257–1285
https://doi.org/10.1214/20-AOP1479

ATYPICAL EXIT EVENTS NEAR A REPELLING EQUILIBRIUM

BY YURI BAKHTIN* AND HONG-BIN CHEN†

*Courant Institute of Mathematical Sciences, New York University, *bakhtin@cims.nyu.edu; †hbchen@cims.nyu.edu*

We consider exit problems for small, white noise perturbations of a dynamical system generated by a vector field and a domain containing a critical point with all positive eigenvalues of linearization. We prove that, in the vanishing noise limit, the probability of exit through a generic set on the boundary is asymptotically polynomial in the noise strength with exponent depending on the mutual position of the set and the flag of the invariant manifolds associated with the top eigenvalues. Furthermore, we compute the limiting exit distributions conditioned on atypical exit events of polynomially small probability and show that the limits are Radon–Nikodym equivalent to volume measures on certain manifolds that we construct. This situation is in sharp contrast with the large deviation picture where the limiting conditional distributions are point masses.

REFERENCES

[1] ALMADA MONTER, S. A. and BAKHTIN, Y. (2011). Normal forms approach to diffusion near hyperbolic equilibria. *Nonlinearity* **24** 1883–1907. MR2802310 https://doi.org/10.1088/0951-7715/24/6/011

[2] ARMBRUSTER, D., STONE, E. and KIRK, V. (2003). Noisy heteroclinic networks. *Chaos* **13** 71–86. MR1964965 https://doi.org/10.1063/1.1539951

[3] AZENCOTT, R. (1980). Grandes déviations et applications. In *Eighth Saint Flour Probability Summer School*—1978 (*Saint Flour*, 1978). *Lecture Notes in Math*. **774** 1–176. Springer, Berlin. MR0590626

[4] BAKHTIN, Y. (2010). Small noise limit for diffusions near heteroclinic networks. *Dyn. Syst*. **25** 413–431. MR2731621 https://doi.org/10.1080/14689367.2010.482520

[5] BAKHTIN, Y. (2011). Noisy heteroclinic networks. *Probab. Theory Related Fields* **150** 1–42. MR2800902 https://doi.org/10.1007/s00440-010-0264-0

[6] BAKHTIN, Y. (2015). On Gumbel limit for the length of reactive paths. *Stoch. Dyn*. **15** 1550001, 11. MR3285318 https://doi.org/10.1142/S021949371550001X

[7] BAKHTIN, Y. and CHEN, H.-B. (2020). Long exit times near a repelling equilibrium. *Ann. Appl. Probab*. To appear. Available at arXiv:1908.11840.

[8] BAKHTIN, Y. and PAJOR-GYULAI, Z. (2019). Malliavin calculus approach to long exit times from an unstable equilibrium. *Ann. Appl. Probab*. **29** 827–850. MR3910018 https://doi.org/10.1214/18-AAP1387

[9] BAKHTIN, Y. and PAJOR-GYULAI, Z. (2020). Tails of exit times from unstable equilibria on the line. *J. Appl. Probab*. **57** 477–496. MR4125460 https://doi.org/10.1017/jpr.2020.16

[10] BASS, R. F. (2011). *Stochastic Processes*. *Cambridge Series in Statistical and Probabilistic Mathematics* **33**. Cambridge Univ. Press, Cambridge. MR2856623 https://doi.org/10.1017/CBO9780511997044

[11] CÉROU, F., GUYADER, A., LELIÈVRE, T. and MALRIEU, F. (2013). On the length of one-dimensional reactive paths. *ALEA Lat. Am. J. Probab. Math. Stat*. **10** 359–389. MR3083930

[12] DAY, M. V. (1995). On the exit law from saddle points. *Stochastic Process. Appl*. **60** 287–311. MR1376805 https://doi.org/10.1016/0304-4149(95)00063-1

[13] EIZENBERG, A. (1984). The exit distributions for small random perturbations of dynamical systems with a repulsive type stationary point. *Stochastics* **12** 251–275. MR0749377 https://doi.org/10.1080/17442508408833304

[14] KATOK, A. and HASSELBLATT, B. (1995). *Introduction to the Modern Theory of Dynamical Systems*. *Encyclopedia of Mathematics and Its Applications* **54**. Cambridge Univ. Press, Cambridge. With a supplementary chapter by Katok and Leonardo Mendoza. MR1326374 https://doi.org/10.1017/CBO9780511809187

MSC2020 subject classifications. 60H07, 60H10, 60J60.

Key words and phrases. Vanishing noise limit, unstable equilibrium, exit problem, polynomial decay, equidistribution, Malliavin calculus.

[15] KIFER, Y. (1981). The exit problem for small random perturbations of dynamical systems with a hyperbolic fixed point. *Israel J. Math.* **40** 74–96. MR0636908 https://doi.org/10.1007/BF02761819

[16] MIKAMI, T. (1995). Large deviations for the first exit time on small random perturbations of dynamical systems with a hyperbolic equilibrium point. *Hokkaido Math. J.* **24** 491–525. MR1357028 https://doi.org/10.14492/hokmj/1380892606

[17] STERNBERG, S. (1957). Local contractions and a theorem of Poincaré. *Amer. J. Math.* **79** 809–824. MR0096853 https://doi.org/10.2307/2372437

[18] STONE, E. and ARMBRUSTER, D. (1999). Noise and $O(1)$ amplitude effects on heteroclinic cycles. *Chaos* **9** 499–506.

[19] STONE, E. and HOLMES, P. (1990). Random perturbations of heteroclinic attractors. *SIAM J. Appl. Math.* **50** 726–743. MR1050910 https://doi.org/10.1137/0150043

The Annals of Probability
2021, Vol. 49, No. 3, 1286–1309
https://doi.org/10.1214/20-AOP1481

THE SMALLEST SINGULAR VALUE OF INHOMOGENEOUS SQUARE RANDOM MATRICES

BY GALYNA V. LIVSHYTS[1,*], KONSTANTIN TIKHOMIROV[1,†] AND ROMAN VERSHYNIN[2]

[1]*Department of Mathematics, Georgia Institute of Technology,* [*]*glivshyts6@math.gatech.edu;* [†]*konstantin.tikhomirov@math.gatech.edu*

[2]*Department of Mathematics, University of California, Irvine, vershyn@uci.edu*

We show that, for an $n \times n$ random matrix A with independent uniformly anticoncentrated entries such that $\mathbb{E}\|A\|_{\mathrm{HS}}^2 \le Kn^2$, the smallest singular value $\sigma_n(A)$ of A satisfies

$$\mathbb{P}\left\{\sigma_n(A) \le \frac{\varepsilon}{\sqrt{n}}\right\} \le C\varepsilon + 2e^{-cn}, \quad \varepsilon \ge 0.$$

This extends earlier results (*Adv. Math.* **218** (2008) 600–633; *Israel J. Math.* **227** (2018) 507–544) by removing the assumption of mean zero and identical distribution of the entries across the matrix as well as the recent result (Livshyts (2018)) where the matrix was required to have i.i.d. rows. Our model covers inhomogeneous matrices allowing different variances of the entries as long as the sum of the second moments is of order $O(n^2)$.

In the past advances, the assumption of i.i.d. rows was required due to lack of Littlewood–Offord-type inequalities for weighted sums of non-i.i.d. random variables. Here, we overcome this problem by introducing *the Randomized Least Common Denominator* (RLCD) which allows to study anticoncentration properties of weighted sums of independent but not identically distributed variables. We construct efficient nets on the sphere with *lattice structure* and show that the lattice points typically have large RLCD. This allows us to derive strong anticoncentration properties for the distance between a fixed column of A and the linear span of the remaining columns and prove the main result.

REFERENCES

[1] ALON, N. and KLARTAG, B. (2017). Optimal compression of approximate inner products and dimension reduction. In 58*th Annual IEEE Symposium on Foundations of Computer Science—FOCS* 2017 639–650. IEEE Comput. Soc., Los Alamitos, CA. MR3734268 https://doi.org/10.1109/FOCS.2017.65

[2] BAI, Z. D. and YIN, Y. Q. (1988). Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. *Ann. Probab.* **16** 1729–1741. MR0958213

[3] BOURGAIN, J., VU, V. H. and WOOD, P. M. (2010). On the singularity probability of discrete random matrices. *J. Funct. Anal.* **258** 559–603. MR2557947 https://doi.org/10.1016/j.jfa.2009.04.016

[4] COOK, N. (2018). Lower bounds for the smallest singular value of structured random matrices. *Ann. Probab.* **46** 3442–3500. MR3857860 https://doi.org/10.1214/17-AOP1251

[5] EDELMAN, A. (1988). Eigenvalues and condition numbers of random matrices. *SIAM J. Matrix Anal. Appl.* **9** 543–560. MR0964668 https://doi.org/10.1137/0609045

[6] ESSEEN, C. G. (1966). On the Kolmogorov–Rogozin inequality for the concentration function. *Z. Wahrsch. Verw. Gebiete* **5** 210–216. MR0205297 https://doi.org/10.1007/BF00533057

[7] JAIN, V. and SILWAL, S. (2020). A note on the universality of ESDs of inhomogeneous random matrices. Preprint. Available at arXiv:2006.05418.

[8] KAHN, J., KOMLÓS, J. and SZEMERÉDI, E. (1995). On the probability that a random ±1-matrix is singular. *J. Amer. Math. Soc.* **8** 223–240. MR1260107 https://doi.org/10.2307/2152887

[9] KANNAN, R. and VEMPALA, S. (1999). Sampling lattice points. In *Proc.* 29*th ACM Symposium on the Theory of Computing* (*STOC'*97) (*El Paso*, *TX*) 696–700. ACM, New York. MR1753411

MSC2020 subject classifications. 60B20.
Key words and phrases. Random matrices, Littlewood–Offord problem.

[10] KLARTAG, B. and LIVSHYTS, G. V. (2020). The lower bound for Koldobsky's slicing inequality via random rounding. In *Geometric Aspects of Functional Analysis* 43–63. Springer, Berlin.

[11] LITTLEWOOD, J. E. and OFFORD, A. C. (1943). On the number of real roots of a random algebraic equation. III. *Math. Sci.* **12** 277–286. MR0009656

[12] LITVAK, A. E., PAJOR, A., RUDELSON, M. and TOMCZAK-JAEGERMANN, N. (2005). Smallest singular value of random matrices and geometry of random polytopes. *Adv. Math.* **195** 491–523. MR2146352 https://doi.org/10.1016/j.aim.2004.08.004

[13] LITVAK, A. E. and RIVASPLATA, O. (2012). Smallest singular value of sparse random matrices. *Studia Math.* **212** 195–218. MR3009072 https://doi.org/10.4064/sm212-3-1

[14] LIVSHYTS, G. V. (2018). The smallest singular value of heavy-tailed not necessarily i.i.d. random matrices via random rounding. Preprint.

[15] MENDELSON, S. and PAOURIS, G. (2014). On the singular values of random matrices. *J. Eur. Math. Soc.* (*JEMS*) **16** 823–834. MR3191978 https://doi.org/10.4171/JEMS/448

[16] PÓLYA, G. and SZEGÖ, G. (1964). *Aufgaben und Lehrsätze aus der Analysis. Band I: Reihen. Integralrechnung. Funktionentheorie. Dritte Berichtigte Auflage. Die Grundlehren der Mathematischen Wissenschaften* **19**. Springer, Berlin. MR0170985

[17] REBROVA, E. and TIKHOMIROV, K. (2018). Coverings of random ellipsoids, and invertibility of matrices with i.i.d. heavy-tailed entries. *Israel J. Math.* **227** 507–544. MR3846333 https://doi.org/10.1007/s11856-018-1732-y

[18] RUDELSON, M. (2008). Invertibility of random matrices: Norm of the inverse. *Ann. of Math.* (2) **168** 575–600. MR2434885 https://doi.org/10.4007/annals.2008.168.575

[19] RUDELSON, M. and VERSHYNIN, R. (2008). The Littlewood–Offord problem and invertibility of random matrices. *Adv. Math.* **218** 600–633. MR2407948 https://doi.org/10.1016/j.aim.2008.01.010

[20] RUDELSON, M. and VERSHYNIN, R. (2009). Smallest singular value of a random rectangular matrix. *Comm. Pure Appl. Math.* **62** 1707–1739. MR2569075 https://doi.org/10.1002/cpa.20294

[21] RUDELSON, M. and VERSHYNIN, R. (2015). Delocalization of eigenvectors of random matrices with independent entries. *Duke Math. J.* **164** 2507–2538. MR3405592 https://doi.org/10.1215/00127094-3129809

[22] SCHÜTT, C. (1984). Entropy numbers of diagonal operators between symmetric Banach spaces. *J. Approx. Theory* **40** 121–128. MR0732693 https://doi.org/10.1016/0021-9045(84)90021-2

[23] SRINIVASAN, A. (1999). Approximation algorithms via randomized rounding: A survey. In *Lectures on Approximation and Randomized Algorithms. Series in Advanced Topics in Mathematics* 9–71. Polish Scientific Publishers PWN, Warsaw.

[24] SZAREK, S. J. (1991). Condition numbers of random matrices. *J. Complexity* **7** 131–149. MR1108773 https://doi.org/10.1016/0885-064X(91)90002-F

[25] TAO, T. and VU, V. (2006). On random ± 1 matrices: Singularity and determinant. *Random Structures Algorithms* **28** 1–23. MR2187480 https://doi.org/10.1002/rsa.20109

[26] TAO, T. and VU, V. (2007). On the singularity probability of random Bernoulli matrices. *J. Amer. Math. Soc.* **20** 603–628. MR2291914 https://doi.org/10.1090/S0894-0347-07-00555-3

[27] TAO, T. and VU, V. (2010). Random matrices: The distribution of the smallest singular values. *Geom. Funct. Anal.* **20** 260–297. MR2647142 https://doi.org/10.1007/s00039-010-0057-8

[28] TAO, T. and VU, V. H. (2009). Inverse Littlewood–Offord theorems and the condition number of random discrete matrices. *Ann. of Math.* (2) **169** 595–632. MR2480613 https://doi.org/10.4007/annals.2009.169.595

[29] TATARKO, K. (2018). An upper bound on the smallest singular value of a square random matrix. *J. Complexity* **48** 119–128. MR3828841 https://doi.org/10.1016/j.jco.2018.06.002

[30] TIKHOMIROV, K. (2015). The limit of the smallest singular value of random matrices with i.i.d. entries. *Adv. Math.* **284** 1–20. MR3391069 https://doi.org/10.1016/j.aim.2015.07.020

[31] TIKHOMIROV, K. (2020). Invertibility via distance for noncentered random matrices with continuous distributions. *Random Structures Algorithms* **57** 526–562. MR4129729 https://doi.org/10.1002/rsa.20920

[32] TIKHOMIROV, K. (2020). Singularity of random Bernoulli matrices. *Ann. of Math.* (2) **191** 593–634. MR4076632 https://doi.org/10.4007/annals.2020.191.2.6

[33] TIKHOMIROV, K. E. (2016). The smallest singular value of random rectangular matrices with no moment assumptions on entries. *Israel J. Math.* **212** 289–314. MR3504328 https://doi.org/10.1007/s11856-016-1287-8

[34] VERSHYNIN, R. (2011). Spectral norm of products of random and deterministic matrices. *Probab. Theory Related Fields* **150** 471–509. MR2824864 https://doi.org/10.1007/s00440-010-0281-z

[35] VON NEUMANN, J. and GOLDSTINE, H. H. (1947). Numerical inverting of matrices of high order. *Bull. Amer. Math. Soc.* **53** 1021–1099. MR0024235 https://doi.org/10.1090/S0002-9904-1947-08909-6

The Annals of Probability
2021, Vol. 49, No. 3, 1310–1346
https://doi.org/10.1214/20-AOP1482

AN OPTIMAL REGULARITY RESULT FOR KOLMOGOROV EQUATIONS AND WEAK UNIQUENESS FOR SOME CRITICAL SPDES

BY ENRICO PRIOLA

Dipartimento di Matematica, Università degli Studi di Pavia, enrico.priola@unipv.it

We show uniqueness in law for the critical SPDE

$$dX_t = AX_t\,dt + (-A)^{1/2}F\big(X(t)\big)\,dt + dW_t, \quad X_0 = x \in H,$$

where $A : \text{dom}(A) \subset H \to H$ is a negative definite self-adjoint operator on a separable Hilbert space H having A^{-1} of trace class and W is a cylindrical Wiener process on H. Here, $F : H \to H$ can be continuous with, at most, linear growth (some functions F which grow more than linearly can also be considered). This leads to new uniqueness results for generalized stochastic Burgers equations and for three-dimensional stochastic Cahn–Hilliard-type equations which have interesting applications. To get weak uniqueness, we use an infinite dimensional localization principle and also establish a new optimal regularity result for the Kolmogorov equation $\lambda u - Lu = f$ associated to the SPDE when $F = 0$ ($\lambda > 0$, $f : H \to \mathbb{R}$ Borel and bounded). In particular, we prove that the first derivative $Du(x)$ belongs to $\text{dom}((-A)^{1/2})$, for any $x \in H$, and $\sup_{x\in H} |(-A)^{1/2}Du(x)|_H = \|(-A)^{1/2}Du\|_0 \le C\|f\|_0$.

REFERENCES

[1] ATHREYA, S. R., BASS, R. F., GORDINA, M. and PERKINS, E. A. (2006). Infinite dimensional stochastic differential equations of Ornstein–Uhlenbeck type. *Stochastic Process. Appl.* **116** 381–406. MR2199555 https://doi.org/10.1016/j.spa.2005.10.001

[2] ATHREYA, S. R., BASS, R. F. and PERKINS, E. A. (2005). Hölder norm estimates for elliptic operators on finite and infinite-dimensional spaces. *Trans. Amer. Math. Soc.* **357** 5001–5029. MR2165395 https://doi.org/10.1090/S0002-9947-05-03638-X

[3] BRZEŹNIAK, Z., LIU, W. and ZHU, J. (2014). Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Anal. Real World Appl.* **17** 283–310. MR3158475 https://doi.org/10.1016/j.nonrwa.2013.12.005

[4] BUTKOVSKY, O. and MYTNIK, L. (2019). Regularization by noise and flows of solutions for a stochastic heat equation. *Ann. Probab.* **47** 165–212. MR3909968 https://doi.org/10.1214/18-AOP1259

[5] CERRAI, S. and LUNARDI, A. (2019). Schauder theorems for Ornstein–Uhlenbeck equations in infinite dimension. *J. Differential Equations* **267** 7462–7482. MR4011050 https://doi.org/10.1016/j.jde.2019.08.005

[6] CHOJNOWSKA-MICHALIK, A. and GOLDYS, B. (2002). Symmetric Ornstein–Uhlenbeck semigroups and their generators. *Probab. Theory Related Fields* **124** 459–486. MR1942319 https://doi.org/10.1007/s004400200222

[7] DA PRATO, G. (2003). A new regularity result for Ornstein–Uhlenbeck generators and applications *J. Evol. Equ.* **3** 485–498. MR2019031 https://doi.org/10.1007/s00028-003-0114-x

[8] DA PRATO, G. and DEBUSSCHE, A. (1996). Stochastic Cahn–Hilliard equation. *Nonlinear Anal.* **26** 241–263. MR1359472 https://doi.org/10.1016/0362-546X(94)00277-O

[9] DA PRATO, G. and DEBUSSCHE, A. (1998). Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton–Jacobi equation. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.* **9** 267–277. MR1722786

[10] DA PRATO, G., FLANDOLI, F., PRIOLA, E. and RÖCKNER, M. (2013). Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** 3306–3344. MR3127884 https://doi.org/10.1214/12-AOP763

MSC2020 subject classifications. Primary 60H15, 35R60; secondary 35R15.
Key words and phrases. Critical SPDEs, weak uniqueness in infinite dimensions, optimal regularity for Kolmogorov operators.

[11] DA PRATO, G., FLANDOLI, F., RÖCKNER, M. and VERETENNIKOV, A. YU. (2016). Strong uniqueness for SDEs in Hilbert spaces with nonregular drift. *Ann. Probab.* **44** 1985–2023. MR3502599 https://doi.org/10.1214/15-AOP1016

[12] DA PRATO, G., KWAPIEŃ, S. and ZABCZYK, J. (1987). Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics* **23** 1–23. MR0920798 https://doi.org/10.1080/17442508708833480

[13] DA PRATO, G. and LUNARDI, A. (1995). On the Ornstein–Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.* **131** 94–114. MR1343161 https://doi.org/10.1006/jfan.1995.1084

[14] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems. London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491 https://doi.org/10.1017/CBO9780511662829

[15] DA PRATO, G. and ZABCZYK, J. (2002). *Second Order Partial Differential Equations in Hilbert Spaces. London Mathematical Society Lecture Note Series* **293**. Cambridge Univ. Press, Cambridge. MR1985790 https://doi.org/10.1017/CBO9780511543210

[16] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge Univ. Press, Cambridge. MR3236753 https://doi.org/10.1017/CBO9781107295513

[17] ELEZOVIĆ, N. and MIKELIĆ, A. (1991). On the stochastic Cahn–Hilliard equation. *Nonlinear Anal.* **16** 1169–1200. MR1111627 https://doi.org/10.1016/0362-546X(91)90204-E

[18] ES-SARHIR, A. and STANNAT, W. (2009). Maximal dissipativity of Kolmogorov operators with Cahn–Hilliard type drift term. *J. Differential Equations* **247** 424–446. MR2523685 https://doi.org/10.1016/j.jde.2009.03.025

[19] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[20] GĄTAREK, D. and GOŁDYS, B. (1994). On weak solutions of stochastic equations in Hilbert spaces. *Stoch. Stoch. Rep.* **46** 41–51. MR1787166 https://doi.org/10.1080/17442509408833868

[21] GĄTAREK, D. and GOŁDYS, B. (1992). On solving stochastic evolution equations by the change of drift with application to optimal control. In *Stochastic Partial Differential Equations and Applications* (*Trento*, 1990). *Pitman Res. Notes Math. Ser.* **268** 180–190. Longman Sci. Tech., Harlow. MR1222696

[22] GRISVARD, P. (1967). Caractérisation de quelques espaces d'interpolation. *Arch. Ration. Mech. Anal.* **25** 40–63. MR0213864 https://doi.org/10.1007/BF00281421

[23] GYÖNGY, I. (1998). Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Process. Appl.* **73** 271–299. MR1608641 https://doi.org/10.1016/S0304-4149(97)00103-8

[24] HAIRER, M. (2009). An Introduction to Stochastic PDEs. Available at arXiv:0907.4178v1.

[25] ISCOE, I., MARCUS, M. B., MCDONALD, D. B., TALAGRAND, M. and ZINN, J. (1990). Continuity of l_2-valued Ornstein-Uhlenbeck processes. *Ann. Probab.* **18** 68–84. MR1043937

[26] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[27] KUNZE, M. C. (2013). On a class of martingale problems on Banach spaces. *Electron. J. Probab.* **18** No. 104, 30. MR3145051 https://doi.org/10.1214/EJP.v18-2924

[28] LORENZI, L. and BERTOLDI, M. (2007). *Analytical Methods for Markov Semigroups. Pure and Applied Mathematics* (*Boca Raton*) **283**. CRC Press/CRC, Boca Raton, FL. MR2313847

[29] LUNARDI, A. (1995). *Analytic Semigroups and Optimal Regularity in Parabolic Problems. Modern Birkhäuser Classics*. Birkhäuser, Basel. MR3012216

[30] LUNARDI, A. (2018). *Interpolation Theory*. Third edition. *Appunti. Scuola Normale Superiore di Pisa* (*Nuova Serie*) [*Lecture Notes. Scuola Normale Superiore di Pisa* (*New Series*)] **16**. Edizioni della Normale, Pisa. MR3753604 https://doi.org/10.1007/978-88-7642-638-4

[31] LUNARDI, A. and RÖCKNER, M. (2019). Schauder theorems for a class of (pseudo-)differential operators on finite and infinite dimensional state spaces. Preprint. Available at https://arxiv.org/abs/1907.06237.

[32] NOVICK-COHEN, A. (1998). The Cahn–Hilliard equation: Mathematical and modeling perspectives. *Adv. Math. Sci. Appl.* **8** 965–985. MR1657208

[33] PRIOLA, E. (2015). On weak uniqueness for some degenerate SDEs by global L^p estimates. *Potential Anal.* **42** 247–281. MR3297995 https://doi.org/10.1007/s11118-014-9432-7

[34] RÖCKNER, M. and SOBOL, Z. (2006). Kolmogorov equations in infinite dimensions: Well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. *Ann. Probab.* **34** 663–727. MR2223955 https://doi.org/10.1214/009117905000000666

[35] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[36] ZAMBOTTI, L. (2000). An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. *Probab. Theory Related Fields* **118** 147–168. MR1790079 https://doi.org/10.1007/s440-000-8012-6

The Annals of Probability
2021, Vol. 49, No. 3, 1347–1401
https://doi.org/10.1214/20-AOP1483

EXTREMAL EIGENVALUES OF CRITICAL ERDŐS–RÉNYI GRAPHS

BY JOHANNES ALT*, RAPHAËL DUCATEZ† AND ANTTI KNOWLES‡

Section of Mathematics, University of Geneva, **johannes.alt@unige.ch;* *†raphael.ducatez@unige.ch;* *‡antti.knowles@unige.ch*

We complete the analysis of the extremal eigenvalues of the adjacency matrix A of the Erdős–Rényi graph $\mathbb{G}(N, d/N)$ in the critical regime $d \asymp \log N$ of the transition uncovered in (*Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020) 2141–2161; *Ann. Probab.* **47** (2019) 1653–1676), where the regimes $d \gg \log N$ and $d \ll \log N$ were studied. We establish a one-to-one correspondence between vertices of degree at least $2d$ and nontrivial (excluding the trivial top eigenvalue) eigenvalues of $A/\sqrt{d}$ outside of the asymptotic bulk $[-2, 2]$. This correspondence implies that the transition characterized by the appearance of the eigenvalues outside of the asymptotic bulk takes place at the critical value $d = d_* = \frac{1}{\log 4 - 1} \log N$. For $d < d_*$, we obtain rigidity bounds on the locations of all eigenvalues outside the interval $[-2, 2]$, and for $d > d_*$, we show that no such eigenvalues exist. All of our estimates are quantitative with polynomial error probabilities.

Our proof is based on a tridiagonal representation of the adjacency matrix and on a detailed analysis of the geometry of the neighbourhood of the large degree vertices. An important ingredient in our estimates is a matrix inequality obtained via the associated nonbacktracking matrix and an Ihara–Bass formula (*Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020) 2141–2161). Our argument also applies to sparse Wigner matrices, defined as the Hadamard product of A and a Wigner matrix, in which case the role of the degrees is replaced by the squares of the ℓ^2-norms of the rows.

REFERENCES

[1] ALON, N. (1998). Spectral techniques in graph algorithms (invited paper). In *LATIN'98: Theoretical Informatics* (*Campinas*, 1998). *Lecture Notes in Computer Science* **1380** 206–215. Springer, Berlin. MR1635529 https://doi.org/10.1007/BFb0054322

[2] ALT, J., DUCATEZ, R. and KNOWLES, A. (2020). Delocalization transition for critical Erdős–Rényi graphs. Preprint. Available at arXiv:2005.14180.

[3] BENAYCH-GEORGES, F., BORDENAVE, C. and KNOWLES, A. (2019). Largest eigenvalues of sparse inhomogeneous Erdős–Rényi graphs. *Ann. Probab.* **47** 1653–1676. MR3945756 https://doi.org/10.1214/18-AOP1293

[4] BENAYCH-GEORGES, F., BORDENAVE, C. and KNOWLES, A. (2020). Spectral radii of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2141–2161. MR4116720 https://doi.org/10.1214/19-AIHP1033

[5] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966 https://doi.org/10.1017/CBO9780511814068

[6] CHUNG, F. R. K. (1997). *Spectral Graph Theory*. *CBMS Regional Conference Series in Mathematics* **92**. Amer. Math. Soc., Providence, RI. MR1421568

[7] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 https://doi.org/10.1007/s00220-012-1527-7

[8] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** 2279–2375. MR3098073 https://doi.org/10.1214/11-AOP734

[9] FEIGE, U. and OFEK, E. (2005). Spectral techniques applied to sparse random graphs. *Random Structures Algorithms* **27** 251–275. MR2155709 https://doi.org/10.1002/rsa.20089

MSC2020 subject classifications. 60B20, 15B52, 05C80.
Key words and phrases. Erdős–Rényi graphs, sparse random matrix, eigenvalue rigidity.

[10] FÜREDI, Z. and KOMLÓS, J. (1981). The eigenvalues of random symmetric matrices. *Combinatorica* **1** 233–241. MR0637828 https://doi.org/10.1007/BF02579329

[11] HOORY, S., LINIAL, N. and WIGDERSON, A. (2006). Expander graphs and their applications. *Bull. Amer. Math. Soc.* (*N.S.*) **43** 439–561. MR2247919 https://doi.org/10.1090/S0273-0979-06-01126-8

[12] HUANG, J., LANDON, B. and YAU, H.-T. (2020). Transition from Tracy–Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 916–962. MR4089498 https://doi.org/10.1214/19-AOP1378

[13] KRIVELEVICH, M. and SUDAKOV, B. (2003). The largest eigenvalue of sparse random graphs. *Combin. Probab. Comput.* **12** 61–72. MR1967486 https://doi.org/10.1017/S0963548302005424

[14] LATAŁA, R., VAN HANDEL, R. and YOUSSEF, P. (2018). The dimension-free structure of nonhomogeneous random matrices. *Invent. Math.* **214** 1031–1080. MR3878726 https://doi.org/10.1007/s00222-018-0817-x

[15] LEE, J. O. and SCHNELLI, K. (2018). Local law and Tracy–Widom limit for sparse random matrices. *Probab. Theory Related Fields* **171** 543–616. MR3800840 https://doi.org/10.1007/s00440-017-0787-8

[16] LOVÁSZ, L. (1993). *Combinatorial Problems and Exercises*, 2nd ed. North-Holland, Amsterdam. MR1265492

[17] TIKHOMIROV, K. and YOUSSEF, P. (2019). Outliers in spectrum of sparse Wigner matrices. Preprint. Available at arXiv:1904.07985.

[18] TRAN, L. V., VU, V. H. and WANG, K. (2013). Sparse random graphs: Eigenvalues and eigenvectors. *Random Structures Algorithms* **42** 110–134. MR2999215 https://doi.org/10.1002/rsa.20406

[19] TROTTER, H. F. (1984). Eigenvalue distributions of large Hermitian matrices; Wigner's semicircle law and a theorem of Kac, Murdock, and Szegő. *Adv. Math.* **54** 67–82. MR0761763 https://doi.org/10.1016/0001-8708(84)90037-9

[20] VU, V. H. (2007). Spectral norm of random matrices. *Combinatorica* **27** 721–736. MR2384414 https://doi.org/10.1007/s00493-007-2190-z

[21] WIGNER, E. P. (1958). On the distribution of the roots of certain symmetric matrices. *Ann. of Math.* (2) **67** 325–327. MR0095527 https://doi.org/10.2307/1970008

The Annals of Probability
2021, Vol. 49, No. 3, 1402–1426
https://doi.org/10.1214/20-AOP1484

GLOBAL WELL POSEDNESS OF THE TWO-DIMENSIONAL STOCHASTIC NONLINEAR WAVE EQUATION ON AN UNBOUNDED DOMAIN

BY LEONARDO TOLOMEO

Mathematical Institute, Hausdorff Center for Mathematics, Universität Bonn, tolomeo@math.uni-bonn.de

We study the two-dimensional wave equation with cubic nonlinearity posed on $\mathbb{R}^2$ with space-time white noise forcing. After a suitable renormalisation of the nonlinearity, we prove global well-posedness for this equation for initial data in $\mathcal{H}^s$, $s > \frac{4}{5}$.

REFERENCES

[1] ALBEVERIO, S., HABA, Z. and RUSSO, F. (1996). Trivial solutions for a non-linear two-space-dimensional wave equation perturbed by space-time white noise. *Stoch. Stoch. Rep.* **56** 127–160. MR1396758 https://doi.org/10.1080/17442509608834039

[2] BOURGAIN, J. (1994). Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166** 1–26. MR1309539

[3] BOURGAIN, J. (2000). Invariant measures for NLS in infinite volume. *Comm. Math. Phys.* **210** 605–620. MR1777342 https://doi.org/10.1007/s002200050792

[4] BURQ, N. and TZVETKOV, N. (2008). Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.* **173** 449–475. MR2425133 https://doi.org/10.1007/s00222-008-0124-z

[5] BURQ, N. and TZVETKOV, N. (2008). Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.* **173** 477–496. MR2425134 https://doi.org/10.1007/s00222-008-0123-0

[6] BURQ, N. and TZVETKOV, N. (2014). Probabilistic well-posedness for the cubic wave equation. *J. Eur. Math. Soc.* (*JEMS*) **16** 1–30. MR3141727 https://doi.org/10.4171/JEMS/426

[7] CACCIAFESTA, F. and DE SUZZONI, A.-S. (2015). Invariant measure for the Schrödinger equation on the real line. *J. Funct. Anal.* **269** 271–324. MR3345610 https://doi.org/10.1016/j.jfa.2015.04.021

[8] CHANDRA, A., MOINAT, A. and WEBER, H. A priori bounds for the Φ^4 equation in the full sub-critical regime. Preprint. Available at arXiv:1910.13854.

[9] FORLANO, J. (2020). Almost sure global well posedness for the BBM equation with infinite L^2 initial data. *Discrete Contin. Dyn. Syst.* **40** 267–318. MR4026960 https://doi.org/10.3934/dcds.2020011

[10] GUBINELLI, M. and HOFMANOVÁ, M. (2019). Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. *Comm. Math. Phys.* **368** 1201–1266. MR3951704 https://doi.org/10.1007/s00220-019-03398-4

[11] GUBINELLI, M., KOCH, H. and OH, T. (2018). Renormalization of the two-dimensional stochastic nonlinear wave equations. *Trans. Amer. Math. Soc.* **370** 7335–7359. MR3841850 https://doi.org/10.1090/tran/7452

[12] GUBINELLI, M., KOCH, H., OH, T. and TOLOMEO, L. Global dynamics for the two-dimensional stochastic nonlinear wave equations. Preprint. Available at arXiv:2005.10570.

[13] HOSHINO, M. (2018). Global well-posedness of complex Ginzburg–Landau equation with a space-time white noise. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 1969–2001. MR3865664 https://doi.org/10.1214/17-AIHP862

[14] MCKEAN, H. P. and VANINSKY, K. L. (1995). Statistical mechanics of nonlinear wave equations. In *Stochastic Analysis* (*Ithaca*, *NY*, 1993). *Proc. Sympos. Pure Math.* **57** 457–463. Amer. Math. Soc., Providence, RI. MR1335489 https://doi.org/10.1090/pspum/057/1335489

[15] MOSINCAT, R., POCOVNICU, O., TOLOMEO, L. and WANG, Y. Well-posedness theory for the three-dimensional stochastic nonlinear beam equation with additive space-time white noise forcing. Preprint.

[16] MOURRAT, J.-C. and WEBER, H. (2017). Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.* **45** 2398–2476. MR3693966 https://doi.org/10.1214/16-AOP1116

MSC2020 subject classifications. 35L71, 60H15.
Key words and phrases. Stochastic nonlinear wave equation, nonlinear wave equation, renormalisation, Wick ordering, Hermite polynomial, white noise.

[17] MOURRAT, J.-C. and WEBER, H. (2017). The dynamic Φ^4_3 model comes down from infinity. *Comm. Math. Phys.* **356** 673–753. MR3719541 https://doi.org/10.1007/s00220-017-2997-4

[18] NELSON, E. (1966). A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles* (*Proc. Conf., Dedham, Mass.*, 1965) 69–73. M.I.T. Press, Cambridge, MA. MR0210416

[19] OBERGUGGENBERGER, M. and RUSSO, F. (1998). Nonlinear stochastic wave equations. *Integral Transforms Spec. Funct.* **6** 71–83. MR1640497 https://doi.org/10.1080/10652469808819152

[20] OBERGUGGENBERGER, M. and RUSSO, F. (2001). Singular limiting behavior in nonlinear stochastic wave equations. In *Stochastic Analysis and Mathematical Physics*. *Progress in Probability* **50** 87–99. Birkhäuser, Boston, MA. MR1886565

[21] OH, T., OKAMOTO, M. and ROBERT, T. (2020). A remark on triviality for the two-dimensional stochastic nonlinear wave equation. *Stochastic Process. Appl.* **130** 5838–5864. MR4127348 https://doi.org/10.1016/j.spa.2020.05.010

[22] OH, T., OKAMOTO, M. and TZVETKOV, N. Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation. Preprint.

[23] OH, T. and POCOVNICU, O. (2017). A remark on almost sure global well-posedness of the energy-critical defocusing nonlinear wave equations in the periodic setting. *Tohoku Math. J.* (2) **69** 455–481. MR3695994 https://doi.org/10.2748/tmj/1505181626

[24] OH, T., POCOVNICU, O. and TZVETKOV, N. Probabilistic local Cauchy theory of the cubic nonlinear wave equation in negative Sobolev spaces. Preprint. Available at arXiv:1904.06792.

[25] ROY, T. (2016). On the interpolation with the potential bound for global solutions of the defocusing cubic wave equation on $\mathbb{T}^2$. *J. Funct. Anal.* **270** 3280–3306. MR3475458 https://doi.org/10.1016/j.jfa.2016.02.018

[26] RUSSO, F. (1994). Colombeau generalized functions and stochastic analysis. In *Stochastic Analysis and Applications in Physics* (*Funchal*, 1993). *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **449** 329–349. Kluwer Academic, Dordrecht. MR1337971

[27] SIMON, B. (1974). *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory*. *Princeton Series in Physics*. Princeton Univ. Press, Princeton, NJ. MR0489552

The Annals of Probability
2021, Vol. 49, No. 3, 1427–1449
https://doi.org/10.1214/20-AOP1485

A NONAMENABLE "FACTOR" OF A EUCLIDEAN SPACE

BY ÁDÁM TIMÁR

Alfréd Rényi Institute of Mathematics, madaramit@gmail.com

Answering a question of Benjamini, we present an isometry-invariant random partition of the Euclidean space $\mathbb{R}^d$, $d \geq 3$, into infinite connected indistinguishable pieces, such that the adjacency graph defined on the pieces is the 3-regular infinite tree. Along the way, it is proved that any finitely generated one-ended amenable Cayley graph can be represented in $\mathbb{R}^d$ as an isometry-invariant random partition of $\mathbb{R}^d$ to bounded polyhedra, and also as an isometry-invariant random partition of $\mathbb{R}^d$ to indistinguishable pieces. A new technique is developed to prove indistinguishability for certain constructions, connecting this notion to factor of IID's.

REFERENCES

[1] ABÉRT, M., GLASNER, Y. and VIRÁG, B. (2014). Kesten's theorem for invariant random subgroups. *Duke Math. J.* **163** 465–488. MR3165420 https://doi.org/10.1215/00127094-2410064

[2] ALDOUS, D. and LYONS, R. (2007). Processes on unimodular random networks. *Electron. J. Probab.* **12** 1454–1508. MR2354165 https://doi.org/10.1214/EJP.v12-463

[3] ANGEL, O., HUTCHCROFT, T., NACHMIAS, A. and RAY, G. (2018). Hyperbolic and parabolic unimodular random maps. *Geom. Funct. Anal.* **28** 879–942. MR3820434 https://doi.org/10.1007/s00039-018-0446-y

[4] BENJAMINI, I. and SCHRAMM, O. (2001). Percolation in the hyperbolic plane. *J. Amer. Math. Soc.* **14** 487–507. MR1815220 https://doi.org/10.1090/S0894-0347-00-00362-3

[5] BENJAMINI, I. and TIMÁR, Á. (2019). Invariant embeddings of unimodular random planar graphs (preprint). arXiv:1910.01614.

[6] PETE, G. (2015). Probability and geometry on groups. Lecture notes for a graduate course, Version of 3 August 2015. http://www.math.bme.hu/~gabor/PGG.pdf.

[7] HOFFMAN, C., HOLROYD, A. E. and PERES, Y. (2006). A stable marriage of Poisson and Lebesgue. *Ann. Probab.* **34** 1241–1272. MR2257646 https://doi.org/10.1214/009117906000000098

[8] LYONS, R. and SCHRAMM, O. (1999). Indistinguishability of percolation clusters. *Ann. Probab.* **27** 1809–1836. MR1742889 https://doi.org/10.1214/aop/1022677549

[9] MARTINEAU, S. (2015). Ergodicity and indistinguishability in percolation theory. *Enseign. Math.* **61** 285–319. MR3539840 https://doi.org/10.4171/LEM/61-3/4-2

[10] ORNSTEIN, D. S. and WEISS, B. (1980). Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc.* (*N.S.*) **2** 161–164. MR0551753 https://doi.org/10.1090/S0273-0979-1980-14702-3

[11] TIMÁR, Á. (2004). Tree and grid factors for general point processes. *Electron. Commun. Probab.* **9** 53–59. MR2081459 https://doi.org/10.1214/ECP.v9-1073

[12] TIMÁR, Á. (2018). Invariant tilings and unimodular decorations of Cayley graphs. In *Unimodularity in Randomly Generated Graphs*. *Contemp. Math.* **719** 43–61. Amer. Math. Soc., Providence, RI. MR3880012 https://doi.org/10.1090/conm/719/14469

[13] TIMÁR, Á. (2019). One-ended spanning trees in amenable unimodular graphs. *Electron. Commun. Probab.* **24** Paper No. 72, 12. MR4040939 https://doi.org/10.1214/19-ecp274

MSC2020 subject classifications. Primary 60D05; secondary 20P99.
Key words and phrases. Random tiling, isometry-invariant tiling, indistinguishability, factor of IID.

The Annals of Probability
2021, Vol. 49, No. 3, 1450–1479
https://doi.org/10.1214/20-AOP1488

ADDITIVE FUNCTIONALS AS ROUGH PATHS

BY JEAN-DOMINIQUE DEUSCHEL[1], TAL ORENSHTEIN[2] AND NICOLAS PERKOWSKI[3]

[1]*Institut für Mathematik, TU Berlin, deuschel@math.tu-berlin.de*
[2]*Institut für Mathematik, TU Berlin and Weierstrass Institute Berlin, orenshtein@wias-berlin.de*
[3]*Institut für Mathematik, FU Berlin, perkowski@math.fu-berlin.de*

We consider additive functionals of stationary Markov processes and show that under Kipnis–Varadhan type conditions they converge in rough path topology to a Stratonovich Brownian motion, with a correction to the Lévy area that can be described in terms of the asymmetry (nonreversibility) of the underlying Markov process. We apply this abstract result to three model problems: First, we study random walks with random conductances under the annealed law. If we consider the Itô rough path, then we see a correction to the iterated integrals even though the underlying Markov process is reversible. If we consider the Stratonovich rough path, then there is no correction. The second example is a nonreversible Ornstein–Uhlenbeck process, while the last example is a diffusion in a periodic environment.

As a technical step, we prove an estimate for the p-variation of stochastic integrals with respect to martingales that can be viewed as an extension of the rough path Burkholder–Davis–Gundy inequality for local martingale rough paths of (In *Séminaire de Probabilités XLI* (2008) 421–438 Springer; In *Probability and Analysis in Interacting Physical Systems* (2019) 17–48 Springer; *J. Differential Equations* **264** (2018) 6226–6301) to the case where only the integrator is a local martingale.

REFERENCES

[1] BAILLEUL, I. and CATELLIER, R. (2017). Rough flows and homogenization in stochastic turbulence. *J. Differential Equations* **263** 4894–4928. MR3680942 https://doi.org/10.1016/j.jde.2017.06.006

[2] BRUNED, Y., CHANDRA, A., CHEVYREV, I. and HAIRER, M. Renormalising spdes in regularity structures. *J. Eur. Math. Soc.* (*JEMS*). To appear.

[3] BRUNED, Y., HAIRER, M. and ZAMBOTTI, L. (2019). Algebraic renormalisation of regularity structures. *Invent. Math.* **215** 1039–1156. MR3935036 https://doi.org/10.1007/s00222-018-0841-x

[4] CHANDRA, A. and HAIRER, M. (2016). An analytic BPHZ theorem for regularity structures. Preprint. Available at arXiv:1612.08138.

[5] CHEVYREV, I., FRIZ, P., KOREPANOV, A., MELBOURNE, I. and ZHANG, H. (2019). Deterministic homogenization for discrete-time fast-slow systems under optimal moment assumptions. Preprint. Available at arXiv:1903.10418.

[6] CHEVYREV, I. and FRIZ, P. K. (2019). Canonical RDEs and general semimartingales as rough paths. *Ann. Probab.* **47** 420–463. MR3909973 https://doi.org/10.1214/18-AOP1264

[7] CHEVYREV, I., FRIZ, P. K., KOREPANOV, A. and MELBOURNE, I. (2020). Superdiffusive limits for deterministic fast-slow dynamical systems. *Probab. Theory Related Fields* **178** 735–770. MR4168387 https://doi.org/10.1007/s00440-020-00988-5

[8] CHEVYREV, I., FRIZ, P. K., KOREPANOV, A., MELBOURNE, I. and ZHANG, H. (2019). Multiscale systems, homogenization, and rough paths. In *Probability and Analysis in Interacting Physical Systems*. *Springer Proc. Math. Stat.* **283** 17–48. Springer, Cham. MR3968507 https://doi.org/10.1007/978-3-030-15338-0_2

[9] FRIZ, P., GASSIAT, P. and LYONS, T. (2015). Physical Brownian motion in a magnetic field as a rough path. *Trans. Amer. Math. Soc.* **367** 7939–7955. MR3391905 https://doi.org/10.1090/S0002-9947-2015-06272-2

MSC2020 subject classifications. 60L20, 60K37, 60F17, 82C41, 82B43.

Key words and phrases. Rough paths, homogenization, additive functionals of Markov processes, random walks among random conductances, Lépingle's Burkholder–Davis–Gundy inequality in p-variation, UCV condition.

[10] FRIZ, P. and VICTOIR, N. (2008). The Burkholder–Davis–Gundy inequality for enhanced martingales. In *Séminaire de Probabilités XLI*. *Lecture Notes in Math*. **1934** 421–438. Springer, Berlin. MR2483743 https://doi.org/10.1007/978-3-540-77913-1_20

[11] FRIZ, P. and ZORIN-KRANICH, P. (2020). Rough semimartingales and p-variation estimates for martingale transforms. Preprint. Available at arXiv:2008.08897.

[12] FRIZ, P. K. and ZHANG, H. (2018). Differential equations driven by rough paths with jumps. *J. Differential Equations* **264** 6226–6301. MR3770049 https://doi.org/10.1016/j.jde.2018.01.031

[13] GLORIA, A., NEUKAMM, S. and OTTO, F. (2015). Quantification of ergodicity in stochastic homogenization: Optimal bounds via spectral gap on Glauber dynamics. *Invent. Math*. **199** 455–515. MR3302119 https://doi.org/10.1007/s00222-014-0518-z

[14] GUBINELLI, M. and PERKOWSKI, N. (2018). Energy solutions of KPZ are unique. *J. Amer. Math. Soc*. **31** 427–471. MR3758149 https://doi.org/10.1090/jams/889

[15] KELLY, D. (2016). Rough path recursions and diffusion approximations. *Ann. Appl. Probab.* **26** 425–461. MR3449323 https://doi.org/10.1214/15-AAP1096

[16] KELLY, D. and MELBOURNE, I. (2016). Smooth approximation of stochastic differential equations. *Ann. Probab.* **44** 479–520. MR3456344 https://doi.org/10.1214/14-AOP979

[17] KELLY, D. and MELBOURNE, I. (2017). Deterministic homogenization for fast-slow systems with chaotic noise. *J. Funct. Anal*. **272** 4063–4102. MR3626033 https://doi.org/10.1016/j.jfa.2017.01.015

[18] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys*. **104** 1–19. MR0834478

[19] KOMOROWSKI, T., LANDIM, C. and OLLA, S. (2012). *Fluctuations in Markov Processes*: *Time Symmetry and Martingale Approximation*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **345**. Springer, Heidelberg. MR2952852 https://doi.org/10.1007/978-3-642-29880-6

[20] KURTZ, T. G. and PROTTER, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070. MR1112406

[21] LEJAY, A. and LYONS, T. (2005). On the importance of the Lévy area for studying the limits of functions of converging stochastic processes. Application to homogenization. In *Current Trends in Potential Theory*. *Theta Ser. Adv. Math*. **4** 63–84. Theta, Bucharest. MR2243956

[22] LÉPINGLE, D. (1975). Sur la variation d'ordre p des martingales locales. *C. R. Acad. Sci. Paris Sér. A-B* **281** A917–A919. MR0418225

[23] LOPUSANSCHI, O. and ORENSHTEIN, T. (2018). Ballistic random walks in random environment as rough paths: Convergence and area anomaly. Preprint. Available at arXiv:1812.01403.

[24] LOPUSANSCHI, O. and SIMON, D. (2018). Lévy area with a drift as a renormalization limit of Markov chains on periodic graphs. *Stochastic Process. Appl*. **128** 2404–2426. MR3804798 https://doi.org/10.1016/j.spa.2017.09.004

[25] MOURRAT, J.-C. (2012). A quantitative central limit theorem for the random walk among random conductances. *Electron. J. Probab.* **17** no. 97, 17. MR2994845 https://doi.org/10.1214/EJP.v17-2414

[26] ORENSHTEIN, T. (2021). Rough invariance principle for delayed regenerative processes. Preprint. Available at arXiv:2101.05222.

[27] PELIGRAD, M. (2020). A new CLT for additive functionals of Markov chains. *Stochastic Process. Appl.* **130** 5695–5708. MR4127343 https://doi.org/10.1016/j.spa.2020.04.004

[28] PERKOWSKI, N. and PRÖMEL, D. J. (2016). Pathwise stochastic integrals for model free finance. *Bernoulli* **22** 2486–2520. MR3498035 https://doi.org/10.3150/15-BEJ735

The Annals of Probability
2021, Vol. 49, No. 3, 1480–1514
https://doi.org/10.1214/20-AOP1489

THE CONTACT PROCESS ON RANDOM HYPERBOLIC GRAPHS: METASTABILITY AND CRITICAL EXPONENTS

BY AMITAI LINKER[1,*], DIETER MITSCHE[1,†], BRUNO SCHAPIRA[2] AND DANIEL VALESIN[3]

[1]*Institut Camille Jordan, Université Lyon, UMR 5209, Univ. Jean Monnet,* [*]*amitailinker@gmail.com;* [†]*dmitsche@unice.fr*
[2]*Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, bruno.schapira@univ-amu.fr*
[3]*Johann Bernoulli Institute, University of Groningen, d.rodrigues.valesin@rug.nl*

We consider the contact process on the model of hyperbolic random graph, in the regime when the degree distribution obeys a power law with exponent $\chi \in (1, 2)$ (so that the degree distribution has finite mean and infinite second moment). We show that the probability of nonextinction as the rate of infection goes to zero decays as a power law with an exponent that only depends on χ and which is the same as in the configuration model, suggesting some universality of this critical exponent. We also consider finite versions of the hyperbolic graph and prove metastability results, as the size of the graph goes to infinity.

REFERENCES

[1] ABDULLAH, M. A., BODE, M. and FOUNTOULAKIS, N. (2017). Typical distances in a geometric model for complex networks. *Internet Math*. 38. MR3708706

[2] ALBERT, R. and BARABÁSI, A.-L. (2002). Statistical mechanics of complex networks. *Rev. Modern Phys*. **74** 47–97. MR1895096 https://doi.org/10.1103/RevModPhys.74.47

[3] BERGER, N., BORGS, C., CHAYES, J. T. and SABERI, A. (2005). On the spread of viruses on the Internet. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms* 301–310. ACM, New York. MR2298278

[4] BERGER, N., BORGS, C., CHAYES, J. T. and SABERI, A. (2014). Asymptotic behavior and distributional limits of preferential attachment graphs. *Ann. Probab.* **42** 1–40. MR3161480 https://doi.org/10.1214/12-AOP755

[5] BHAMIDI, S., NAM, D., NGUYEN, O. and SLY, A. (2020). Survival and extinction of epidemics on random graphs with general degrees. *Ann. Probab.* to appear.

[6] BODE, M., FOUNTOULAKIS, N. and MÜLLER, T. (2015). On the largest component of a hyperbolic model of complex networks. *Electron. J. Combin*. **22** Paper 3.24, 46. MR3386525 https://doi.org/10.37236/4958

[7] BODE, M., FOUNTOULAKIS, N. and MÜLLER, T. (2016). The probability of connectivity in a hyperbolic model of complex networks. *Random Structures Algorithms* **49** 65–94. MR3521274 https://doi.org/10.1002/rsa.20626

[8] BOGUÑÁ, M., PAPADOPOULOS, F. and KRIOUKOV, D. (2010). Sustaining the Internet with hyperbolic mapping. *Nat. Commun*. **1** 62. https://doi.org/10.1038/ncomms1063

[9] BRINGMANN, K., KEUSCH, R. and LENGLER, J. (2019). Geometric inhomogeneous random graphs. *Theoret. Comput. Sci*. **760** 35–54. MR3913223 https://doi.org/10.1016/j.tcs.2018.08.014

[10] CAN, V. H. (2017). Metastability for the contact process on the preferential attachment graph. *Internet Math*. 45. MR3683430 https://doi.org/10.24166/im.08.2017

[11] CAN, V. H. (2019). Exponential extinction time of the contact process on rank-one inhomogeneous random graphs. *J. Theoret. Probab.* **32** 106–130. MR3908908 https://doi.org/10.1007/s10959-017-0786-9

[12] CAN, V. H. and SCHAPIRA, B. (2015). Metastability for the contact process on the configuration model with infinite mean degree. *Electron. J. Probab.* **20** no. 26, 22. MR3325096 https://doi.org/10.1214/EJP.v20-3859

[13] CANDELLERO, E. and FOUNTOULAKIS, N. (2016). Clustering and the hyperbolic geometry of complex networks. *Internet Math*. **12** 2–53. MR3474052 https://doi.org/10.1080/15427951.2015.1067848

MSC2020 subject classifications. Primary 82C22; secondary 05C80.
Key words and phrases. Contact process, random hyperbolic graphs.

[14] CASSANDRO, M., GALVES, A., OLIVIERI, E. and VARES, M. E. (1984). Metastable behavior of stochastic dynamics: A pathwise approach. *J. Stat. Phys.* **35** 603–634. MR0749840 https://doi.org/10.1007/BF01010826

[15] CHATTERJEE, S. and DURRETT, R. (2009). Contact processes on random graphs with power law degree distributions have critical value 0. *Ann. Probab.* **37** 2332–2356. MR2573560 https://doi.org/10.1214/09-AOP471

[16] CRANSTON, M., MOUNTFORD, T., MOURRAT, J.-C. and VALESIN, D. (2014). The contact process on finite homogeneous trees revisited. *ALEA Lat. Am. J. Probab. Math. Stat.* **11** 385–408. MR3249416

[17] DEIJFEN, M., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2013). Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 817–838. MR3112435 https://doi.org/10.1214/12-AIHP480

[18] DIEL, R. and MITSCHE, D. On the largest component of subcritical random hyperbolic graphs. Preprint available at https://arxiv.org/pdf/2003.02156.pdf.

[19] DURRETT, R. and SCHONMANN, R. H. (1988). The contact process on a finite set. II. *Ann. Probab.* **16** 1570–1583. MR0958203

[20] FOUNTOULAKIS, N. (2015). On a geometrization of the Chung–Lu model for complex networks. *J. Complex Netw.* **3** 361–387. MR3449864 https://doi.org/10.1093/comnet/cnu049

[21] FOUNTOULAKIS, N. and MÜLLER, T. (2018). Law of large numbers for the largest component in a hyperbolic model of complex networks. *Ann. Appl. Probab.* **28** 607–650. MR3770885 https://doi.org/10.1214/17-AAP1314

[22] FOUNTOULAKIS, N., VAN DER HOORN, P., MÜLLER, T. and SCHEPERS, M. Clustering in a hyperbolic model of complex networks. Preprint available at https://arxiv.org/pdf/2003.05525.pdf.

[23] FOUNTOULAKIS, N. and YUKICH, J. (2020). Limit theory for isolated and extreme points in hyperbolic random geometric graphs. *Electron. J. Probab.* **25** 1–51. MR4186260 https://doi.org/10.1214/20-ejp531

[24] GUGELMANN, L., PANAGIOTOU, K. and PETER, U. (2012). Random hyperbolic graphs: Degree sequence and clustering. In *Automata, Languages, and Programming—39th International Colloquium—ICALP Part II* **7392** 573–585.

[25] HUANG, X. and DURRETT, R. (2020). The contact process on periodic trees. *Electron. Commun. Probab.* **25** Paper No. 24, 12. MR4089731 https://doi.org/10.1214/20-ecp305

[26] JACOB, E., LINKER, A. and MÖRTERS, P. (2019). Metastability of the contact process on fast evolving scale-free networks. *Ann. Appl. Probab.* **29** 2654–2699. MR4019872 https://doi.org/10.1214/18-AAP1460

[27] KIWI, M. and MITSCHE, D. (2018). Spectral gap of random hyperbolic graphs and related parameters. *Ann. Appl. Probab.* **28** 941–989. MR3784493 https://doi.org/10.1214/17-AAP1323

[28] KIWI, M. and MITSCHE, D. (2019). On the second largest component of random hyperbolic graphs. *SIAM J. Discrete Math.* **33** 2200–2217. MR4032854 https://doi.org/10.1137/18M121201X

[29] KRIOUKOV, D., PAPADOPOULOS, F., KITSAK, M., VAHDAT, A. and BOGUÑÁ, M. (2010). Hyperbolic geometry of complex networks. *Phys. Rev. E* (3) **82** 036106, 18. MR2787998 https://doi.org/10.1103/PhysRevE.82.036106

[30] LALLEY, S. and SU, W. (2017). Contact processes on random regular graphs. *Ann. Appl. Probab.* **27** 2061–2097. MR3693520 https://doi.org/10.1214/16-AAP1249

[31] LAST, G. and PENROSE, M. (2018). *Lectures on the Poisson Process. Institute of Mathematical Statistics Textbooks* **7**. Cambridge Univ. Press, Cambridge. MR3791470

[32] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346 https://doi.org/10.1007/978-3-662-03990-8

[33] MOUNTFORD, T., MOURRAT, J.-C., VALESIN, D. and YAO, Q. (2016). Exponential extinction time of the contact process on finite graphs. *Stochastic Process. Appl.* **126** 1974–2013. MR3483744 https://doi.org/10.1016/j.spa.2016.01.001

[34] MOUNTFORD, T., VALESIN, D. and YAO, Q. (2013). Metastable densities for the contact process on power law random graphs. *Electron. J. Probab.* **18** No. 103, 36. MR3145050 https://doi.org/10.1214/EJP.v18-2512

[35] MOUNTFORD, T. S. (1993). A metastable result for the finite multidimensional contact process. *Canad. Math. Bull.* **36** 216–226. MR1222537 https://doi.org/10.4153/CMB-1993-031-3

[36] MOURRAT, J. C. and VALESIN, D. (2018). Phase transition of the contact process on random regular graphs. *Ann. Appl. Probab.* **28** 751–789.

[37] MÜLLER, T. and STAPS, M. (2019). The diameter of KPKVB random graphs. *Adv. in Appl. Probab.* **51** 358–377. MR3989518 https://doi.org/10.1017/apr.2019.23

[38] SCHAPIRA, B. and VALESIN, D. (2017). Extinction time for the contact process on general graphs. *Probab. Theory Related Fields* **169** 871–899. MR3719058 https://doi.org/10.1007/s00440-016-0742-0

[39] STACEY, A. (2001). The contact process on finite homogeneous trees. *Probab. Theory Related Fields* **121** 551–576. MR1872428 https://doi.org/10.1007/s004400100149

The Annals of Probability
2021, Vol. 49, No. 3, 1515–1553
https://doi.org/10.1214/20-AOP1490

ANTI-CONCENTRATION FOR SUBGRAPH COUNTS IN RANDOM GRAPHS

BY JACOB FOX[1,*], MATTHEW KWAN[1,†] AND LISA SAUERMANN[2]

[1]*Department of Mathematics, Stanford University,* [*]*jacobfox@stanford.edu;* [†]*mattkwan@stanford.edu*
[2]*School of Mathematics, Institute for Advanced Study, lsauerma@stanford.edu*

Fix a graph H and some $p \in (0, 1)$, and let X_H be the number of copies of H in a random graph $\mathbb{G}(n, p)$. Random variables of this form have been intensively studied since the foundational work of Erdős and Rényi. There has been a great deal of progress over the years on the large-scale behaviour of X_H, but the more challenging problem of understanding the small-ball probabilities has remained poorly understood until now. More precisely, how likely can it be that X_H falls in some small interval or is equal to some particular value? In this paper, we prove the almost-optimal result that if H is connected then for any $x \in \mathbb{N}$ we have $\Pr(X_H = x) \le n^{1-v(H)+o(1)}$. Our proof proceeds by iteratively breaking X_H into different components which fluctuate at "different scales", and relies on a new anti-concentration inequality for random vectors that behave "almost linearly."

REFERENCES

[1] ALON, N. and SPENCER, J. H. (2016). *The Probabilistic Method*, 4th ed. *Wiley Series in Discrete Mathematics and Optimization*. Wiley, Hoboken, NJ. MR3524748

[2] AUGERI, F. (2020). Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 2404–2448. MR4152647 https://doi.org/10.1214/20-AOP1427

[3] BARBOUR, A. D. (1982). Poisson convergence and random graphs. *Math. Proc. Cambridge Philos. Soc.* **92** 349–359. MR0671189 https://doi.org/10.1017/S0305004100059995

[4] BARBOUR, A. D., KAROŃSKI, M. and RUCIŃSKI, A. (1989). A central limit theorem for decomposable random variables with applications to random graphs. *J. Combin. Theory Ser. B* **47** 125–145. MR1047781 https://doi.org/10.1016/0095-8956(89)90014-2

[5] BASAK, A. and BASU, R. (2019). Upper tail large deviations of regular subgraph counts in Erdős–Rényi graphs in the full localized regime. Preprint. Available at arXiv:1912.11410.

[6] BERKOWITZ, R. (2016). A quantitative local limit theorem for triangles in random graphs. Preprint. Available at arXiv:1610.01281.

[7] BERKOWITZ, R. (2018). A local limit theorem for cliques in $G(n, p)$. Preprint. Available at arXiv:1811.03527.

[8] BHATTACHARYA, B. B., GANGULY, S., LUBETZKY, E. and ZHAO, Y. (2017). Upper tails and independence polynomials in random graphs. *Adv. Math.* **319** 313–347. MR3695877 https://doi.org/10.1016/j.aim.2017.08.003

[9] BOLLOBÁS, B. (1981). Threshold functions for small subgraphs. *Math. Proc. Cambridge Philos. Soc.* **90** 197–206. MR0620729 https://doi.org/10.1017/S0305004100058655

[10] BOLLOBÁS, B. and WIERMAN, J. C. (1989). Subgraph counts and containment probabilities of balanced and unbalanced subgraphs in a large random graph. In *Graph Theory and Its Applications*: *East and West* (*Jinan*, 1986). *Ann. New York Acad. Sci.* **576** 63–70. New York Acad. Sci., New York. MR1110801 https://doi.org/10.1111/j.1749-6632.1989.tb16383.x

[11] CHATTERJEE, S. (2017). *Large Deviations for Random Graphs*. *Lecture Notes in Math.* **2197**. Springer, Cham. Lecture notes from the 45th Probability Summer School held in Saint-Flour, June 2015, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. MR3700183 https://doi.org/10.1007/978-3-319-65816-2

[12] CHATTERJEE, S. and DEMBO, A. (2016). Nonlinear large deviations. *Adv. Math.* **299** 396–450. MR3519474 https://doi.org/10.1016/j.aim.2016.05.017

[13] COOK, N. and DEMBO, A. (2020). Large deviations of subgraph counts for sparse Erdős–Rényi graphs. *Adv. Math.* **373** 107289, 53. MR4130460 https://doi.org/10.1016/j.aim.2020.107289

MSC2020 subject classifications. 05C80, 60C05.
Key words and phrases. Random graphs, anti-concentration.

[14] DEMARCO, B., KAHN, J. and REDLICH, A. (2015). Modular statistics for subgraph counts in sparse random graphs. *Electron. J. Combin.* **22** Paper 1.37, 6. MR3315479 https://doi.org/10.37236/4094

[15] DEMARCO, B. and REDLICH, A. (2016). Graph decomposition and parity. *J. Graph Theory* **82** 374–386. MR3515574 https://doi.org/10.1002/jgt.21907

[16] ELDAN, R. (2018). Gaussian-width gradient complexity, reverse log-Sobolev inequalities and nonlinear large deviations. *Geom. Funct. Anal.* **28** 1548–1596. MR3881829 https://doi.org/10.1007/s00039-018-0461-z

[17] ERDÖS, P. (1945). On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.* **51** 898–902. MR0014608 https://doi.org/10.1090/S0002-9904-1945-08454-7

[18] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magy. Tud. Akad. Mat. Kut. Intéz. Közl.* **5** 17–61. MR0125031

[19] FOX, J., KWAN, M. and SAUERMANN, L. Combinatorial anti-concentration inequalities, with applications. *Math. Proc. Cambridge Philos. Soc.* To appear.

[20] GILMER, J. and KOPPARTY, S. (2016). A local central limit theorem for triangles in a random graph. *Random Structures Algorithms* **48** 732–750. MR3508725 https://doi.org/10.1002/rsa.20604

[21] HALÁSZ, G. (1977). Estimates for the concentration function of combinatorial number theory and probability. *Period. Math. Hungar.* **8** 197–211. MR0494478 https://doi.org/10.1007/BF02018403

[22] HAREL, M., MOUSSET, F. and SAMOTIJ, W. (2019). Upper tails via high moments and entropic stability. Preprint. Available at arXiv:1904.08212.

[23] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization*. Wiley Interscience, New York. MR1782847 https://doi.org/10.1002/9781118032718

[24] KAROŃSKI, M. (1984). *Balanced Subgraphs of Large Random Graphs. Seria Matematyka* [*Mathematics Series*] **7**. Uniwersytet im. Adama Mickiewicza w Poznaniu, Poznań. MR0779093

[25] KAROŃSKI, M. and RUCIŃSKI, A. (1983). On the number of strictly balanced subgraphs of a random graph. In *Graph Theory* (*Łagów*, 1981). *Lecture Notes in Math.* **1018** 79–83. Springer, Berlin. MR0730636 https://doi.org/10.1007/BFb0071616

[26] KOLAITIS, P. G. and KOPPARTY, S. (2013). Random graphs and the parity quantifier. *J. ACM* **60** Art. 37, 34. MR3124686 https://doi.org/10.1145/2528402

[27] LOEBL, M., MATOUŠEK, J. and PANGRÁC, O. (2004). Triangles in random graphs. *Discrete Math.* **289** 181–185. MR2106042 https://doi.org/10.1016/j.disc.2004.08.008

[28] LUBELL, D. (1966). A short proof of Sperner's lemma. *J. Combin. Theory* **1** 299. MR0194348

[29] LUBETZKY, E. and ZHAO, Y. (2017). On the variational problem for upper tails in sparse random graphs. *Random Structures Algorithms* **50** 420–436. MR3632418 https://doi.org/10.1002/rsa.20658

[30] MEKA, R., NGUYEN, O. and VU, V. (2016). Anti-concentration for polynomials of independent random variables. *Theory Comput.* **12** Paper No. 11, 16. MR3542863 https://doi.org/10.4086/toc.2016.v012a011

[31] NGUYEN, H. H. and VU, V. H. (2013). Small ball probability, inverse theorems, and applications. In *Erdös Centennial. Bolyai Soc. Math. Stud.* **25** 409–463. János Bolyai Math. Soc., Budapest. MR3203607 https://doi.org/10.1007/978-3-642-39286-3_16

[32] NOWICKI, K. and WIERMAN, J. C. (1988). Subgraph counts in random graphs using incomplete U-statistics methods. In *Proceedings of the First Japan Conference on Graph Theory and Applications* (*Hakone*, 1986) **72** 299–310. MR0975550 https://doi.org/10.1016/0012-365X(88)90220-8

[33] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[34] RUCIŃSKI, A. (1988). When are small subgraphs of a random graph normally distributed? *Probab. Theory Related Fields* **78** 1–10. MR0940863 https://doi.org/10.1007/BF00718031

[35] RUCIŃSKI, A. and VINCE, A. (1985). Balanced graphs and the problem of subgraphs of random graphs. In *Proceedings of the Sixteenth Southeastern International Conference on Combinatorics, Graph Theory and Computing* (*Boca Raton, Fla.*, 1985) **49** 181–190. MR0830741

[36] SAH, A. and SAWHNEY, M. (2020). Local limit theorems for subgraph counts. Preprint. Available at arXiv:2006.11369.

[37] SCHÜRGER, K. (1979). Limit theorems for complete subgraphs of random graphs. *Period. Math. Hungar.* **10** 47–53. MR0505825 https://doi.org/10.1007/BF02018372

[38] TAO, T. and VU, V. (2012). The Littlewood–Offord problem in high dimensions and a conjecture of Frankl and Füredi. *Combinatorica* **32** 363–372. MR2965282 https://doi.org/10.1007/s00493-012-2716-x

The Annals of Probability
2021, Vol. 49, No. 3, 1554–1606
https://doi.org/10.1214/20-AOP1494

THE ANNEALED SPECTRAL SAMPLE OF VORONOI PERCOLATION

BY HUGO VANNEUVILLE

ICJ, Université Lyon 1, vanneuville@math.univ-lyon1.fr

In this paper, we introduce and study the annealed spectral sample of Voronoi percolation, which is a continuous and finite point process in $\mathbb{R}^2$ whose definition is mostly inspired by the spectral sample of Bernoulli percolation introduced in (*Acta Math.* **205** (2010) 19–104) by Garban, Pete and Schramm. We show a clustering effect as well as estimates on the full lower tail of this spectral object.

Our main motivation is the study of two models of dynamical critical Voronoi percolation in the plane. In the first model, the Voronoi tiling does not evolve in time while the colors of the cells are resampled at rate 1. In the second model, the centers of the cells move according to (independent) long range stable Lévy processes but the colors do not evolve in time. We prove that for these two dynamical processes there exist almost surely exceptional times with an unbounded monochromatic component.

REFERENCES

[1] AHLBERG, D. and BALDASSO, R. (2018). Noise sensitivity and Voronoi percolation. *Electron. J. Probab.* **23** Paper No. 108, 21. MR3878133 https://doi.org/10.1214/18-ejp233

[2] AHLBERG, D., BROMAN, E., GRIFFITHS, S. and MORRIS, R. (2014). Noise sensitivity in continuum percolation. *Israel J. Math.* **201** 847–899. MR3265306 https://doi.org/10.1007/s11856-014-1038-y

[3] AHLBERG, D., GRIFFITHS, S., MORRIS, R. and TASSION, V. (2016). Quenched Voronoi percolation. *Adv. Math.* **286** 889–911. MR3415699 https://doi.org/10.1016/j.aim.2015.09.005

[4] BENJAMINI, I., KALAI, G. and SCHRAMM, O. (1999). Noise sensitivity of Boolean functions and applications to percolation. *Publ. Math. Inst. Hautes Études Sci.* **90** 5–43 (2001). MR1813223

[5] BENJAMINI, I. and SCHRAMM, O. (1998). Conformal invariance of Voronoi percolation. *Comm. Math. Phys.* **197** 75–107. MR1646475 https://doi.org/10.1007/s002200050443

[6] BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[7] BOLLOBÁS, B. and RIORDAN, O. (2006). The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields* **136** 417–468. MR2257131 https://doi.org/10.1007/s00440-005-0490-z

[8] BROMAN, E. I., GARBAN, C. and STEIF, J. E. (2013). Exclusion sensitivity of Boolean functions. *Probab. Theory Related Fields* **155** 621–663. MR3034789 https://doi.org/10.1007/s00440-011-0409-9

[9] DALEY, D. J. and VERE-JONES, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I: Elementary Theory and Methods*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1950431

[10] DUMINIL-COPIN, H., RAOUFI, A. and TASSION, V. (2019). Exponential decay of connection probabilities for subcritical Voronoi percolation in $\mathbb{R}^d$. *Probab. Theory Related Fields* **173** 479–490. MR3916112 https://doi.org/10.1007/s00440-018-0838-9

[11] GARBAN, C., PETE, G. and SCHRAMM, O. (2010). The Fourier spectrum of critical percolation. *Acta Math.* **205** 19–104. MR2736153 https://doi.org/10.1007/s11511-010-0051-x

[12] GARBAN, C. and STEIF, J. E. (2015). *Noise Sensitivity of Boolean Functions and Percolation. Institute of Mathematical Statistics Textbooks* **5**. Cambridge Univ. Press, New York. MR3468568 https://doi.org/10.1017/CBO9781139924160

[13] GARBAN, C. and VANNEUVILLE, H. (2019). Exceptional times for percolation under exclusion dynamics. *Ann. Sci. Éc. Norm. Supér.* (4) **52** 1–57. MR3940906 https://doi.org/10.24033/asens.2383

[14] HÄGGSTRÖM, O., PERES, Y. and STEIF, J. E. (1997). Dynamical percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 497–528. MR1465800 https://doi.org/10.1016/S0246-0203(97)80103-3

MSC2020 subject classifications. Primary 60K35, 60K37; secondary 82C43.
Key words and phrases. Percolation, Voronoi, annealed, Fourier spectrum, exceptional times.

[15] SCHRAMM, O. and STEIF, J. E. (2010). Quantitative noise sensitivity and exceptional times for percolation. *Ann. of Math.* (2) **171** 619–672. MR2630053 https://doi.org/10.4007/annals.2010.171.619

[16] TASSION, V. (2016). Crossing probabilities for Voronoi percolation. *Ann. Probab.* **44** 3385–3398. MR3551200 https://doi.org/10.1214/15-AOP1052

[17] VAN DEN BERG, J., MEESTER, R. and WHITE, D. G. (1997). Dynamic Boolean models. *Stochastic Process. Appl.* **69** 247–257. MR1472953 https://doi.org/10.1016/S0304-4149(97)00044-6

[18] VANNEUVILLE, H. (2018). Quantitative quenched Voronoi percolation and applications. arXiv preprint, arXiv:1806.08448.

[19] VANNEUVILLE, H. (2019). Annealed scaling relations for Voronoi percolation. *Electron. J. Probab.* **24** Paper No. 39, 71. MR3940769 https://doi.org/10.1214/19-EJP293

[20] ZVAVITCH, A. (1996). The critical probability for Voronoi percolation. MSc. thesis, Weizmann Inst. of Science.

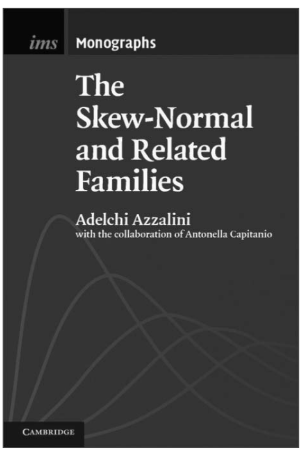
The Annals of Probability

Vol. 49 **July 2021** **No. 4**

Articles

The Institute of Mathematical Statistics presents

IMS MONOGRAPHS



The Skew-Normal and Related Families

Adelchi Azzalini
in collaboration with Antonella Capitanio

Interest in the skew-normal and related families of distributions has grown enormously over recent years, as theory has advanced, challenges of data have grown, and computational tools have made substantial progress. This comprehensive treatment, blending theory and practice, will be the standard resource for statisticians and applied researchers. Assuming only basic knowledge of (non-measure-theoretic) probability and statistical inference, the book is accessible to the wide range of researchers who use statistical modelling techniques. Guiding readers through the main concepts and results, it covers both the probability and the statistics sides of the subject, in the univariate and multivariate settings. The theoretical development is complemented by numerous illustrations and applications to a range of fields including quantitative finance, medical statistics, environmental risk studies, and industrial and business efficiency.

The author's freely available R package sn, available from CRAN, equips readers to put the methods into action with their own data.

IMS member? Claim your 40% discount:
www.cambridge.org/ims

Hardback price US$48.00
(non-member price $80.00)

Cambridge University Press, in conjunction with the Institute of Mathematical Statistics, established the IMS Monographs and IMS Textbooks series of high-quality books. The Series Editors are Xiao-Li Meng, Susan Holmes, Ben Hambly, D. R. Cox and Alan Agresti.