ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 49, No. 4—July 2021

THE ANNALS OF PROBABILITY Vol. 49, No. 4, pp. 1607–2140 July 2021

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 49, Number 4, July 2021. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Probability
2021, Vol. 49, No. 4, 1607–1632
https://doi.org/10.1214/20-AOP1463

COLOR-POSITION SYMMETRY IN INTERACTING PARTICLE SYSTEMS

BY ALEXEI BORODIN[1] AND ALEXEY BUFETOV[2]

[1]*Department of Mathematics, MIT, borodin@math.mit.edu*
[2]*Hausdorff Center for Mathematics & Institute for Applied Mathematics, University of Bonn, alexey.bufetov@gmail.com*

We prove a color-position symmetry for a class of ASEP-like interacting particle systems with discrete time on the one-dimensional lattice. The full space-time inhomogeneity of our systems allows to apply the result to colored (or multi-species) ASEP and stochastic vertex models for a certain class of initial/boundary conditions, generalizing previous results of Amir–Angel–Valko and Borodin–Wheeler. We are also able to use the symmetry, together with previously known results for uncolored models, to find novel asymptotic behavior of the second-class particles in several situations.

REFERENCES

[1] AGGARWAL, A. (2018). Current fluctuations of the stationary ASEP and six-vertex model. *Duke Math. J.* **167** 269–384. MR3754630 https://doi.org/10.1215/00127094-2017-0029
[2] AGGARWAL, A. (2020). Limit shapes and local statistics for the stochastic six-vertex model. *Comm. Math. Phys.* **376** 681–746. MR4093866 https://doi.org/10.1007/s00220-019-03643-w
[3] AGGARWAL, A. and BORODIN, A. (2019). Phase transitions in the ASEP and stochastic six-vertex model. *Ann. Probab.* **47** 613–689. MR3916931 https://doi.org/10.1214/17-AOP1253
[4] AMIR, G., ANGEL, O. and VALKÓ, B. (2011). The TASEP speed process. *Ann. Probab.* **39** 1205–1242. MR2857238 https://doi.org/10.1214/10-AOP561
[5] ANDJEL, E. D. and VARES, M. E. (1987). Hydrodynamic equations for attractive particle systems on $\mathbb{Z}$. *J. Stat. Phys.* **47** 265–288. MR0892931 https://doi.org/10.1007/BF01009046
[6] ANGEL, O., HOLROYD, A. and ROMIK, D. (2009). The oriented swap process. *Ann. Probab.* **37** 1970–1998. MR2561438 https://doi.org/10.1214/09-AOP456
[7] BAHADORAN, C. (1998). Hydrodynamical limit for spatially heterogeneous simple exclusion processes. *Probab. Theory Related Fields* **110** 287–331. MR1616563 https://doi.org/10.1007/s004400050150
[8] BENASSI, A. and FOUQUE, J.-P. (1987). Hydrodynamical limit for the asymmetric simple exclusion process. *Ann. Probab.* **15** 546–560. MR0885130
[9] BORODIN, A. (2018). Stochastic higher spin six vertex model and Macdonald measures. *J. Math. Phys.* **59** 023301. MR3760963 https://doi.org/10.1063/1.5000046
[10] BORODIN, A., BUFETOV, A. and WHEELER, M. Between the stochastix six vertex model and Hall–Littlewood processes. *J. Combin. Theory Ser. A*. To appear. Available at arXiv:1611.09486.
[11] BORODIN, A. and CORWIN, I. (2014). Macdonald processes. *Probab. Theory Related Fields* **158** 225–400. MR3152785 https://doi.org/10.1007/s00440-013-0482-3
[12] BORODIN, A., CORWIN, I. and GORIN, V. (2016). Stochastic six-vertex model. *Duke Math. J.* **165** 563–624. MR3466163 https://doi.org/10.1215/00127094-3166843
[13] BORODIN, A. and FERRARI, P. L. (2008). Large time asymptotics of growth models on space-like paths. I. PushASEP. *Electron. J. Probab.* **13** 1380–1418. MR2438811 https://doi.org/10.1214/EJP.v13-541
[14] BORODIN, A., FERRARI, P. L. and SASAMOTO, T. (2008). Transition between Airy_1 and Airy_2 processes and TASEP fluctuations. *Comm. Pure Appl. Math.* **61** 1603–1629. MR2444377 https://doi.org/10.1002/cpa.20234
[15] BORODIN, A. and GORIN, V. (2019). A stochastic telegraph equation from the six-vertex model. *Ann. Probab.* **47** 4137–4194. MR4038051 https://doi.org/10.1214/19-aop1356
[16] BORODIN, A. and OLSHANSKI, G. (2017). The ASEP and determinantal point processes. *Comm. Math. Phys.* **353** 853–903. MR3649488 https://doi.org/10.1007/s00220-017-2858-1
[17] BORODIN, A. and PETROV, L. (2016). Lectures on integrable probability: Stochastic vertex models and symmetric functions. Lecture Notes of the Les Houches Summer School 104. Available at arXiv:1605.01349.

MSC2020 subject classifications. 60K35.
Key words and phrases. Exclusion process, second-class particle, dissolving shock.

[18] BORODIN, A. and WHEELER, M. Coloured stochastic vertex models and their spectral theory. Preprint. Available at arXiv:1808.01866.

[19] BUFETOV, A. and MATVEEV, K. (2018). Hall–Littlewood RSK field. *Selecta Math.* (*N.S.*) **24** 4839–4884. MR3874706 https://doi.org/10.1007/s00029-018-0442-y

[20] BUFETOV, A. and PETROV, L. (2019). Yang–Baxter field for spin Hall–Littlewood symmetric functions. *Forum Math. Sigma* **7** Paper No. e39. MR4031106 https://doi.org/10.1017/fms.2019.36

[21] CATOR, E. and PIMENTEL, L. P. R. (2013). Busemann functions and the speed of a second class particle in the rarefaction fan. *Ann. Probab.* **41** 2401–2425. MR3112921 https://doi.org/10.1214/11-AOP709

[22] CORWIN, I. and DIMITROV, E. (2018). Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall–Littlewood Gibbsian line ensembles. *Comm. Math. Phys.* **363** 435–501. MR3851820 https://doi.org/10.1007/s00220-018-3139-3

[23] CORWIN, I., GHOSAL, P., SHEN, H. and TSAI, L.-C. (2020). Stochastic PDE limit of the six vertex model. *Comm. Math. Phys.* **375** 1945–2038. MR4091508 https://doi.org/10.1007/s00220-019-03678-z

[24] DERRIDA, B., LEBOWITZ, J. L. and SPEER, E. R. (1997). Shock profiles for the asymmetric simple exclusion process in one dimension. *J. Stat. Phys.* **89** 135–167. MR1492490 https://doi.org/10.1007/BF02770758

[25] DIMITROV, E. (2018). KPZ and Airy limits of Hall–Littlewood random plane partitions. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 640–693. MR3795062 https://doi.org/10.1214/16-AIHP817

[26] FERRARI, P. A. (1992). Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Related Fields* **91** 81–101. MR1142763 https://doi.org/10.1007/BF01194491

[27] FERRARI, P. A., GONÇALVES, P. and MARTIN, J. B. (2009). Collision probabilities in the rarefaction fan of asymmetric exclusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 1048–1064. MR2572163 https://doi.org/10.1214/08-AIHP303

[28] FERRARI, P. A. and KIPNIS, C. (1995). Second class particles in the rarefaction fan. *Ann. Inst. Henri Poincaré Probab. Stat.* **31** 143–154. MR1340034

[29] FERRARI, P. A., KIPNIS, C. and SAADA, E. (1991). Microscopic structure of travelling waves in the asymmetric simple exclusion process. *Ann. Probab.* **19** 226–244. MR1085334

[30] FERRARI, P. L. (2010). From interacting particle systems to random matrices. *J. Stat. Mech. Theory Exp.* **10** P10016. MR2800495 https://doi.org/10.1088/1742-5468/2010/10/p10016

[31] FERRARI, P. L. and NEJJAR, P. (2015). Anomalous shock fluctuations in TASEP and last passage percolation models. *Probab. Theory Related Fields* **161** 61–109. MR3304747 https://doi.org/10.1007/s00440-013-0544-6

[32] FERRARI, P. L. and NEJJAR, P. (2017). Fluctuations of the competition interface in presence of shocks. *ALEA Lat. Am. J. Probab. Math. Stat.* **14** 299–325. MR3637438 https://doi.org/10.30757/alea.v14-17

[33] GEORGIOU, N., KUMAR, R. and SEPPÄLÄINEN, T. (2010). TASEP with discontinuous jump rates. *ALEA Lat. Am. J. Probab. Math. Stat.* **7** 293–318. MR2732897

[34] GHOSAL, P., SAENZ, A. and ZELL, E. Limiting speed of a second class particle in ASEP. Preprint. Available at arXiv:1903.09615.

[35] GWA, L.-H. and SPOHN, H. (1992). Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. *Phys. Rev. Lett.* **68** 725–728. MR1147356 https://doi.org/10.1103/PhysRevLett.68.725

[36] HARRIS, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Adv. Math.* **9** 66–89. MR0307392 https://doi.org/10.1016/0001-8708(72)90030-8

[37] HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378. MR0488377 https://doi.org/10.1214/aop/1176995523

[38] HOLLEY, R. (1970). A class of interactions in an infinite particle system. *Adv. Math.* **5** 291–309. MR0268960 https://doi.org/10.1016/0001-8708(70)90035-6

[39] JOHANSSON, K. (2003). Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.* **242** 277–329. MR2018275 https://doi.org/10.1007/s00220-003-0945-y

[40] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[41] KUAN, J. (2018). An algebraic construction of duality functions for the stochastic $\mathcal{U}_q(A_n^{(1)})$ vertex model and its degenerations. *Comm. Math. Phys.* **359** 121–187. MR3781448 https://doi.org/10.1007/s00220-018-3108-x

[42] KUNIBA, A., MANGAZEEV, V. V., MARUYAMA, S. and OKADO, M. (2016). Stochastic R matrix for $U_q(A_n^{(1)})$. *Nuclear Phys. B* **913** 248–277. MR3575300 https://doi.org/10.1016/j.nuclphysb.2016.09.016

[43] LIGGETT, T. M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481. MR0309218 https://doi.org/10.2307/1995898

[44] LIGGETT, T. M. (1975). Ergodic theorems for the asymmetric simple exclusion process. *Trans. Amer. Math. Soc.* **213** 237–261. MR0410986 https://doi.org/10.2307/1998046

[45] LIGGETT, T. M. (1977). Ergodic theorems for the asymmetric simple exclusion process. II. *Ann. Probab.* **5** 795–801. MR0445644 https://doi.org/10.1214/aop/1176995721

[46] LIGGETT, T. M. (1985). *Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[47] MACDONALD, I. G. (1999). *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford Univ. Press, London.

[48] QUASTEL, J. and RAHMAN, M. (2020). TASEP fluctuations with soft-shock initial data. *Ann. Henri Lebesgue* **3** 999–1021. MR4149832 https://doi.org/10.5802/ahl.52

[49] RESHETIKHIN, N. and SRIDHAR, A. (2018). Limit shapes of the stochastic six vertex model. *Comm. Math. Phys.* **363** 741–765. MR3858820 https://doi.org/10.1007/s00220-018-3253-2

[50] ROST, H. (1981). Nonequilibrium behaviour of a many particle process: Density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete* **58** 41–53. MR0635270 https://doi.org/10.1007/BF00536194

[51] SHEN, H. and TSAI, L.-C. (2019). Stochastic telegraph equation limit for the stochastic six vertex model. *Proc. Amer. Math. Soc.* **147** 2685–2705. MR3951443 https://doi.org/10.1090/proc/14415

The Annals of Probability
2021, Vol. 49, No. 4, 1633–1676
https://doi.org/10.1214/20-AOP1486

EXTERNAL DIFFUSION-LIMITED AGGREGATION ON A SPANNING-TREE-WEIGHTED RANDOM PLANAR MAP

BY EWAIN GWYNNE[1] AND JOSHUA PFEFFER[2]

[1]*Department of Mathematics, University of Cambridge, eg558@cam.ac.uk*
[2]*Department of Mathematics, Massachusetts Institute of Technology, pfeffer@mit.edu*

Let M be the infinite spanning-tree-weighted random planar map, which is the local limit of finite random planar maps sampled with probability proportional to the number of spanning trees they admit. We show that a.s. the M-graph-distance diameter of the external diffusion-limited aggregation (DLA) cluster on M run for m steps is of order $m^{2/d+o_m(1)}$, where d is the metric ball volume growth exponent for M (which was shown to exist by Ding and Gwynne (*Comm. Math. Phys.* **374** (2020) 1877–1934). By known bounds for d, one has $0.55051\cdots \le 2/d \le 0.563315\ldots$.

Along the way, we also prove that loop-erased random walk (LERW) on M typically travels graph distance $m^{2/d+o_m(1)}$ in m units of time and that the graph-distance diameter of a finite spanning-tree-weighted random planar map with n edges, with or without boundary, is of order $n^{1/d+o_n(1)}$ except on an event with probability decaying faster than any negative power of n.

Our proofs are based on a special relationship between DLA and LERW on spanning-tree-weighted random planar maps as well as estimates for distances in such maps which come from the theory of Liouville quantum gravity.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[2] ALBENQUE, M., MÉNARD, L. and SCHAEFFER, G. (2021). Local convergence of large random triangulations coupled with an Ising model. *Trans. Amer. Math. Soc.* **374** 175–217. MR4188181 https://doi.org/10.1090/tran/8150

[3] ANGEL, O. (2003). Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.* **13** 935–974. MR2024412 https://doi.org/10.1007/s00039-003-0436-5

[4] ANGEL, O. and SCHRAMM, O. (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241** 191–213. MR2013797 https://doi.org/10.1007/978-1-4419-9675-6_16

[5] ARU, J. (2017). Gaussian multiplicative chaos through the lens of the 2D Gaussian free field. ArXiv e-prints.

[6] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (2001). Uniform spanning forests. *Ann. Probab.* **29** 1–65. MR1825141 https://doi.org/10.1214/aop/1008956321

[7] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** no. 23. MR1873300 https://doi.org/10.1214/EJP.v6-96

[8] BENJAMINI, I. and YADIN, A. (2017). Upper bounds on the growth rate of diffusion limited aggregation. ArXiv e-prints.

[9] BENOIST, S. (2018). Natural parametrization of SLE: The Gaussian free field point of view. *Electron. J. Probab.* **23** Paper No. 103. MR3870446 https://doi.org/10.1214/18-ejp232

[10] BERESTYCKI, N. (2017). An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.* **22** Paper No. 27. MR3652040 https://doi.org/10.1214/17-ECP58

[11] BERNARDI, O. (2007). Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.* **14** Research Paper 9. MR2285813

[12] BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216. MR0399402 https://doi.org/10.1007/BF01425510

MSC2020 subject classifications. 82B24, 82B41, 60J67.
Key words and phrases. Diffusion-limited aggregation, loop-erased random walk, random planar maps, Schramm–Loewner evolution, Liouville quantum gravity.

[13] BOROT, G., BOUTTIER, J. and GUITTER, E. (2012). A recursive approach to the $O(n)$ model on random maps via nested loops. *J. Phys. A* **45** 045002. MR2874232 https://doi.org/10.1088/1751-8113/45/4/045002

[14] BOUTTIER, J., DI FRANCESCO, P. and GUITTER, E. (2004). Planar maps as labeled mobiles. *Electron. J. Combin.* **11** Research Paper 69. MR2097335

[15] CARACENI, A. and CURIEN, N. (2019). Self-avoiding walks on the UIPQ. In *Sojourns in Probability Theory and Statistical Physics III* 138–165. Springer, Singapore.

[16] CARLESON, L. and MAKAROV, N. (2001). Aggregation in the plane and Loewner's equation. *Comm. Math. Phys.* **216** 583–607. MR1815718 https://doi.org/10.1007/s002200000340

[17] CHEN, L. (2017). Basic properties of the infinite critical-FK random map. *Ann. Inst. Henri Poincaré D* **4** 245–271. MR3713017 https://doi.org/10.4171/AIHPD/40

[18] CURIEN, N. (2015). A glimpse of the conformal structure of random planar maps. *Comm. Math. Phys.* **333** 1417–1463. MR3302638 https://doi.org/10.1007/s00220-014-2196-5

[19] DING, J., DUBÉDAT, J., DUNLAP, A. and FALCONET, H. (2020). Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *Publ. Math. Inst. Hautes Études Sci.* **132** 353–403. MR4179836 https://doi.org/10.1007/s10240-020-00121-1

[20] DING, J. and GWYNNE, E. (2020). The fractal dimension of Liouville quantum gravity: Universality, monotonicity, and bounds. *Comm. Math. Phys.* **374** 1877–1934. MR4076090 https://doi.org/10.1007/s00220-019-03487-4

[21] DING, J., ZEITOUNI, O. and ZHANG, F. (2019). Heat kernel for Liouville Brownian motion and Liouville graph distance. *Comm. Math. Phys.* **371** 561–618. MR4019914 https://doi.org/10.1007/s00220-019-03467-8

[22] DUBÉDAT, J. (2009). Duality of Schramm–Loewner evolutions. *Ann. Sci. Éc. Norm. Supér.* (4) **42** 697–724. MR2571956 https://doi.org/10.24033/asens.2107

[23] DUBÉDAT, J., FALCONET, H., GWYNNE, E., PFEFFER, J. and SUN, X. (2020). Weak LQG metrics and Liouville first passage percolation. *Probab. Theory Related Fields* **178** 369–436. MR4146541 https://doi.org/10.1007/s00440-020-00979-6

[24] DUPLANTIER, B. and KOSTOV, I. (1988). Conformal spectra of polymers on a random surface. *Phys. Rev. Lett.* **61** 1433–1437. MR0960093 https://doi.org/10.1103/PhysRevLett.61.1433

[25] DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2014). Liouville quantum gravity as a mating of trees. ArXiv e-prints.

[26] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 https://doi.org/10.1007/s00222-010-0308-1

[27] FREDES, L. and SEPÚLVEDA, A. (2020). Tree-decorated planar maps. *Electron. J. Combin.* **27** Paper No. 1.66. MR4093167 https://doi.org/10.37236/8635

[28] GUREL-GUREVICH, O. and NACHMIAS, A. (2013). Recurrence of planar graph limits. *Ann. of Math.* (2) **177** 761–781. MR3010812 https://doi.org/10.4007/annals.2013.177.2.10

[29] GWYNNE, E., HOLDEN, N. and SUN, X. (2019). A distance exponent for Liouville quantum gravity. *Probab. Theory Related Fields* **173** 931–997. MR3936149 https://doi.org/10.1007/s00440-018-0846-9

[30] GWYNNE, E., HOLDEN, N. and SUN, X. (2020). A mating-of-trees approach for graph distances in random planar maps. *Probab. Theory Related Fields* **177** 1043–1102. MR4126936 https://doi.org/10.1007/s00440-020-00969-8

[31] GWYNNE, E. and HUTCHCROFT, T. (2020). Anomalous diffusion of random walk on random planar maps. *Probab. Theory Related Fields* **178** 567–611. MR4146545 https://doi.org/10.1007/s00440-020-00986-7

[32] GWYNNE, E. and MILLER, J. (2016). Convergence of the self-avoiding walk on random quadrangulations to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity. *Ann. Sci. Éc. Norm. Supér.* To appear.

[33] GWYNNE, E. and MILLER, J. (2017). Convergence of percolation on uniform quadrangulations with boundary to SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity. ArXiv e-prints.

[34] GWYNNE, E. and MILLER, J. (2017). Random walk on random planar maps: Spectral dimension, resistance, and displacement. ArXiv e-prints.

[35] GWYNNE, E. and MILLER, J. (2019). Local metrics of the Gaussian free field. *Ann. Inst. Fourier* (*Grenoble*). To appear.

[36] GWYNNE, E. and MILLER, J. (2019). Conformal covariance of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. ArXiv e-prints.

[37] GWYNNE, E. and MILLER, J. (2020). Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$. *Ann. Probab.* **48** 1861–1901. MR4124527 https://doi.org/10.1214/19-AOP1409

[38] GWYNNE, E. and MILLER, J. (2021). Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Invent. Math.* **223** 213–333. MR4199443 https://doi.org/10.1007/s00222-020-00991-6

[39] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2019). Harmonic functions on mated-CRT maps. *Electron. J. Probab.* **24** Paper No. 58. MR3978208 https://doi.org/10.1214/19-EJP325

[40] GWYNNE, E. and PFEFFER, J. (2019). KPZ formulas for the Liouville quantum gravity metric. *Trans. Amer. Math. Soc.* To appear.

[41] GWYNNE, E. and PFEFFER, J. (2019). Bounds for distances and geodesic dimension in Liouville first passage percolation. *Electron. Commun. Probab.* **24** Paper No. 56. MR4003130 https://doi.org/10.1214/19-ecp248

[42] HALSEY, T. C. (2000). Diffusion-limited aggregation: A model for pattern formation. *Phys. Today* **53** 11.

[43] HASTINGS, M. and LEVITOV, L. (1998). Laplacian growth as one-dimensional turbulence. *Phys. D* **116** 244–252.

[44] IMHOF, J.-P. (1984). Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.* **21** 500–510. MR0752015 https://doi.org/10.2307/3213612

[45] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[46] KENYON, R., MILLER, J., SHEFFIELD, S. and WILSON, D. B. (2019). Bipolar orientations on planar maps and SLE_{12}. *Ann. Probab.* **47** 1240–1269. MR3945746 https://doi.org/10.1214/18-AOP1282

[47] KESTEN, H. (1987). How long are the arms in DLA? *J. Phys. A* **20** L29–L33. MR0873177 https://doi.org/10.1088/0305-4470/20/1/007

[48] KESTEN, H. (1990). Upper bounds for the growth rate of DLA. *Phys. A* **168** 529–535. MR1077203 https://doi.org/10.1016/0378-4371(90)90405-H

[49] KNIZHNIK, V. G., POLYAKOV, A. M. and ZAMOLODCHIKOV, A. B. (1988). Fractal structure of 2D-quantum gravity. *Modern Phys. Lett. A* **3** 819–826. MR0947880 https://doi.org/10.1142/S0217732388000982

[50] KRIKUN, M. (2005). Local structure of random quadrangulations. ArXiv Mathematics E-prints.

[51] LAWLER, G. F. (2005). *Conformally Invariant Processes in the Plane*. *Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588 https://doi.org/10.1090/surv/114

[52] LAWLER, G. F. (2006). The Laplacian-b random walk and the Schramm–Loewner evolution. *Illinois J. Math.* **50** 701–746. MR2247843

[53] MEAKIN, P. (1986). Universality, nonuniversality, and the effects of anisotropy on diffusion-limited aggregation. *Phys. Rev. A* **33** 3371–3382.

[54] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[55] MILLER, J. and SHEFFIELD, S. (2016). Quantum Loewner evolution. *Duke Math. J.* **165** 3241–3378. MR3572845 https://doi.org/10.1215/00127094-3627096

[56] MILLER, J. and SHEFFIELD, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2

[57] MULLIN, R. C. (1967). On the enumeration of tree-rooted maps. *Canad. J. Math.* **19** 174–183. MR0205882 https://doi.org/10.4153/CJM-1967-010-x

[58] NIEMEYER, L., PIETRONERO, L. and WIESMANN, H. J. (1984). Fractal dimension of dielectric breakdown. *Phys. Rev. Lett.* **52** 1033–1036. MR0736820 https://doi.org/10.1103/PhysRevLett.52.1033

[59] NORRIS, J., SILVESTRI, V. and TURNER, A. (2019). Scaling limits for planar aggregation with subcritical fluctuations. ArXiv e-prints.

[60] NORRIS, J. and TURNER, A. (2012). Hastings–Levitov aggregation in the small-particle limit. *Comm. Math. Phys.* **316** 809–841. MR2993934 https://doi.org/10.1007/s00220-012-1552-6

[61] PITMAN, J. W. (1975). One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. in Appl. Probab.* **7** 511–526. MR0375485 https://doi.org/10.2307/1426125

[62] PROCACCIA, E. B. and ZHANG, Y. (2019). Stationary harmonic measure and DLA in the upper half plane. *J. Stat. Phys.* **176** 946–980. MR3990220 https://doi.org/10.1007/s10955-019-02327-y

[63] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218

[64] ROHDE, S. and ZINSMEISTER, M. (2005). Some remarks on Laplacian growth. *Topology Appl.* **152** 26–43. MR2160804 https://doi.org/10.1016/j.topol.2004.08.013

[65] SANDER, L. M. (2000). Diffusion-limited aggregation: A kinetic critical phenomenon? *Contemp. Phys.* **41** 203–218.

[66] SCHAEFFER, G. (1997). Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees. *Electron. J. Combin.* **4** Research Paper 20. MR1465581

[67] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084 https://doi.org/10.1007/BF02803524

[68] SCHRAMM, O. and SHEFFIELD, S. (2013). A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields* **157** 47–80. MR3101840 https://doi.org/10.1007/s00440-012-0449-9

[69] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 https://doi.org/10.1007/s00440-006-0050-1

[70] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[71] SHEFFIELD, S. (2016). Quantum gravity and inventory accumulation. *Ann. Probab.* **44** 3804–3848. MR3572324 https://doi.org/10.1214/15-AOP1061

[72] SOLA, A., TURNER, A. and VIKLUND, F. (2019). One-dimensional scaling limits in a planar Laplacian random growth model. *Comm. Math. Phys.* **371** 285–329. MR4015346 https://doi.org/10.1007/s00220-019-03460-1

[73] SUDAKOV, V. N. and CIREL'SON, B. S. (1974). Extremal properties of half-spaces for spherically invariant measures. Problems in the theory of probability distributions, II. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **41** 14–24, 165. MR0365680

[74] WILSON, D. B. (1996). Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing* (*Philadelphia*, *PA*, 1996) 296–303. ACM, New York. MR1427525 https://doi.org/10.1145/237814.237880

[75] WITTEN, T. A. and SANDER, L. M. (1981). Diffusion-limited aggregation, a kinetic critical phenomenon. *Phys. Rev. Lett.* **47** 1400.

[76] WITTEN, T. A. and SANDER, L. M. (1983). Diffusion-limited aggregation. *Phys. Rev. B* **27** 5686–5697. MR0704464 https://doi.org/10.1103/physrevb.27.5686

[77] ZAITSEV, A. Y. (1998). Multidimensional version of the results of Komlós, Major and Tusnády for vectors with finite exponential moments. *ESAIM Probab. Stat.* **2** 41–108. MR1616527 https://doi.org/10.1051/ps:1998103

[78] ZHAN, D. (2008). Duality of chordal SLE. *Invent. Math.* **174** 309–353. MR2439609 https://doi.org/10.1007/s00222-008-0132-z

[79] ZHAN, D. (2010). Duality of chordal SLE, II. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 740–759. MR2682265 https://doi.org/10.1214/09-AIHP340

The Annals of Probability
2021, Vol. 49, No. 4, 1677–1717
https://doi.org/10.1214/20-AOP1487

THE TUTTE EMBEDDING OF THE MATED-CRT MAP CONVERGES TO LIOUVILLE QUANTUM GRAVITY

BY EWAIN GWYNNE[1,*], JASON MILLER[1,†] AND SCOTT SHEFFIELD[2]

[1]*Department of Mathematics, University of Cambridge,* [*]*eg558@cam.ac.uk;* [†]*jpmiller@statslab.cam.ac.uk*
[2]*Department of Mathematics, Massachusetts Institute of Technology, sheffield@math.mit.edu*

We prove that the Tutte embeddings (a.k.a. harmonic/barycentric embeddings) of certain random planar maps converge to γ-Liouville quantum gravity (γ-LQG). Specifically, we treat mated-CRT maps, which are discretized matings of correlated continuum random trees, and γ ranges from 0 to 2 as one varies the correlation parameter. We also show that the associated space-filling path on the embedded map converges to space-filling SLE_κ for $\kappa = 16/\gamma^2$ (in the annealed sense) and that simple random walk on the embedded map converges to Brownian motion (in the quenched sense).

This work constitutes the first proof that a discrete conformal embedding of a random planar map converges to LQG. Many more such statements have been conjectured. Since the mated-CRT map can be viewed as a coarse-grained approximation to other random planar maps (the UIPT, tree-weighted maps, bipolar-oriented maps, etc.), our results indicate a potential approach for proving that embeddings of these maps converge to LQG as well.

To prove the main result, we establish several (independently interesting) theorems about LQG surfaces decorated by space-filling SLE. There is a natural way to use the SLE curve to divide the plane into "cells" corresponding to vertices of the mated-CRT map. We study the law of the *shape* of the origin-containing cell, in particular proving moments for the ratio of its squared diameter to its area. We also give bounds on the degree of the origin-containing cell and establish a form of ergodicity for the entire configuration. Ultimately, we use these properties to show (with the help of a general theorem proved in a separate paper) that random walk on these cells converges to a time change of Brownian motion, which in turn leads to the Tutte embedding result.

REFERENCES

[1] AIZENMAN, M. and BURCHARD, A. (1999). Hölder regularity and dimension bounds for random curves. *Duke Math. J.* **99** 419–453. MR1712629 https://doi.org/10.1215/S0012-7094-99-09914-3
[2] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326
[3] ALDOUS, D. (1991). The continuum random tree. II. An overview. In *Stochastic Analysis* (*Durham*, 1990). *London Mathematical Society Lecture Note Series* **167** 23–70. Cambridge Univ. Press, Cambridge. MR1166406 https://doi.org/10.1017/CBO9780511662980.003
[4] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226
[5] ANG, M. and GWYNNE, E. (2019). Liouville quantum gravity surfaces with boundary as matings of trees. *Ann. Inst. Henri Poincaré*. To appear.
[6] ANGEL, O., BARLOW, M. T., GUREL-GUREVICH, O. and NACHMIAS, A. (2016). Boundaries of planar graphs, via circle packings. *Ann. Probab.* **44** 1956–1984. MR3502598 https://doi.org/10.1214/15-AOP1014
[7] ANGEL, O., HUTCHCROFT, T., NACHMIAS, A. and RAY, G. (2016). Unimodular hyperbolic triangulations: Circle packing and random walk. *Invent. Math.* **206** 229–268. MR3556528 https://doi.org/10.1007/s00222-016-0653-9
[8] ANGEL, O., HUTCHCROFT, T., NACHMIAS, A. and RAY, G. (2018). Hyperbolic and parabolic unimodular random maps. *Geom. Funct. Anal.* **28** 879–942. MR3820434 https://doi.org/10.1007/s00039-018-0446-y

MSC2020 subject classifications. 60J67, 60D05, 60J65.
Key words and phrases. Random planar maps, mated-CRT map, Tutte embedding, Liouville quantum gravity, Schramm–Loewner evolution, random walk.

[9] BEN AROUS, G. and FRIBERGH, A. (2016). Biased random walks on random graphs. In *Probability and Statistical Physics in St. Petersburg*. *Proc. Sympos. Pure Math.* **91** 99–153. Amer. Math. Soc., Providence, RI. MR3526827 https://doi.org/10.1214/10-aop620

[10] BENJAMINI, I. and CURIEN, N. (2013). Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity via pioneer points. *Geom. Funct. Anal.* **23** 501–531. MR3053754 https://doi.org/10.1007/s00039-013-0212-0

[11] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** no. 23, 13. MR1873300 https://doi.org/10.1214/EJP.v6-96

[12] BERESTYCKI, N. (2015). Diffusion in planar Liouville quantum gravity. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 947–964. MR3365969 https://doi.org/10.1214/14-AIHP605

[13] BERESTYCKI, N. and GWYNNE, E. (2020). Random walks on mated-CRT planar maps and Liouville Brownian motion. ArXiv e-prints.

[14] BERNARDI, O. (2007). Bijective counting of Kreweras walks and loopless triangulations. *J. Combin. Theory Ser. A* **114** 931–956. MR2333142 https://doi.org/10.1016/j.jcta.2006.09.009

[15] BERNARDI, O. (2007). Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.* **14** Research Paper 9, 36. MR2285813

[16] BERNARDI, O., HOLDEN, N. and SUN, X. (2018). Percolation on triangulations: A bijective path to Liouville quantum gravity. ArXiv e-prints.

[17] BISKUP, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373. MR2861133 https://doi.org/10.1214/11-PS190

[18] BISKUP, M., DING, J. and GOSWAMI, S. (2020). Return probability and recurrence for the random walk driven by two-dimensional Gaussian free field. *Comm. Math. Phys.* **373** 45–106. MR4050092 https://doi.org/10.1007/s00220-019-03589-z

[19] BROOKS, R. L., SMITH, C. A. B., STONE, A. H. and TUTTE, W. T. (1940). The dissection of rectangles into squares. *Duke Math. J.* **7** 312–340. MR0003040

[20] CANNON, J. W. (1994). The combinatorial Riemann mapping theorem. *Acta Math.* **173** 155–234. MR1301392 https://doi.org/10.1007/BF02398434

[21] CARMESIN, J. and GEORGAKOPOULOS, A. (2020). Every planar graph with the Liouville property is amenable. *Random Structures Algorithms* **57** 706–729. MR4144081 https://doi.org/10.1002/rsa.20936

[22] CURIEN, N. (2015). A glimpse of the conformal structure of random planar maps. *Comm. Math. Phys.* **333** 1417–1463. MR3302638 https://doi.org/10.1007/s00220-014-2196-5

[23] CURIEN, N., HUTCHCROFT, T. and NACHMIAS, A. (2020). Geometric and spectral properties of causal maps. *J. Eur. Math. Soc.* (*JEMS*) **22** 3997–4024. MR4176785 https://doi.org/10.4171/jems/1001

[24] DING, J. and GWYNNE, E. (2020). The fractal dimension of Liouville quantum gravity: Universality, monotonicity, and bounds. *Comm. Math. Phys.* **374** 1877–1934. MR4076090 https://doi.org/10.1007/s00220-019-03487-4

[25] DUBEJKO, T. (1997). Random walks on circle packings. In *Lipa's Legacy* (*New York*, 1995). *Contemp. Math.* **211** 169–182. Amer. Math. Soc., Providence, RI. MR1476986 https://doi.org/10.1090/conm/211/02819

[26] DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2014). Liouville quantum gravity as a mating of trees. *Astérisque*. To appear.

[27] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 https://doi.org/10.1007/s00222-010-0308-1

[28] GARBAN, C., HOLDEN, N., SEPÚLVEDA, A. and SUN, X. (2018). Negative moments for Gaussian multiplicative chaos on fractal sets. *Electron. Commun. Probab.* **23** Paper No. 100, 10. MR3896838 https://doi.org/10.1214/18-ECP168

[29] GARBAN, C., RHODES, R. and VARGAS, V. (2016). Liouville Brownian motion. *Ann. Probab.* **44** 3076–3110. MR3531686 https://doi.org/10.1214/15-AOP1042

[30] GEORGAKOPOULOS, A. (2016). The boundary of a square tiling of a graph coincides with the Poisson boundary. *Invent. Math.* **203** 773–821. MR3461366 https://doi.org/10.1007/s00222-015-0601-0

[31] GILL, J. T. and ROHDE, S. (2013). On the Riemann surface type of random planar maps. *Rev. Mat. Iberoam.* **29** 1071–1090. MR3090146 https://doi.org/10.4171/RMI/749

[32] GUREL-GUREVICH, O. and NACHMIAS, A. (2013). Recurrence of planar graph limits. *Ann. of Math.* (2) **177** 761–781. MR3010812 https://doi.org/10.4007/annals.2013.177.2.10

[33] GWYNNE, E., HOLDEN, N. and MILLER, J. (2020). An almost sure KPZ relation for SLE and Brownian motion. *Ann. Probab.* **48** 527–573. MR4089487 https://doi.org/10.1214/19-AOP1385

[34] GWYNNE, E., HOLDEN, N., MILLER, J. and SUN, X. (2017). Brownian motion correlation in the peanosphere for $\kappa > 8$. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1866–1889. MR3729638 https://doi.org/10.1214/16-AIHP774

[35] GWYNNE, E., HOLDEN, N. and SUN, X. (2016). Joint scaling limit of a bipolar-oriented triangulation and its dual in the peanosphere sense. ArXiv e-prints.

[36] GWYNNE, E., HOLDEN, N. and SUN, X. (2019). A distance exponent for Liouville quantum gravity. *Probab. Theory Related Fields* **173** 931–997. MR3936149 https://doi.org/10.1007/s00440-018-0846-9

[37] GWYNNE, E., HOLDEN, N. and SUN, X. (2020). A mating-of-trees approach for graph distances in random planar maps. *Probab. Theory Related Fields* **177** 1043–1102. MR4126936 https://doi.org/10.1007/s00440-020-00969-8

[38] GWYNNE, E. and HUTCHCROFT, T. (2020). Anomalous diffusion of random walk on random planar maps. *Probab. Theory Related Fields* **178** 567–611. MR4146545 https://doi.org/10.1007/s00440-020-00986-7

[39] GWYNNE, E., KASSEL, A., MILLER, J. and WILSON, D. B. (2018). Active spanning trees with bending energy on planar maps and SLE-decorated Liouville quantum gravity for $\kappa > 8$. *Comm. Math. Phys.* **358** 1065–1115. MR3778352 https://doi.org/10.1007/s00220-018-3104-1

[40] GWYNNE, E. and MILLER, J. (2016). Convergence of the self-avoiding walk on random quadrangulations to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity. *Ann. Sci. Ec. Norm. Super.* To appear.

[41] GWYNNE, E. and MILLER, J. (2017). Characterizations of SLE_κ for $\kappa \in (4,8)$ on Liouville quantum gravity. *Astérisque*. To appear.

[42] GWYNNE, E. and MILLER, J. (2017). Convergence of percolation on uniform quadrangulations with boundary to SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity. *Astérisque*. To appear.

[43] GWYNNE, E. and MILLER, J. (2017). Random walk on random planar maps: Spectral dimension, resistance, and displacement. *Ann. Probab.* To appear.

[44] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2018). An invariance principle for ergodic scale-free random environments. ArXiv e-prints.

[45] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2019). Harmonic functions on mated-CRT maps. *Electron. J. Probab.* **24** Paper No. 58, 55. MR3978208 https://doi.org/10.1214/19-EJP325

[46] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2020). The Tutte embedding of the Poisson–Voronoi tessellation of the Brownian disk converges to $\sqrt{8/3}$-Liouville quantum gravity. *Comm. Math. Phys.* **374** 735–784. MR4072229 https://doi.org/10.1007/s00220-019-03610-5

[47] GWYNNE, E. and PFEFFER, J. (2019). External diffusion limited aggregation on a spanning-tree-weighted random planar map. *Ann. Probab.* To appear.

[48] HOLDEN, N., LAWLER, G. F., LI, X. and SUN, X. (2018). Minkowski content of Brownian cut points. ArXiv e-prints.

[49] HOLDEN, N., LI, X. and SUN, X. (2018). Natural parametrization of percolation interface and pivotal points. ArXiv e-prints.

[50] HOLDEN, N. and SUN, X. (2018). SLE as a mating of trees in Euclidean geometry. *Comm. Math. Phys.* **364** 171–201. MR3861296 https://doi.org/10.1007/s00220-018-3149-1

[51] HUANG, Y., RHODES, R. and VARGAS, V. (2018). Liouville quantum gravity on the unit disk. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 1694–1730. MR3825895 https://doi.org/10.1214/17-AIHP852

[52] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[53] KENYON, R., MILLER, J., SHEFFIELD, S. and WILSON, D. B. (2019). Bipolar orientations on planar maps and SLE_{12}. *Ann. Probab.* **47** 1240–1269. MR3945746 https://doi.org/10.1214/18-AOP1282

[54] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 33–58. MR0402883 https://doi.org/10.1007/BF00532688

[55] LE GALL, J.-F. (2013). Uniqueness and universality of the Brownian map. *Ann. Probab.* **41** 2880–2960. MR3112934 https://doi.org/10.1214/12-AOP792

[56] LEE, J. R. (2017). Conformal growth rates and spectral geometry on distributional limits of graphs. ArXiv e-prints.

[57] LEE, J. R. (2018). Discrete uniformizing metrics on distributional limits of sphere packings. *Geom. Funct. Anal.* **28** 1091–1130. MR3820440 https://doi.org/10.1007/s00039-018-0442-2

[58] LI, Y., SUN, X. and WATSON, S. S. (2017). Schnyder Woods, SLE(16), and Liouville Quantum Gravity. ArXiv e-prints.

[59] LUPU, T. (2019). Convergence of the two-dimensional random walk loop-soup clusters to CLE. *J. Eur. Math. Soc.* (*JEMS*) **21** 1201–1227. MR3941462 https://doi.org/10.4171/JEMS/859

[60] MIERMONT, G. (2013). The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.* **210** 319–401. MR3070569 https://doi.org/10.1007/s11511-013-0096-8

[61] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[62] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry II: Reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. *Ann. Probab.* **44** 1647–1722. MR3502592 https://doi.org/10.1214/14-AOP943

[63] Miller, J. and Sheffield, S. (2016). Imaginary geometry III: Reversibility of SLE_κ for $\kappa \in (4, 8)$. *Ann. of Math.* (2) **184** 455–486. MR3548530 https://doi.org/10.4007/annals.2016.184.2.3
[64] Miller, J. and Sheffield, S. (2016). Liouville quantum gravity and the Brownian map II: Geodesics and continuity of the embedding. *Ann. Probab.* To appear.
[65] Miller, J. and Sheffield, S. (2016). Liouville quantum gravity and the Brownian map III: The conformal structure is determined. *Probab. Theory Related Fields*. To appear.
[66] Miller, J. and Sheffield, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2
[67] Miller, J. and Sheffield, S. (2019). Liouville quantum gravity spheres as matings of finite-diameter trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1712–1750. MR4010949 https://doi.org/10.1214/18-aihp932
[68] Miller, J. and Sheffield, S. (2020). Liouville quantum gravity and the Brownian map I: The $\mathrm{QLE}(8/3, 0)$ metric. *Invent. Math.* **219** 75–152. MR4050102 https://doi.org/10.1007/s00222-019-00905-1
[69] Moore, R. L. (1925). Concerning upper semi-continuous collections of continua. *Trans. Amer. Math. Soc.* **27** 416–428. MR1501320 https://doi.org/10.2307/1989234
[70] Mullin, R. C. (1967). On the enumeration of tree-rooted maps. *Canad. J. Math.* **19** 174–183. MR0205882 https://doi.org/10.4153/CJM-1967-010-x
[71] Nakayama, Y. (2004). Liouville field theory: A decade after the revolution. *Internat. J. Modern Phys. A* **19** 2771–2930. MR2073993 https://doi.org/10.1142/S0217751X04019500
[72] Gurel-Gurevich O., Jerison, D. C. and Nachmias, A. (2019). A combinatorial criterion for macroscopic circles in planar triangulations. ArXiv e-prints.
[73] Rhodes, R. and Vargas, V. (2011). KPZ formula for log-infinitely divisible multifractal random measures. *ESAIM Probab. Stat.* **15** 358–371. MR2870520 https://doi.org/10.1051/ps/2010007
[74] Rhodes, R. and Vargas, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218
[75] Rohde, S. and Schramm, O. (2005). Basic properties of SLE. *Ann. of Math.* (2) **161** 883–924. MR2153402 https://doi.org/10.4007/annals.2005.161.883
[76] Schramm, O. (1993). Square tilings with prescribed combinatorics. *Israel J. Math.* **84** 97–118. MR1244661 https://doi.org/10.1007/BF02761693
[77] Schramm, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084 https://doi.org/10.1007/BF02803524
[78] Schramm, O. and Sheffield, S. (2009). Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.* **202** 21–137. MR2486487 https://doi.org/10.1007/s11511-009-0034-y
[79] Schramm, O. and Sheffield, S. (2013). A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields* **157** 47–80. MR3101840 https://doi.org/10.1007/s00440-012-0449-9
[80] Sheffield, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 https://doi.org/10.1007/s00440-006-0050-1
[81] Sheffield, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055
[82] Sheffield, S. (2016). Quantum gravity and inventory accumulation. *Ann. Probab.* **44** 3804–3848. MR3572324 https://doi.org/10.1214/15-AOP1061
[83] Shimura, M. (1985). Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.* **25** 433–443. MR0807490 https://doi.org/10.1215/kjm/1250521064
[84] Smirnov, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 239–244. MR1851632 https://doi.org/10.1016/S0764-4442(01)01991-7
[85] Stephenson, K. (2003). Circle packing: A mathematical tale. *Notices Amer. Math. Soc.* **50** 1376–1388. MR2011604
[86] Zaitsev, A. Y. (1998). Multidimensional version of the results of Komlós, Major and Tusnády for vectors with finite exponential moments. *ESAIM Probab. Stat.* **2** 41–108. MR1616527 https://doi.org/10.1051/ps:1998103

The Annals of Probability
2021, Vol. 49, No. 4, 1718–1737
https://doi.org/10.1214/20-AOP1491

BROWNIAN ABSOLUTE CONTINUITY OF THE KPZ FIXED POINT WITH ARBITRARY INITIAL CONDITION

BY SOURAV SARKAR[1] AND BÁLINT VIRÁG[2]

[1]*Department of Mathematics, University of Toronto, ssarkar@math.toronto.edu*
[2]*Departments of Mathematics and Statistics, University of Toronto, balint@math.toronto.edu*

We show that the law of the KPZ fixed point starting from arbitrary initial condition is absolutely continuous with respect to the law of Brownian motion B on every compact interval. In particular, the Airy_1 process is absolutely continuous with respect to B on any compact interval.

REFERENCES

[1] ADLER, M. and VAN MOERBEKE, P. (2005). PDEs for the joint distributions of the Dyson, Airy and sine processes. *Ann. Probab.* **33** 1326–1361. MR2150191 https://doi.org/10.1214/009117905000000107
[2] CALVERT, J., HAMMOND, A. and HEGDE, M. (2019). Brownian structure in the KPZ fixed point. Available at arXiv:1912.00992.
[3] CATOR, E. and PIMENTEL, L. P. R. (2015). On the local fluctuations of last-passage percolation models. *Stochastic Process. Appl.* **125** 538–551. MR3293293 https://doi.org/10.1016/j.spa.2014.08.009
[4] CORWIN, I. and HAMMOND, A. (2014). Brownian Gibbs property for Airy line ensembles. *Invent. Math.* **195** 441–508. MR3152753 https://doi.org/10.1007/s00222-013-0462-3
[5] DAUVERGNE, D., ORTMANN, J. and VIRÁG, B. (2018). The directed landscape. Available at arXiv:1812.00309.
[6] HÄGG, J. (2008). Local Gaussian fluctuations in the Airy and discrete PNG processes. *Ann. Probab.* **36** 1059–1092. MR2408583 https://doi.org/10.1214/07-AOP353
[7] HAMMOND, A. (2019). A patchwork quilt sewn from Brownian fabric: Regularity of polymer weight profiles in Brownian last passage percolation. *Forum Math. Pi* **7** e2, 69. MR3987302 https://doi.org/10.1017/fmp.2019.2
[8] HAMMOND, A. (2021). Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation. *Mem. Amer. Math. Soc.* To appear. Available at arXiv:1609.02971.
[9] JOHANSSON, K. (2003). Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.* **242** 277–329. MR2018275 https://doi.org/10.1007/s00220-003-0945-y
[10] KARDAR, M., PARISI, G. and ZHANG, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892.
[11] MATETSKI, K., QUASTEL, J. and REMENIK, D. (2016). The KPZ fixed point. Available at arXiv:1701.00018.
[12] NICA, M., QUASTEL, J. and REMENIK, D. (2020). One-sided reflected Brownian motions and the KPZ fixed point. Available at arXiv:2002.02922.
[13] O'CONNELL, N. and YOR, M. (2002). A representation for non-colliding random walks. *Electron. Commun. Probab.* **7** 1–12. MR1887169 https://doi.org/10.1214/ECP.v7-1042
[14] PIMENTEL, L. P. R. (2018). Local behaviour of Airy processes. *J. Stat. Phys.* **173** 1614–1638. MR3880997 https://doi.org/10.1007/s10955-018-2147-1
[15] PIMENTEL, L. P. R. (2019). Brownian aspects of the KPZ fixed point. Available at arXiv:1912.11712.
[16] PITMAN, J. W. (1975). One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. in Appl. Probab.* **7** 511–526. MR0375485 https://doi.org/10.2307/1426125
[17] PRÄHOFER, M. and SPOHN, H. (2002). Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* **108** 1071–1106. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR1933446 https://doi.org/10.1023/A:1019791415147
[18] QUASTEL, J. and REMENIK, D. (2011). Local Brownian property of the narrow wedge solution of the KPZ equation. *Electron. Commun. Probab.* **16** 712–719. MR2861435 https://doi.org/10.1214/ECP.v16-1678

MSC2020 subject classifications. 82B23, 82C22, 60H15.
Key words and phrases. KPZ fixed point, locally Brownian, absolute continuity, Brownian on compacts, finitary initial conditions.

[19] QUASTEL, J. and REMENIK, D. (2013). Local behavior and hitting probabilities of the Airy_1 process. *Probab. Theory Related Fields* **157** 605–634. MR3129799 https://doi.org/10.1007/s00440-012-0466-8

The Annals of Probability
2021, Vol. 49, No. 4, 1738–1777
https://doi.org/10.1214/20-AOP1492

BULK PROPERTIES OF THE AIRY LINE ENSEMBLE

BY DUNCAN DAUVERGNE[1] AND BÁLINT VIRÁG[2]

[1] *Department of Mathematics, Princeton University, dd18@math.princeton.edu*
[2] *Department of Mathematics, University of Toronto, balint@math.toronto.edu*

The Airy line ensemble is a central object in random matrix theory and last passage percolation defined by a determinantal formula. The goal of this paper is to provide a set of tools, which allow for precise probabilistic analysis of the Airy line ensemble. The two main theorems are a representation in terms of independent Brownian bridges connecting a fine grid of points, and a modulus of continuity result for all lines. Along the way, we give tail bounds and moduli of continuity for nonintersecting Brownian ensembles, and a quick proof of tightness for Dyson's Brownian motion converging to the Airy line ensemble.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A. (1972). *Handbook of Mathematical Functions*: *With Formulas*, *Graphs*, *and Mathematical Tables*, **55**. Dover Publications, New York.

[2] ADLER, M. and VAN MOERBEKE, P. (2005). PDEs for the joint distributions of the Dyson, Airy and sine processes. *Ann*. *Probab*. **33** 1326–1361. MR2150191 https://doi.org/10.1214/009117905000000107

[3] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[4] BASU, R., HOFFMAN, C. and SLY, A. (2018). Nonexistence of bigeodesics in integrable models of last passage percolation. Preprint. Available at arXiv:1811.04908.

[5] BASU, R., SARKAR, S. and SLY, A. (2017). Invariant measures for TASEP with a slow bond. Preprint. Available at arXiv:1704.07799.

[6] BASU, R., SARKAR, S. and SLY, A. (2019). Coalescence of geodesics in exactly solvable models of last passage percolation. *J*. *Math*. *Phys*. **60** 093301, 22. MR4002528 https://doi.org/10.1063/1.5093799

[7] BASU, R., SIDORAVICIUS, V. and SLY, A. (2014). Last passage percolation with a defect line and the solution of the slow bond problem. Preprint. Available at arXiv:1408.3464.

[8] CALVERT, J., HAMMOND, A. and HEGDE, M. (2019). Brownian structure in the KPZ fixed point. Preprint. Available at arXiv:1912.00992.

[9] CORWIN, I. and HAMMOND, A. (2014). Brownian Gibbs property for Airy line ensembles. *Invent*. *Math*. **195** 441–508. MR3152753 https://doi.org/10.1007/s00222-013-0462-3

[10] CORWIN, I. and HAMMOND, A. (2016). KPZ line ensemble. *Probab*. *Theory Related Fields* **166** 67–185. MR3547737 https://doi.org/10.1007/s00440-015-0651-7

[11] CORWIN, I. and SUN, X. (2014). Ergodicity of the Airy line ensemble. *Electron*. *Commun*. *Probab*. **19** no. 49, 11. MR3246968 https://doi.org/10.1214/ECP.v19-3504

[12] DAUVERGNE, D., ORTMANN, J. and VIRÁG, B. (2018). The directed landscape. Preprint. Available at arXiv:1812.00309.

[13] DAUVERGNE, D. and VIRÁG, B. (2020). The scaling limit of the longest increasing subsequence. In *Preparation*.

[14] DIMITROV, E. and MATETSKI, K. (2020). Characterization of Brownian Gibbsian line ensembles. Preprint. Available at arXiv:2002.00684.

[15] DYSON, F. J. (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J*. *Math*. *Phys*. **3** 1191–1198. MR0148397 https://doi.org/10.1063/1.1703862

[16] GEORGIOU, N., RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2017). Geodesics and the competition interface for the corner growth model. *Probab*. *Theory Related Fields* **169** 223–255. MR3704769 https://doi.org/10.1007/s00440-016-0734-0

MSC2020 subject classifications. Primary 60K35; secondary 60B20.
Key words and phrases. Airy line ensemble, Airy process, KPZ universality class, last passage percolation, Brownian Gibbs property, modulus of continuity, Dyson's Brownian motion.

[17] GRABINER, D. J. (1999). Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **35** 177–204. MR1678525 https://doi.org/10.1016/S0246-0203(99)80010-7

[18] GUSTAVSSON, J. (2005). Gaussian fluctuations of eigenvalues in the GUE. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 151–178. MR2124079 https://doi.org/10.1016/j.anihpb.2004.04.002

[19] HAMMOND, A. (2016). Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation. Preprint. Available at arXiv:1609.02971.

[20] HAMMOND, A. (2019). Modulus of continuity of polymer weight profiles in Brownian last passage percolation. *Ann. Probab.* **47** 3911–3962. MR4038045 https://doi.org/10.1214/19-aop1350

[21] HAMMOND, A. (2019). A patchwork quilt sewn from Brownian fabric: Regularity of polymer weight profiles in Brownian last passage percolation. *Forum Math. Pi* **7** e2, 69. MR3987302 https://doi.org/10.1017/fmp.2019.2

[22] HAMMOND, A. (2020). Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *Proc. Lond. Math. Soc.* (3) **120** 370–433. MR4008374 https://doi.org/10.1112/plms.12292

[23] HAMMOND, A. and SARKAR, S. (2020). Modulus of continuity for polymer fluctuations and weight profiles in Poissonian last passage percolation. *Electron. J. Probab.* **25** Paper No. 29, 38. MR4073690 https://doi.org/10.1214/20-ejp430

[24] HORN, R. A. and JOHNSON, C. R. (2012). *Matrix Analysis*, Cambridge Univ. Press, Cambridge. MR2978290

[25] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. *University Lecture Series* **51**. Amer. Math. Soc., Providence, RI. MR2552864 https://doi.org/10.1090/ulect/051

[26] KALLENBERG, O. (2006). *Foundations of Modern Probability*. Springer Science & Business Media.

[27] LEDOUX, M. (2007). Deviation inequalities on largest eigenvalues. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **1910** 167–219. Springer, Berlin. MR2349607 https://doi.org/10.1007/978-3-540-72053-9_10

[28] LEDOUX, M. and RIDER, B. (2010). Small deviations for beta ensembles. *Electron. J. Probab.* **15** 1319–1343. MR2678393 https://doi.org/10.1214/EJP.v15-798

[29] O'CONNELL, N. (2003). Random matrices, non-colliding processes and queues. In *Séminaire de Probabilités, XXXVI*. *Lecture Notes in Math.* **1801** 165–182. Springer, Berlin. MR1971584 https://doi.org/10.1007/978-3-540-36107-7_3

[30] PIMENTEL, L. P. R. (2016). Duality between coalescence times and exit points in last-passage percolation models. *Ann. Probab.* **44** 3187–3206. MR3551194 https://doi.org/10.1214/15-AOP1044

[31] PRÄHOFER, M. and SPOHN, H. (2002). Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* **108** 1071–1106. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR1933446 https://doi.org/10.1023/A:1019791415147

[32] SOSHNIKOV, A. B. (2000). Gaussian fluctuation for the number of particles in Airy, Bessel, sine, and other determinantal random point fields. *J. Stat. Phys.* **100** 491–522. MR1788476 https://doi.org/10.1023/A:1018672622921

The Annals of Probability
2021, Vol. 49, No. 4, 1778–1846
https://doi.org/10.1214/20-AOP1493

EIGENVECTOR STATISTICS OF LÉVY MATRICES

BY AMOL AGGARWAL[1], PATRICK LOPATTO[2] AND JAKE MARCINEK[3]

[1]*Department of Mathematics, Columbia University, amolaggarwal@math.columbia.edu*
[2]*School of Mathematics, Institute for Advanced Study, lopatto@ias.edu*
[3]*Department of Mathematics, Harvard University, marcinek@math.harvard.edu*

We analyze statistics for eigenvector entries of heavy-tailed random symmetric matrices (also called Lévy matrices) whose associated eigenvalues are sufficiently small. We show that the limiting law of any such entry is non-Gaussian, given by the product of a normal distribution with another random variable that depends on the location of the corresponding eigenvalue. Although the latter random variable is typically nonexplicit, for the median eigenvector it is given by the inverse of a one-sided stable law. Moreover, we show that different entries of the same eigenvector are asymptotically independent, but that there are nontrivial correlations between eigenvectors with nearby eigenvalues. Our findings contrast sharply with the known eigenvector behavior for Wigner matrices and sparse random graphs.

REFERENCES

[1] ABOU-CHACRA, R. and THOULESS, D. (1974). Self-consistent theory of localization. II. Localization near the band edges. *J. Phys. C, Solid State Phys.* **7** 65.

[2] ABOU-CHACRA, R., THOULESS, D. and ANDERSON, P. (1973). A selfconsistent theory of localization. *J. Phys. C, Solid State Phys.* **6** 1734.

[3] ABRAHAMS, E., ed. (2010). 50 *Years of Anderson Localization*. World Scientific, Hackensack, NJ. MR2848302 https://doi.org/10.1142/9789814299084

[4] AGGARWAL, A. (2019). Bulk universality for generalized Wigner matrices with few moments. *Probab. Theory Related Fields* **173** 375–432. MR3916110 https://doi.org/10.1007/s00440-018-0836-y

[5] AGGARWAL, A., LOPATTO, P. and YAU, H.-T. (2018). GOE statistics for Lévy matrices. Preprint. Available at arXiv:1806.07363.

[6] AIZENMAN, M. and WARZEL, S. (2011). Absence of mobility edge for the Anderson random potential on tree graphs at weak disorder. *Europhys. Lett.* **96** 37004.

[7] AIZENMAN, M. and WARZEL, S. (2013). Resonant delocalization for random Schrödinger operators on tree graphs. *J. Eur. Math. Soc.* (*JEMS*) **15** 1167–1222. MR3055759 https://doi.org/10.4171/JEMS/389

[8] AJANKI, O., ERDŐS, L. and KRÜGER, T. (2017). Singularities of solutions to quadratic vector equations on the complex upper half-plane. *Comm. Pure Appl. Math.* **70** 1672–1705. MR3684307 https://doi.org/10.1002/cpa.21639

[9] AJANKI, O. H., ERDŐS, L. and KRÜGER, T. (2016). Local spectral statistics of Gaussian matrices with correlated entries. *J. Stat. Phys.* **163** 280–302. MR3478311 https://doi.org/10.1007/s10955-016-1479-y

[10] AJANKI, O. H., ERDŐS, L. and KRÜGER, T. (2017). Universality for general Wigner-type matrices. *Probab. Theory Related Fields* **169** 667–727. MR3719056 https://doi.org/10.1007/s00440-016-0740-2

[11] ALDOUS, D. (1992). Asymptotics in the random assignment problem. *Probab. Theory Related Fields* **93** 507–534. MR1183889 https://doi.org/10.1007/BF01192719

[12] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*. *Encyclopaedia Math. Sci.* **110** 1–72. Springer, Berlin. MR2023650 https://doi.org/10.1007/978-3-662-09444-0_1

[13] ANDERSON, P. W. (1958). Absence of diffusion in certain random lattices. *Phys. Rev.* **109** 1492.

[14] ANDERSON, P. W. (1978). Local moments and localized states. *Rev. Modern Phys.* **50** 191.

[15] AUFFINGER, A., BEN AROUS, G. and PÉCHÉ, S. (2009). Poisson convergence for the largest eigenvalues of heavy tailed random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 589–610. MR2548495 https://doi.org/10.1214/08-AIHP188

MSC2020 subject classifications. 15B52, 60B20.
Key words and phrases. Lévy matrices, eigenvector statistics, eigenvector moment flow, resolvent, Poisson weighted infinite tree.

[16] AUFFINGER, A. and TANG, S. (2016). Extreme eigenvalues of sparse, heavy tailed random matrices. *Stochastic Process. Appl.* **126** 3310–3330. MR3549709 https://doi.org/10.1016/j.spa.2016.04.029

[17] BAPST, V. (2014). The large connectivity limit of the Anderson model on tree graphs. *J. Math. Phys.* **55** 092101, 20. MR3390777 https://doi.org/10.1063/1.4894055

[18] BAUERSCHMIDT, R., HUANG, J., KNOWLES, A. and YAU, H.-T. (2017). Bulk eigenvalue statistics for random regular graphs. *Ann. Probab.* **45** 3626–3663. MR3729611 https://doi.org/10.1214/16-AOP1145

[19] BELINSCHI, S., DEMBO, A. and GUIONNET, A. (2009). Spectral measure of heavy tailed band and covariance random matrices. *Comm. Math. Phys.* **289** 1023–1055. MR2511659 https://doi.org/10.1007/s00220-009-0822-4

[20] BEN AROUS, G. and GUIONNET, A. (2008). The spectrum of heavy tailed random matrices. *Comm. Math. Phys.* **278** 715–751. MR2373441 https://doi.org/10.1007/s00220-007-0389-x

[21] BENAYCH-GEORGES, F. and GUIONNET, A. (2014). Central limit theorem for eigenvectors of heavy tailed matrices. *Electron. J. Probab.* **19** no. 54, 27. MR3227063 https://doi.org/10.1214/EJP.v19-3093

[22] BENAYCH-GEORGES, F. and KNOWLES, A. (2016). Lectures on the local semicircle law for Wigner matrices. Preprint. Available at arXiv:1601.04055.

[23] BENIGNI, L. (2019). Fermionic eigenvector moment flow. Preprint. Available at arXiv:1908.10855.

[24] BENIGNI, L. (2020). Eigenvectors distribution and quantum unique ergodicity for deformed Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2822–2867. MR4164858 https://doi.org/10.1214/20-AIHP1060

[25] BIANE, P. (1997). On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.* **46** 705–718. MR1488333 https://doi.org/10.1512/iumj.1997.46.1467

[26] BIROLI, G., BOUCHAUD, J.-P. and POTTERS, M. (2007). Extreme value problems in random matrix theory and other disordered systems. *J. Stat. Mech. Theory Exp.* **7** P07019, 15. MR2335697 https://doi.org/10.1088/1742-5468/2007/07/p07019

[27] BIROLI, G., BOUCHAUD, J.-P. and POTTERS, M. (2007). On the top eigenvalue of heavy-tailed random matrices. *Europhys. Lett. EPL* **78** Art. 10001, 5. MR2371333 https://doi.org/10.1209/0295-5075/78/10001

[28] BORDENAVE, C., CAPUTO, P. and CHAFAÏ, D. (2011). Spectrum of large random reversible Markov chains: Heavy-tailed weights on the complete graph. *Ann. Probab.* **39** 1544–1590. MR2857250 https://doi.org/10.1214/10-AOP587

[29] BORDENAVE, C. and GUIONNET, A. (2013). Localization and delocalization of eigenvectors for heavy-tailed random matrices. *Probab. Theory Related Fields* **157** 885–953. MR3129806 https://doi.org/10.1007/s00440-012-0473-9

[30] BORDENAVE, C. and GUIONNET, A. (2017). Delocalization at small energy for heavy-tailed random matrices. *Comm. Math. Phys.* **354** 115–159. MR3656514 https://doi.org/10.1007/s00220-017-2914-x

[31] BOUCHAUD, J. (1995). More Lévy distributions in physics. In *Lévy Flights and Related Topics in Physics* 237–250. Springer, Berlin.

[32] BOUCHAUD, J.-P. and POTTERS, M. (2011). Financial applications of random matrix theory: A short review. In *The Oxford Handbook of Random Matrix Theory* 824–850. Oxford Univ. Press, Oxford. MR2932660

[33] BOUCHAUD, J.-P., SORNETTE, D. and POTTERS, M. (1997). Option pricing in the presence of extreme fluctuations. In *Mathematics of Derivative Securities* (*Cambridge*, 1995). *Publ. Newton Inst.* **15** 112–125. Cambridge Univ. Press, Cambridge. MR1491371

[34] BOUCHAUD, J.-P., SORNETTE, D., WALTER, C. and AGUILAR, J. (1998). Taming large events: Optimal portfolio theory for strongly fluctuating assets. *Int. J. Theor. Appl. Finance* **1** 25–41.

[35] BOURGADE, P. (2018). Random band matrices. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro* 2018. *Vol. IV. Invited Lectures* 2759–2784. World Sci. Publ., Hackensack, NJ. MR3966510

[36] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2016). Fixed energy universality for generalized Wigner matrices. *Comm. Pure Appl. Math.* **69** 1815–1881. MR3541852 https://doi.org/10.1002/cpa.21624

[37] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2017). Universality for a class of random band matrices. *Adv. Theor. Math. Phys.* **21** 739–800. MR3695802 https://doi.org/10.4310/ATMP.2017.v21.n3.a5

[38] BOURGADE, P., HUANG, J. and YAU, H.-T. (2017). Eigenvector statistics of sparse random matrices. *Electron. J. Probab.* **22** Paper No. 64, 38. MR3690289 https://doi.org/10.1214/17-EJP81

[39] BOURGADE, P., YANG, F., YAU, H.-T. and YIN, J. (2019). Random band matrices in the delocalized phase, II: Generalized resolvent estimates. *J. Stat. Phys.* **174** 1189–1221. MR3934695 https://doi.org/10.1007/s10955-019-02229-z

[40] BOURGADE, P. and YAU, H.-T. (2017). The eigenvector moment flow and local quantum unique ergodicity. *Comm. Math. Phys.* **350** 231–278. MR3606475 https://doi.org/10.1007/s00220-016-2627-6

[41] BOURGADE, P., YAU, H.-T. and YIN, J. (2020). Random band matrices in the delocalized phase I: Quantum unique ergodicity and universality. *Comm. Pure Appl. Math.* **73** 1526–1596. MR4156609 https://doi.org/10.1002/cpa.21895

[42] BUN, J., BOUCHAUD, J.-P. and POTTERS, M. (2017). Cleaning large correlation matrices: Tools from random matrix theory. *Phys. Rep.* **666** 1–109. MR3590056 https://doi.org/10.1016/j.physrep.2016.10.005

[43] BURDA, Z., JURKIEWICZ, J., NOWAK, M. A., PAPP, G. and ZAHED, I. (2005). Random Lévy matrices. II. *Acta Phys. Polon. B* **36** 2635–2640. MR2188085

[44] CHE, Z. (2017). Universality of random matrices with correlated entries. *Electron. J. Probab.* **22** Paper No. 30, 38. MR3629874 https://doi.org/10.1214/17-EJP46

[45] CIZEAU, P. and BOUCHAUD, J. (1993). Mean field theory of dilute spin-glasses with power-law interactions. *J. Phys. A: Math. Gen.* **26** L187.

[46] CIZEAU, P. and BOUCHAUD, J.-P. (1994). Theory of Lévy matrices. *Phys. Rev. E* **50** 1810.

[47] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 https://doi.org/10.1007/s00220-012-1527-7

[48] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** 2279–2375. MR3098073 https://doi.org/10.1214/11-AOP734

[49] ERDŐS, L., PÉCHÉ, S., RAMÍREZ, J. A., SCHLEIN, B. and YAU, H.-T. (2010). Bulk universality for Wigner matrices. *Comm. Pure Appl. Math.* **63** 895–925. MR2662426 https://doi.org/10.1002/cpa.20317

[50] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2011). Universality of random matrices and local relaxation flow. *Invent. Math.* **185** 75–119. MR2810797 https://doi.org/10.1007/s00222-010-0302-7

[51] ERDŐS, L., SCHLEIN, B., YAU, H.-T. and YIN, J. (2012). The local relaxation flow approach to universality of the local statistics for random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 1–46. MR2919197 https://doi.org/10.1214/10-AIHP388

[52] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory. Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York. MR3699468

[53] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Bulk universality for generalized Wigner matrices. *Probab. Theory Related Fields* **154** 341–407. MR2981427 https://doi.org/10.1007/s00440-011-0390-3

[54] GALLUCCIO, S., BOUCHAUD, J.-P. and POTTERS, M. (1998). Rational decisions, random matrices and spin glasses. *Phys. A, Stat. Mech. Appl.* **259** 449–456.

[55] HEINY, J. and MIKOSCH, T. (2019). The eigenstructure of the sample covariance matrices of high-dimensional stochastic volatility models with heavy tails. *Bernoulli* **25** 3590–3622. MR4010966 https://doi.org/10.3150/18-bej1103

[56] HUANG, J. and LANDON, B. (2020). Spectral statistics of sparse Erdős–Rényi graph Laplacians. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 120–154. MR4058984 https://doi.org/10.1214/19-AIHP957

[57] HUANG, J., LANDON, B. and YAU, H.-T. (2015). Bulk universality of sparse random matrices. *J. Math. Phys.* **56** 123301, 19. MR3429490 https://doi.org/10.1063/1.4936139

[58] JOHANSSON, K. (2001). Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Comm. Math. Phys.* **215** 683–705. MR1810949 https://doi.org/10.1007/s002200000328

[59] JOHANSSON, K. (2012). Universality for certain Hermitian Wigner matrices under weak moment conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 47–79. MR2919198 https://doi.org/10.1214/11-AIHP429

[60] KNOWLES, A. and YIN, J. (2013). Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields* **155** 543–582. MR3034787 https://doi.org/10.1007/s00440-011-0407-y

[61] LALOUX, L., CIZEAU, P., BOUCHAUD, J.-P. and POTTERS, M. (1999). Noise dressing of financial correlation matrices. *Phys. Rev. Lett.* **83** 1467.

[62] LALOUX, L., CIZEAU, P., POTTERS, M. and BOUCHAUD, J.-P. (2000). Random matrix theory and financial correlations. *Int. J. Theor. Appl. Finance* **3** 391–397.

[63] LANDON, B. and YAU, H.-T. (2017). Convergence of local statistics of Dyson Brownian motion. *Comm. Math. Phys.* **355** 949–1000. MR3687212 https://doi.org/10.1007/s00220-017-2955-1

[64] LEE, J. O. and YIN, J. (2014). A necessary and sufficient condition for edge universality of Wigner matrices. *Duke Math. J.* **163** 117–173. MR3161313 https://doi.org/10.1215/00127094-2414767

[65] MAHONEY, M. and MARTIN, C. (2019). Traditional and heavy tailed self regularization in neural network models. In *International Conference on Machine Learning* 4284–4293.

[66] MARTIN, C. H. and MAHONEY, M. W. (2018). Implicit self-regularization in deep neural networks: Evidence from random matrix theory and implications for learning. Preprint. Available at arXiv:1810.01075.

[67] MARTIN, C. H. and MAHONEY, M. W. (2019). Heavy-tailed universality predicts trends in test accuracies for very large pre-trained deep neural networks. Preprint. Available at arXiv:1901.08278.

[68] MONTHUS, C. (2016). Localization transition in random Lévy matrices: Multifractality of eigenvectors in the localized phase and at criticality. *J. Stat. Mech. Theory Exp.* **9** 093304, 22. MR3562241 https://doi.org/10.1088/1742-5468/2016/09/093304

[69] MOTT, N. (1987). The mobility edge since 1967. *J. Phys. C, Solid State Phys.* **20** 3075.

[70] MOTT, N. F. (1949). The basis of the electron theory of metals, with special reference to the transition metals. *Proc. R. Soc. Edinb. A* **62** 416.

[71] SHOHAT, J. A. and TAMARKIN, J. D. (1943). *The Problem of Moments. American Mathematical Society Mathematical Surveys, Vol. I.* Amer. Math. Soc., New York. MR0008438

[72] SORNETTE, D. (2006). *Critical Phenomena in Natural Sciences: Chaos, Fractals, Selforganization and Disorder: Concepts and Tools*. Springer Science & Business Media.

[73] SOSHNIKOV, A. (2004). Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Commun. Probab.* **9** 82–91. MR2081462 https://doi.org/10.1214/ECP.v9-1112

[74] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses. Volume I: Basic Examples. Ergebnisse der Mathematik und Ihrer Grenzgebiete.* 3. *Folge. A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas.* 3*rd Series. A Series of Modern Surveys in Mathematics*] **54**. Springer, Berlin. MR2731561 https://doi.org/10.1007/978-3-642-15202-3

[75] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math.* **206** 127–204. MR2784665 https://doi.org/10.1007/s11511-011-0061-3

[76] TAO, T. and VU, V. (2012). Random matrices: Universal properties of eigenvectors. *Random Matrices Theory Appl.* **1** 1150001, 27. MR2930379 https://doi.org/10.1142/S2010326311500018

[77] TARQUINI, E., BIROLI, G. and TARZIA, M. (2016). Level statistics and localization transitions of Lévy matrices. *Phys. Rev. Lett.* **116** 010601, 5. MR3555687 https://doi.org/10.1103/PhysRevLett.116.010601

[78] WIGNER, E. P. (1955). Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math.* (2) **62** 548–564. MR0077805 https://doi.org/10.2307/1970079

[79] YANG, F. and YIN, J. (2018). Random band matrices in the delocalized phase, III: Averaging fluctuations. Preprint. Available at arXiv:1807.02447.

The Annals of Probability
2021, Vol. 49, No. 4, 1847–1885
https://doi.org/10.1214/20-AOP1495

SPECTRAL EDGE IN SPARSE RANDOM GRAPHS: UPPER AND LOWER TAIL LARGE DEVIATIONS

BY BHASWAR B. BHATTACHARYA[1], SOHOM BHATTACHARYA[2] AND SHIRSHENDU GANGULY[3]

[1] *Department of Statistics, University of Pennsylvania, bhaswar@wharton.upenn.edu*
[2] *Department of Statistics, Stanford University, sohomb@stanford.edu*
[3] *Department of Statistics, University of California, Berkeley, sganguly@berkeley.edu*

In this paper, we consider the problem of estimating the joint upper and lower tail large deviations of the edge eigenvalues of an Erdős–Rényi random graph $\mathcal{G}_{n,p}$, in the regime of p where the edge of the spectrum is no longer governed by global observables, such as the number of edges, but rather by localized statistics, such as high degree vertices. Going beyond the recent developments in mean-field approximations of related problems, this paper provides a comprehensive treatment of the large deviations of the spectral edge in this entire regime, which notably includes the well-studied case of constant average degree. In particular, for $r \geq 1$ fixed, we pin down the asymptotic probability that the top r eigenvalues are jointly greater/less than their typical values by multiplicative factors bigger/smaller than 1, in the regime mentioned above. The proof for the upper tail relies on a novel structure theorem, obtained by building on estimates in (*Combin. Probab. Comput.* **12** (2003) 61–72), followed by an iterative cycle removal process, which shows, conditional on the upper tail large deviation event, with high probability the graph admits a decomposition in to a disjoint union of stars and a spectrally negligible part. On the other hand, the key ingredient in the proof of the lower tail is a Ramsey-type result which shows that if the Kth largest degree of a graph is not atypically small (for some large K depending on r), then either the top eigenvalue or the rth largest eigenvalue is larger than that allowed by the lower tail event on the top r eigenvalues, thus forcing a contradiction. The above arguments reduce the problems to developing a large deviation theory for the extremal degrees which could be of independent interest.

REFERENCES

[1] ALON, N., KRIVELEVICH, M. and VU, V. H. (2002). On the concentration of eigenvalues of random symmetric matrices. *Israel J. Math.* **131** 259–267. MR1942311 https://doi.org/10.1007/BF02785860
[2] ALT, J., DUCATEZ, R. and KNOWLES, A. (2019). Extremal eigenvalues of critical Erdős–Rényi graphs. Preprint. Available at arXiv:1905.03243.
[3] ASH, R. B. (1990). *Information Theory*. Dover, New York. MR1088248
[4] AUGERI, F. (2016). Large deviations principle for the largest eigenvalue of Wigner matrices without Gaussian tails. *Electron. J. Probab.* **21** Paper No. 32, 49. MR3492936 https://doi.org/10.1214/16-EJP4146
[5] AUGERI, F. (2020). Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 2404–2448. MR4152647 https://doi.org/10.1214/20-AOP1427
[6] AUSTIN, T. (2019). The structure of low-complexity Gibbs measures on product spaces. *Ann. Probab.* **47** 4002–4023. MR4038047 https://doi.org/10.1214/19-aop1352
[7] BASAK, A. and BASU, R. (2019). Upper tail large deviations of regular subgraph counts in Erdős–Rényi graphs in the full localized regime. Preprint. Available at arXiv:1912.11410.
[8] BASAK, A. and MUKHERJEE, S. (2017). Universality of the mean-field for the Potts model. *Probab. Theory Related Fields* **168** 557–600. MR3663625 https://doi.org/10.1007/s00440-016-0718-0
[9] BEN AROUS, G., DEMBO, A. and GUIONNET, A. (2001). Aging of spherical spin glasses. *Probab. Theory Related Fields* **120** 1–67. MR1856194 https://doi.org/10.1007/PL00008774

MSC2020 subject classifications. 60F10, 05C80, 60B20, 15A18.
Key words and phrases. Erdős–Rényi random graph, large deviations, random matrices, extreme eigenvalues.

[10] Ben Arous, G. and Guionnet, A. (1997). Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields* **108** 517–542. MR1465640 https://doi.org/10.1007/s004400050119
[11] Benaych-Georges, F., Bordenave, C. and Knowles, A. (2019). Largest eigenvalues of sparse inhomogeneous Erdős–Rényi graphs. *Ann. Probab.* **47** 1653–1676. MR3945756 https://doi.org/10.1214/18-AOP1293
[12] Benaych-Georges, F., Bordenave, C. and Knowles, A. (2020). Spectral radii of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2141–2161. MR4116720 https://doi.org/10.1214/19-AIHP1033
[13] Bhattacharya, B. B. and Ganguly, S. (2020). Upper tails for edge eigenvalues of random graphs. *SIAM J. Discrete Math.* **34** 1069–1083. MR4083586 https://doi.org/10.1137/18M1230852
[14] Bhattacharya, B. B., Ganguly, S., Lubetzky, E. and Zhao, Y. (2017). Upper tails and independence polynomials in random graphs. *Adv. Math.* **319** 313–347. MR3695877 https://doi.org/10.1016/j.aim.2017.08.003
[15] Bollobás, B. (1985). *Random Graphs*. Academic Press [Harcourt Brace Jovanovich, Publishers], London. MR0809996
[16] Bordenave, C. and Caputo, P. (2014). A large deviation principle for Wigner matrices without Gaussian tails. *Ann. Probab.* **42** 2454–2496. MR3265172 https://doi.org/10.1214/13-AOP866
[17] Bordenave, C. and Caputo, P. (2015). Large deviations of empirical neighborhood distribution in sparse random graphs. *Probab. Theory Related Fields* **163** 149–222. MR3405616 https://doi.org/10.1007/s00440-014-0590-8
[18] Bordenave, C., Sen, A. and Virág, B. (2017). Mean quantum percolation. *J. Eur. Math. Soc.* (*JEMS*) **19** 3679–3707. MR3730511 https://doi.org/10.4171/JEMS/750
[19] Chatterjee, S. and Dembo, A. (2016). Nonlinear large deviations. *Adv. Math.* **299** 396–450. MR3519474 https://doi.org/10.1016/j.aim.2016.05.017
[20] Chatterjee, S. and Varadhan, S. R. S. (2011). The large deviation principle for the Erdős–Rényi random graph. *European J. Combin.* **32** 1000–1017. MR2825532 https://doi.org/10.1016/j.ejc.2011.03.014
[21] Chatterjee, S. and Varadhan, S. R. S. (2012). Large deviations for random matrices. *Commun. Stoch. Anal.* **6** 1–13. MR2890846 https://doi.org/10.31390/cosa.6.1.02
[22] Conlon, D., Fox, J. and Sudakov, B. (2015). Recent developments in graph Ramsey theory. In *Surveys in Combinatorics* 2015. *London Mathematical Society Lecture Note Series* **424** 49–118. Cambridge Univ. Press, Cambridge. MR3497267
[23] Cook, N. and Dembo, A. (2020). Large deviations of subgraph counts for sparse Erdős–Rényi graphs. *Adv. Math.* **373** 107289, 53. MR4130460 https://doi.org/10.1016/j.aim.2020.107289
[24] Dean, D. S. and Majumdar, S. N. (2006). Large deviations of extreme eigenvalues of random matrices. *Phys. Rev. Lett.* **97** 160201, 4. MR2274338 https://doi.org/10.1103/PhysRevLett.97.160201
[25] Eldan, R. (2018). Gaussian-width gradient complexity, reverse log-Sobolev inequalities and nonlinear large deviations. *Geom. Funct. Anal.* **28** 1548–1596. MR3881829 https://doi.org/10.1007/s00039-018-0461-z
[26] Erdős, L., Knowles, A., Yau, H.-T. and Yin, J. (2012). Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 https://doi.org/10.1007/s00220-012-1527-7
[27] Erdős, L., Knowles, A., Yau, H.-T. and Yin, J. (2013). Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** 2279–2375. MR3098073 https://doi.org/10.1214/11-AOP734
[28] Feige, U. and Ofek, E. (2005). Spectral techniques applied to sparse random graphs. *Random Structures Algorithms* **27** 251–275. MR2155709 https://doi.org/10.1002/rsa.20089
[29] Füredi, Z. and Komlós, J. (1981). The eigenvalues of random symmetric matrices. *Combinatorica* **1** 233–241. MR0637828 https://doi.org/10.1007/BF02579329
[30] Grimmett, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6
[31] Guionnet, A. and Husson, J. (2020). Large deviations for the largest eigenvalue of Rademacher matrices. *Ann. Probab.* **48** 1436–1465. MR4112720 https://doi.org/10.1214/19-AOP1398
[32] Harel, M., Mousset, F. and Samotij, W. (2019). Upper tails via high moments and entropic stability. Preprint. Available at arXiv:1904.08212.
[33] Krivelevich, M. and Sudakov, B. (2003). The largest eigenvalue of sparse random graphs. *Combin. Probab. Comput.* **12** 61–72. MR1967486 https://doi.org/10.1017/S0963548302005424
[34] Le, C. M., Levina, E. and Vershynin, R. (2017). Concentration and regularization of random graphs. *Random Structures Algorithms* **51** 538–561. MR3689343 https://doi.org/10.1002/rsa.20713

[35] LUBETZKY, E. and ZHAO, Y. (2015). On replica symmetry of large deviations in random graphs. *Random Structures Algorithms* **47** 109–146. MR3366814 https://doi.org/10.1002/rsa.20536

[36] LUBETZKY, E. and ZHAO, Y. (2017). On the variational problem for upper tails in sparse random graphs. *Random Structures Algorithms* **50** 420–436. MR3632418 https://doi.org/10.1002/rsa.20658

[37] LUGOSI, G., MENDELSON, S. and ZHIVOTOVSKIY, N. (2020). Concentration of the spectral norm of Erdős–Rényi random graphs. *Bernoulli* **26** 2253–2274. MR4091108 https://doi.org/10.3150/19-BEJ1192

[38] REIMER, D. (2000). Proof of the van den Berg–Kesten conjecture. *Combin. Probab. Comput.* **9** 27–32. MR1751301 https://doi.org/10.1017/S0963548399004113

[39] SPIELMAN, D. A. (2019). Spectral and algebraic graph theory. Available at http://cs-www.cs.yale.edu/homes/spielman/sagt.

[40] VIVO, P., MAJUMDAR, S. N. and BOHIGAS, O. (2007). Large deviations of the maximum eigenvalue in Wishart random matrices. *J. Phys. A* **40** 4317–4337. MR2316708 https://doi.org/10.1088/1751-8113/40/16/005

[41] VU, V. H. (2007). Spectral norm of random matrices. *Combinatorica* **27** 721–736. MR2384414 https://doi.org/10.1007/s00493-007-2190-z

[42] YAN, J. (2020). Nonlinear large deviations: Beyond the hypercube. *Ann. Appl. Probab.* **30** 812–846. MR4108123 https://doi.org/10.1214/19-AAP1516

The Annals of Probability
2021, Vol. 49, No. 4, 1886–1916
https://doi.org/10.1214/20-AOP1496

ON WORDS OF NON-HERMITIAN RANDOM MATRICES

BY GUILLAUME DUBACH[1] AND YUVAL PELED[2]

[1] *IST Austria, guillaume.dubach@ist.ac.at*
[2] *Courant Institute, New York University, yuval.peled@cims.nyu.edu*

We consider words $G_{i_1} \cdots G_{i_m}$ involving i.i.d. complex Ginibre matrices and study tracial expressions of their eigenvalues and singular values. We show that the limit distribution of the squared singular values of every word of length m is a Fuss–Catalan distribution with parameter $m+1$. This generalizes previous results concerning powers of a complex Ginibre matrix and products of independent Ginibre matrices. In addition, we find other combinatorial parameters of the word that determine the second-order limits of the spectral statistics. For instance, the so-called coperiod of a word characterizes the fluctuations of the eigenvalues. We extend these results to words of general non-Hermitian matrices with i.i.d. entries under moment-matching assumptions, band matrices, and sparse matrices.

These results rely on the moments method and genus expansion, relating Gaussian matrix integrals to the counting of compact orientable surfaces of a given genus. This allows us to derive a central limit theorem for the trace of any word of complex Ginibre matrices and their conjugate transposes, where all parameters are defined topologically.

REFERENCES

[1] AKEMANN, G. and IPSEN, J. R. (2015). Recent exact and asymptotic results for products of independent random matrices. *Acta Phys. Polon. B* **46** 1747–1784. MR3403839 https://doi.org/10.5506/APhysPolB.46.1747

[2] ALEXEEV, N. Gaussian random matrices and genus expansion. Available at http://www.pdmi.ras.ru/EIMI/2011/NTS/presentations/alexeev.pdf.

[3] ALEXEEV, N., GÖTZE, F. and TIKHOMIROV, A. (2010). Asymptotic distribution of singular values of powers of random matrices. *Lith. Math. J.* **50** 121–132. MR2653641 https://doi.org/10.1007/s10986-010-9074-4

[4] AU, B. (2018). Traffic distributions of random band matrices. *Electron. J. Probab.* **23** Paper No. 77. MR3858905 https://doi.org/10.1214/18-EJP205

[5] AU, B. (2020). Finite-rank perturbations of random band matrices via infinitesimal free probability. *Comm. Pure Appl. Math.* https://doi.org/10.1002/cpa.21924

[6] BRÉZIN, E., ITZYKSON, C., PARISI, G. and ZUBER, J. (1993). Planar diagrams. In *The Large N Expansion in Quantum Field Theory and Statistical Physics: From Spin Systems to 2-Dimensional Gravity* 567–583. World Scientific, Singapore.

[7] BURDA, Z., NOWAK, M. A. and SWIECH, A. (2012). Spectral relations between products and powers of isotropic random matrices. *Phys. Rev. E* **86** 061137.

[8] BURDA, Z. and SWIECH, A. (2015). Quaternionic R transform and non-Hermitian random matrices. *Phys. Rev. E* **92** 052111.

[9] CAPITAINE, M. and CASALIS, M. (2006). Cumulants for random matrices as convolutions on the symmetric group. *Probab. Theory Related Fields* **136** 19–36. MR2240781 https://doi.org/10.1007/s00440-005-0474-z

[10] CAPITAINE, M. and CASALIS, M. (2007). Cumulants for random matrices as convolutions on the symmetric group. II. *J. Theoret. Probab.* **20** 505–533. MR2337139 https://doi.org/10.1007/s10959-007-0077-y

[11] DIACONIS, P. and SHAHSHAHANI, M. (1994). On the eigenvalues of random matrices. *J. Appl. Probab.* **31A** 49–62. MR1274717 https://doi.org/10.2307/3214948

MSC2020 subject classifications. 60B20, 57K20.
Key words and phrases. Complex Ginibre ensemble, Fuss–Catalan distribution, genus expansion, mixed moments of non-Hermitian matrices, second order freeness, words of random matrices.

[12] DUBACH, G. (2018). Powers of Ginibre eigenvalues. *Electron. J. Probab.* **23** Paper No. 111. MR3878136 https://doi.org/10.1214/18-ejp234

[13] FEYNMAN, R. (2005). Space-time approach to non-relativistic quantum mechanics. In *Feynman's Thesis—a New Approach to Quantum Theory* 71–109. World Scientific, Singapore.

[14] GÖTZE, F., KÖSTERS, H. and TIKHOMIROV, A. (2015). Asymptotic spectra of matrix-valued functions of independent random matrices and free probability. *Random Matrices Theory Appl.* **4** 1550005. MR3356883 https://doi.org/10.1142/S2010326315500057

[15] GUIONNET, A. (2006). Random matrices and enumeration of maps. In *International Congress of Mathematicians. Vol. III* 623–636. Eur. Math. Soc., Zürich. MR2275699

[16] GUIONNET, A. and MAÏDA, M. (2005). Character expansion method for the first order asymptotics of a matrix integral. *Probab. Theory Related Fields* **132** 539–578. MR2198201 https://doi.org/10.1007/s00440-004-0403-6

[17] GUIONNET, A. and MAUREL-SEGALA, E. (2007). Second order asymptotics for matrix models. *Ann. Probab.* **35** 2160–2212. MR2353386 https://doi.org/10.1214/009117907000000141

[18] HARER, J. and ZAGIER, D. (1986). The Euler characteristic of the moduli space of curves. *Invent. Math.* **85** 457–485. MR0848681 https://doi.org/10.1007/BF01390325

[19] IPSEN, J. R. (2015). Products of independent Gaussian random matrices. Ph.D. thesis. Available at arXiv:1510.06128.

[20] JANA, I. (2019). CLT for non-Hermitian random band matrices with variance profiles. Available at arXiv:1904.11098.

[21] KEMP, T. and SPEICHER, R. (2007). Strong Haagerup inequalities for free $\mathscr{R}$-diagonal elements. *J. Funct. Anal.* **251** 141–173. MR2353703 https://doi.org/10.1016/j.jfa.2007.03.011

[22] KIEBURG, M. and KÖSTERS, H. (2016). Exact relation between singular value and eigenvalue statistics. *Random Matrices Theory Appl.* **5** 1650015. MR3569843 https://doi.org/10.1142/S2010326316500155

[23] KIEBURG, M. and KÖSTERS, H. (2019). Products of random matrices from polynomial ensembles. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 98–126. MR3901642 https://doi.org/10.1214/17-aihp877

[24] LAMBERT, G. (2018). Limit theorems for biorthogonal ensembles and related combinatorial identities. *Adv. Math.* **329** 590–648. MR3783424 https://doi.org/10.1016/j.aim.2017.12.025

[25] LANDO, S. and ZVONKIN, A. (2013). *Graphs on Surfaces and Their Applications* **141**. Springer, Berlin.

[26] MAGEE, M. and PUDER, D. (2019). Matrix group integrals, surfaces, and mapping class groups I: $\mathcal{U}(n)$. *Invent. Math.* **218** 341–411. MR4011702 https://doi.org/10.1007/s00222-019-00891-4

[27] MALE, C. (2017). The limiting distributions of large heavy Wigner and arbitrary random matrices. *J. Funct. Anal.* **272** 1–46. MR3567500 https://doi.org/10.1016/j.jfa.2016.10.001

[28] MINGO, J. A. and NICA, A. (2004). Annular noncrossing permutations and partitions, and second-order asymptotics for random matrices. *Int. Math. Res. Not.* **28** 1413–1460. MR2052516 https://doi.org/10.1155/S1073792804133023

[29] MINGO, J. A., ŚNIADY, P. and SPEICHER, R. (2007). Second order freeness and fluctuations of random matrices. II. Unitary random matrices. *Adv. Math.* **209** 212–240. MR2294222 https://doi.org/10.1016/j.aim.2006.05.003

[30] MINGO, J. A. and SPEICHER, R. (2006). Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces. *J. Funct. Anal.* **235** 226–270. MR2216446 https://doi.org/10.1016/j.jfa.2005.10.007

[31] MINGO, J. A. and SPEICHER, R. (2017). *Free Probability and Random Matrices. Fields Institute Monographs* **35**. Springer, New York. MR3585560 https://doi.org/10.1007/978-1-4939-6942-5

[32] NICA, A. and SPEICHER, R. (2006). *Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Note Series* **335**. Cambridge Univ. Press, Cambridge. MR2266879 https://doi.org/10.1017/CBO9780511735127

[33] NOWAK, M. A. and TARNOWSKI, W. (2018). Probing non-orthogonality of eigenvectors in non-Hermitian matrix models: Diagrammatic approach. *J. High Energy Phys.* **6** 152. MR3831715 https://doi.org/10.1007/jhep06(2018)152

[34] PENSON, K. A. and ŻYCZKOWSKI, K. (2011). Product of Ginibre matrices: Fuss–Catalan and Raney distributions. *Phys. Rev. E* **83** 061118.

[35] RĂDULESCU, F. (2006). Combinatorial aspects of Connes's embedding conjecture and asymptotic distribution of traces of products of unitaries. *Operator Theory* **20** 197–205.

[36] REDELMEIER, C. E. I. (2011). Genus expansion for real Wishart matrices. *J. Theoret. Probab.* **24** 1044–1062. MR2851244 https://doi.org/10.1007/s10959-010-0278-7

[37] REDELMEIER, C. E. I. (2014). Real second-order freeness and the asymptotic real second-order freeness of several real matrix models. *Int. Math. Res. Not. IMRN* **12** 3353–3395. MR3217665 https://doi.org/10.1093/imrn/rnt043

[38] Redelmeier, C. E. I. (2016). Explicit multi-matrix topological expansion for quaternionic random matrices. *J. Math. Phys.* **57** 023511. MR3455672 https://doi.org/10.1063/1.4940338

[39] Rider, B. and Virág, B. (2007). The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN* **2** Art. ID rnm006. MR2361453 https://doi.org/10.1093/imrn/rnm006

[40] Rudelson, M. and Tikhomirov, K. (2019). The sparse circular law under minimal assumptions. *Geom. Funct. Anal.* **29** 561–637. MR3945840 https://doi.org/10.1007/s00039-019-00492-6

[41] 't Hooft, G. (1993). A planar diagram theory for strong interactions. In *The Large N Expansion in Quantum Field Theory and Statistical Physics*: *From Spin Systems to* 2-*Dimensional Gravity* 80–92. World Scientific, Singapore.

[42] Tutte, W. T. (1962). A census of planar triangulations. *Canad. J. Math.* **14** 21–38. MR0130841 https://doi.org/10.4153/CJM-1962-002-9

[43] Tutte, W. T. (1963). A census of planar maps. *Canad. J. Math.* **15** 249–271. MR0146823 https://doi.org/10.4153/CJM-1963-029-x

[44] Walters, M. and Starr, S. (2015). A note on mixed matrix moments for the complex Ginibre ensemble. *J. Math. Phys.* **56** 013301. MR3390837 https://doi.org/10.1063/1.4904451

[45] Zvonkin, A. (1997). Matrix integrals and map enumeration: An accessible introduction. *Math. Comput. Modelling* **26** 281–304. MR1492512 https://doi.org/10.1016/S0895-7177(97)00210-0

The Annals of Probability
2021, Vol. 49, No. 4, 1917–1964
https://doi.org/10.1214/20-AOP1497

ASYMPTOTICS OF THE EIGENVALUES OF THE ANDERSON HAMILTONIAN WITH WHITE NOISE POTENTIAL IN TWO DIMENSIONS

BY KHALIL CHOUK[1] AND WILLEM VAN ZUIJLEN[2]

[1]*School of Mathematics, University of Edinburgh, khalil.chouk@gmail.com*
[2]*Weierstrass Institute for Applied Analysis and Stochastics, Berlin, vanzuijlen@wias-berlin.de*

In this paper we consider the Anderson Hamiltonian with white noise potential on the box $[0, L]^2$ with Dirichlet boundary conditions. We show that all of the eigenvalues divided by $\log L$, converge as $L \to \infty$, almost surely to the same deterministic constant which is given by a variational formula.

REFERENCES

[1] ALLEZ, R. and CHOUK, K. The continuous Anderson Hamiltonian in dimension two. Preprint. Available at https://arxiv.org/abs/1511.02718.
[2] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7
[3] BISKUP, M. and KÖNIG, W. (2001). Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29** 636–682. MR1849173 https://doi.org/10.1214/aop/1008956688
[4] CANNIZZARO, G., FRIZ, P. K. and GASSIAT, P. (2017). Malliavin calculus for regularity structures: The case of gPAM. *J. Funct. Anal.* **272** 363–419. MR3567508 https://doi.org/10.1016/j.jfa.2016.09.024
[5] CARMONA, R. A. and MOLCHANOV, S. A. (1995). Stationary parabolic Anderson model and intermittency. *Probab. Theory Related Fields* **102** 433–453. MR1346261 https://doi.org/10.1007/BF01198845
[6] CHEN, X. (2010). *Random Walk Intersections*: *Large Deviations and Related Topics*. *Mathematical Surveys and Monographs* **157**. Amer. Math. Soc., Providence, RI. MR2584458 https://doi.org/10.1090/surv/157
[7] CHEN, X. (2014). Quenched asymptotics for Brownian motion in generalized Gaussian potential. *Ann. Probab.* **42** 576–622. MR3178468 https://doi.org/10.1214/12-AOP830
[8] CHOUK, K., GAIRING, J. and PERKOWSKI, N. (2017). An invariance principle for the two-dimensional parabolic Anderson model with small potential. *Stoch. Partial Differ. Equ. Anal. Comput.* **5** 520–558. MR3736653 https://doi.org/10.1007/s40072-017-0096-3
[9] CONWAY, J. B. (1990). *A Course in Functional Analysis*, 2nd ed. *Graduate Texts in Mathematics* **96**. Springer, New York. MR1070713
[10] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications*. *Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. Corrected reprint of the second (1998) edition. MR2571413 https://doi.org/10.1007/978-3-642-03311-7
[11] DEUSCHEL, J.-D. and STROOCK, D. W. (1989). *Large Deviations*. *Pure and Applied Mathematics* **137**. Academic Press, Boston, MA. MR0997938
[12] DONOGHUE, W. F. JR. (1969). *Distributions and Fourier Transforms*. *Pure and Applied Mathematics* **32**. Academic Press, New York. MR3363413
[13] DUMAZ, L. and LABBÉ, C. (2020). Localization of the continuous Anderson Hamiltonian in 1-D. *Probab. Theory Related Fields* **176** 353–419. MR4055192 https://doi.org/10.1007/s00440-019-00920-6
[14] FUKUSHIMA, M. and NAKAO, S. (1976/77). On spectra of the Schrödinger operator with a white Gaussian noise potential. *Z. Wahrsch. Verw. Gebiete* **37** 267–274. MR0438481 https://doi.org/10.1007/BF00537493
[15] GÄRTNER, J. and KÖNIG, W. (2000). Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10** 192–217. MR1765208 https://doi.org/10.1214/aoap/1019737669
[16] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6, 75. MR3406823 https://doi.org/10.1017/fmp.2015.2

MSC2020 subject classifications. Primary 60H25, 60F15, 35J10, 35J15; secondary 60F10.
Key words and phrases. Anderson Hamiltonian, white noise, paracontrolled distributions, operators with Dirichlet boundary conditions.

[17] GUBINELLI, M., UGURCAN, B. and ZACHHUBER, I. (2020). Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** 82–149. MR4058957 https://doi.org/10.1007/s40072-019-00143-9

[18] JANSON, S. (1997). *Gaussian Hilbert Spaces*. *Cambridge Tracts in Mathematics* **129**. Cambridge Univ. Press, Cambridge. MR1474726 https://doi.org/10.1017/CBO9780511526169

[19] KALLENBERG, O. (1997). *Foundations of Modern Probability*. *Probability and Its Applications (New York)*. Springer, New York. MR1464694

[20] KÖNIG, W. (2016). *The Parabolic Anderson Model: Random Walk in Random Potential*. *Pathways in Mathematics*. Birkhäuser/Springer, Cham. MR3526112 https://doi.org/10.1007/978-3-319-33596-4

[21] LABBÉ, C. (2019). The continuous Anderson Hamiltonian in $d \leq 3$. *J. Funct. Anal.* **277** 3187–3235. MR3997634 https://doi.org/10.1016/j.jfa.2019.05.027

[22] LADYŽENSKAJA, O. A. (1958). Solution "in the large" to the boundary-value problem for the Navier–Stokes equations in two space variables. *Soviet Physics. Dokl.* **123** 1128–1131. MR0108963

[23] LAX, P. D. (2002). *Functional Analysis*. *Pure and Applied Mathematics (New York)*. Wiley Interscience, New York. MR1892228

[24] NIRENBERG, L. (1959). On elliptic partial differential equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (3) **13** 115–162. MR0109940

[25] NUALART, D. (2009). *Malliavin Calculus and Its Applications*. *CBMS Regional Conference Series in Mathematics* **110**. Amer. Math. Soc., Providence, RI. MR2498953 https://doi.org/10.1090/cbms/110

[26] PAPOULIS, A. (1965). *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, New York. MR0176501

[27] PERKOWSKI, N. and ROSATI, T. A rough super-Brownian motion. Preprint. Available at http://arxiv.org/abs/1905.05825.

[28] PRÖMEL, D. J. and TRABS, M. (2016). Rough differential equations driven by signals in Besov spaces. *J. Differential Equations* **260** 5202–5249. MR3448778 https://doi.org/10.1016/j.jde.2015.12.012

[29] REED, M. and SIMON, B. (1975). *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. Academic Press, New York. MR0493420

[30] REED, M. and SIMON, B. (1978). *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York. MR0493421

[31] SCHMEISSER, H.-J. and TRIEBEL, H. (1987). *Topics in Fourier Analysis and Function Spaces*. *A Wiley-Interscience Publication*. Wiley, Chichester. MR0891189

[32] SIMON, B. (1983). Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: Asymptotic expansions. *Ann. Inst. Henri Poincaré A, Phys. Théor.* **38** 295–308. MR0708966

[33] STEIN, E. M. and SHAKARCHI, R. (2003). *Fourier Analysis*. *Princeton Lectures in Analysis: An Introduction* **1**. Princeton Univ. Press, Princeton, NJ. MR1970295

[34] TRIEBEL, H. (1983). *Theory of Function Spaces*. *Monographs in Mathematics* **78**. Birkhäuser, Basel. MR0781540 https://doi.org/10.1007/978-3-0346-0416-1

[35] VERAAR, M. C. (2011). Regularity of Gaussian white noise on the d-dimensional torus. In *Marcinkiewicz Centenary Volume*. *Banach Center Publ.* **95** 385–398. Polish Acad. Sci. Inst. Math., Warsaw. MR2918350 https://doi.org/10.4064/bc95-0-24

The Annals of Probability
2021, Vol. 49, No. 4, 1965–2011
https://doi.org/10.1214/20-AOP1498

DOMINO TILINGS OF THE AZTEC DIAMOND WITH DOUBLY PERIODIC WEIGHTINGS

BY TOMAS BERGGREN

Department of Mathematics, University of Michigan, berggren@umich.edu

In this paper we consider domino tilings of the Aztec diamond with doubly periodic weightings. In particular, a family of models which, for any $k \in \mathbb{N}$, includes models with k smooth regions is analyzed as the size of the Aztec diamond tends to infinity. We use a nonintersecting paths formulation and give a double integral formula for the correlation kernel of the Aztec diamond of finite size. By a classical steepest descent analysis of the correlation kernel, we obtain the local behavior in the smooth and rough regions, as the size of the Aztec diamond tends to infinity. From the mentioned limit the macroscopic picture, such as the arctic curves and, in particular, the number of smooth regions, is deduced. Moreover, we compute the limit of the height function, and, as a consequence, we confirm in the setting of this paper that the limit in the rough region fulfills the complex Burgers' equation, as stated by Kenyon and Okounkov.

REFERENCES

[1] BEFFARA, V., CHHITA, S. and JOHANSSON, K. (2018). Airy point process at the liquid-gas boundary. *Ann. Probab.* **46** 2973–3013. MR3846843 https://doi.org/10.1214/17-AOP1244

[2] BERGGREN, T. and DUITS, M. (2019). Correlation functions for determinantal processes defined by infinite block Toeplitz minors. *Adv. Math.* **356** 106766, 48. MR3998766 https://doi.org/10.1016/j.aim.2019.106766

[3] BUFETOV, A. and GORIN, V. (2018). Fluctuations of particle systems determined by Schur generating functions. *Adv. Math.* **338** 702–781. MR3861715 https://doi.org/10.1016/j.aim.2018.07.009

[4] BUFETOV, A. and GORIN, V. (2019). Fourier transform on high-dimensional unitary groups with applications to random tilings. *Duke Math. J.* **168** 2559–2649. MR4007600 https://doi.org/10.1215/00127094-2019-0023

[5] CHARLIER, C., DUITS, M., KUIJLAARS, A. B. J. and LENELLS, J. (2020). A periodic hexagon tiling model and non-Hermitian orthogonal polynomials. *Comm. Math. Phys.* **378** 401–466. MR4124992 https://doi.org/10.1007/s00220-020-03779-0

[6] CHHITA, S. and JOHANSSON, K. (2016). Domino statistics of the two-periodic Aztec diamond. *Adv. Math.* **294** 37–149. MR3479561 https://doi.org/10.1016/j.aim.2016.02.025

[7] CHHITA, S., JOHANSSON, K. and YOUNG, B. (2015). Asymptotic domino statistics in the Aztec diamond. *Ann. Appl. Probab.* **25** 1232–1278. MR3325273 https://doi.org/10.1214/14-AAP1021

[8] CHHITA, S. and YOUNG, B. (2014). Coupling functions for domino tilings of Aztec diamonds. *Adv. Math.* **259** 173–251. MR3197656 https://doi.org/10.1016/j.aim.2014.01.023

[9] COHN, H., ELKIES, N. and PROPP, J. (1996). Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.* **85** 117–166. MR1412441 https://doi.org/10.1215/S0012-7094-96-08506-3

[10] COHN, H., KENYON, R. and PROPP, J. (2001). A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** 297–346. MR1815214 https://doi.org/10.1090/S0894-0347-00-00355-6

[11] DI FRANCESCO, P. and SOTO-GARRIDO, R. (2014). Arctic curves of the octahedron equation. *J. Phys. A* **47** 285204, 34. MR3228361 https://doi.org/10.1088/1751-8113/47/28/285204

[12] DUITS, M. (2018). On global fluctuations for non-colliding processes. *Ann. Probab.* **46** 1279–1350. MR3785589 https://doi.org/10.1214/17-AOP1185

[13] DUITS, M. and KUIJLAARS, A. (2021). The two-periodic Aztec diamond and matrix valued orthogonal polynomials. *J. Eur. Math. Soc.* (*JEMS*) **23** 1075–1131. MR4228276 https://doi.org/10.4171/jems/1029

[14] ELKIES, N., KUPERBERG, G., LARSEN, M. and PROPP, J. (1992). Alternating-sign matrices and domino tilings. I and II. *J. Algebraic Combin.* **1** 111–132, 219–234.

MSC2020 subject classifications. Primary 60G55; secondary 60B10, 52C20, 30E20.
Key words and phrases. Determinantal point processes, periodically weighted random tilings.

[15] JOCKUSCH, W., PROPP, J. and SHOR, P. (1995). Random domino tilings and the arctic circle theorem. Preprint. Available at arXiv:math/9801068.

[16] JOHANSSON, K. (2002). Non-intersecting paths, random tilings and random matrices. *Probab. Theory Related Fields* **123** 225–280. MR1900323 https://doi.org/10.1007/s004400100187

[17] JOHANSSON, K. (2005). The arctic circle boundary and the Airy process. *Ann. Probab.* **33** 1–30. MR2118857 https://doi.org/10.1214/009117904000000937

[18] JOHANSSON, K. (2006). Random matrices and determinantal processes. In *Mathematical Statistical Physics* 1–55. Elsevier, Amsterdam. MR2581882 https://doi.org/10.1016/S0924-8099(06)80038-7

[19] JOHANSSON, K. and NORDENSTAM, E. (2006). Eigenvalues of GUE minors. *Electron. J. Probab.* **11** 1342–1371. Erratum: *Electron. J. Probab.* **12** (2007) 1048–1051. MR2268547 https://doi.org/10.1214/EJP.v11-370

[20] KENYON, R. and OKOUNKOV, A. (2007). Limit shapes and the complex Burgers equation. *Acta Math.* **199** 263–302. MR2358053 https://doi.org/10.1007/s11511-007-0021-0

[21] KENYON, R., OKOUNKOV, A. and SHEFFIELD, S. (2006). Dimers and amoebae. *Ann. of Math.* (2) **163** 1019–1056. MR2215138 https://doi.org/10.4007/annals.2006.163.1019

[22] LAM, T. and PYLYAVSKYY, P. (2012). Total positivity in loop groups, I: Whirls and curls. *Adv. Math.* **230** 1222–1271. MR2921179 https://doi.org/10.1016/j.aim.2012.03.012

[23] NIENHUIS, B., HILHORST, H. J. and BLÖTE, H. W. J. (1984). Triangular SOS models and cubic-crystal shapes. *J. Phys. A* **17** 3559–3581. MR0772340

[24] OKOUNKOV, A. and RESHETIKHIN, N. (2003). Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. *J. Amer. Math. Soc.* **16** 581–603. MR1969205 https://doi.org/10.1090/S0894-0347-03-00425-9

[25] THURSTON, W. P. (1990). Conway's tiling groups. *Amer. Math. Monthly* **97** 757–773. MR1072815 https://doi.org/10.2307/2324578

The Annals of Probability
2021, Vol. 49, No. 4, 2012–2030
https://doi.org/10.1214/20-AOP1499

EMERGENCE OF EXTENDED STATES AT ZERO IN THE SPECTRUM OF SPARSE RANDOM GRAPHS

BY SIMON COSTE[1] AND JUSTIN SALEZ[2]

[1]*INRIA Paris, simon.coste@inria.fr*
[2]*CEREMADE, University Paris-Dauphine & PSL, justin.salez@dauphine.psl.eu*

We confirm the long-standing prediction that $c = e \approx 2.718$ is the threshold for the emergence of a nonvanishing absolutely continuous part (extended states) at zero in the limiting spectrum of the Erdős–Rényi random graph with average degree c. This is achieved by a detailed second-order analysis of the resolvent $(A - z)^{-1}$ near the singular point $z = 0$, where A is the adjacency operator of the Poisson–Galton–Watson tree with mean offspring c. More generally, our method applies to arbitrary unimodular Galton–Watson trees, yielding explicit criteria for the presence or absence of extended states at zero in the limiting spectral measure of a variety of random graph models, in terms of the underlying degree distribution.

REFERENCES

[1] ABÉRT, M., THOM, A. and VIRÁG, B. (2016). Benjamini–Schramm convergence and pointwise convergence of the spectral measure. In preparation.

[2] AIZENMAN, M., SIMS, R. and WARZEL, S. (2006). Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. *Probab. Theory Related Fields* **136** 363–394. MR2257129 https://doi.org/10.1007/s00440-005-0486-8

[3] ALDOUS, D. and LYONS, R. (2007). Processes on unimodular random networks. *Electron. J. Probab.* **12** 1454–1508. MR2354165 https://doi.org/10.1214/EJP.v12-463

[4] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*. *Encyclopaedia Math. Sci.* **110** 1–72. Springer, Berlin. MR2023650 https://doi.org/10.1007/978-3-662-09444-0_1

[5] BAUER, M. and GOLINELLI, O. (2001). Exactly solvable model with two conductor–insulator transitions driven by impurities. *Phys. Rev. Lett.* **86** 2621–2624. https://doi.org/10.1103/PhysRevLett.86.2621

[6] BAUER, M. and GOLINELLI, O. (2001). Random incidence matrices: Moments of the spectral density. *J. Stat. Phys.* **103** 301–337. MR1828732 https://doi.org/10.1023/A:1004879905284

[7] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** no. 23, 13. MR1873300 https://doi.org/10.1214/EJP.v6-96

[8] BORDENAVE, C. (2017). Spectrum of random graphs. In *Advanced Topics in Random Matrices*. *Panor. Synthèses* **53** 91–150. Soc. Math. France, Paris. MR3792625

[9] BORDENAVE, C. and LELARGE, M. (2010). Resolvent of large random graphs. *Random Structures Algorithms* **37** 332–352. MR2724665 https://doi.org/10.1002/rsa.20313

[10] BORDENAVE, C., LELARGE, M. and SALEZ, J. (2011). The rank of diluted random graphs. *Ann. Probab.* **39** 1097–1121. MR2789584 https://doi.org/10.1214/10-AOP567

[11] BORDENAVE, C., SEN, A. and VIRÁG, B. (2017). Mean quantum percolation. *J. Eur. Math. Soc.* (*JEMS*) **19** 3679–3707. MR3730511 https://doi.org/10.4171/JEMS/750

[12] ENRIQUEZ, N. and MÉNARD, L. (2016). Spectra of large diluted but bushy random graphs. *Random Structures Algorithms* **49** 160–184. MR3521277 https://doi.org/10.1002/rsa.20618

[13] JUNG, P. and LEE, J. (2018). Delocalization and limiting spectral distribution of Erdős–Rényi graphs with constant expected degree. *Electron. Commun. Probab.* **23** Paper No. 92, 13. MR3896830 https://doi.org/10.1214/18-ECP198

[14] KELLER, M. (2012). Absolutely continuous spectrum for multi-type Galton Watson trees. *Ann. Henri Poincaré* **13** 1745–1766. MR2994759 https://doi.org/10.1007/s00023-012-0169-9

MSC2020 subject classifications. 05C80, 60B20, 47A10.
Key words and phrases. Spectrum, sparse Erdős–Rényi random graphs, unimodular Galton–Watson trees, extended states.

[15] KHORUNZHY, O., SHCHERBINA, M. and VENGEROVSKY, V. (2004). Eigenvalue distribution of large weighted random graphs. *J. Math. Phys.* **45** 1648–1672. MR2043849 https://doi.org/10.1063/1.1667610

[16] MCKAY, B. D. (1981). The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl.* **40** 203–216. MR0629617 https://doi.org/10.1016/0024-3795(81)90150-6

[17] SALEZ, J. (2011). Some implications of local weak convergence for sparse random graphs. Thesis, Université Pierre et Marie Curie—Paris VI; Ecole Normale Supérieure de Paris—ENS Paris.

[18] SALEZ, J. (2015). Every totally real algebraic integer is a tree eigenvalue. *J. Combin. Theory Ser. B* **111** 249–256. MR3315609 https://doi.org/10.1016/j.jctb.2014.09.001

[19] SALEZ, J. (2020). Spectral atoms of unimodular random trees. *J. Eur. Math. Soc.* (*JEMS*) **22** 345–363. MR4049219 https://doi.org/10.4171/jems/923

[20] TRAN, L. V., VU, V. H. and WANG, K. (2013). Sparse random graphs: Eigenvalues and eigenvectors. *Random Structures Algorithms* **42** 110–134. MR2999215 https://doi.org/10.1002/rsa.20406

[21] WIGNER, E. P. (1996). *The Collected Works of Eugene Paul Wigner. Part A. The Scientific Papers. Vol. II. Nuclear Physics*. Springer, Berlin. MR1366418

[22] ZAKHAREVICH, I. (2006). A generalization of Wigner's law. *Comm. Math. Phys.* **268** 403–414. MR2259200 https://doi.org/10.1007/s00220-006-0074-5

The Annals of Probability
2021, Vol. 49, No. 4, 2031–2075
https://doi.org/10.1214/20-AOP1501

AGE EVOLUTION IN THE MEAN FIELD FOREST FIRE MODEL VIA MULTITYPE BRANCHING PROCESSES

BY EDWARD CRANE[1], BALÁZS RÁTH[2] AND DOMINIC YEO[3]

[1]*School of Mathematics, University of Bristol, Edward.Crane@bristol.ac.uk*
[2]*MTA-BME Stochastics Research Group, Budapest University of Technology and Economics, rathb@math.bme.hu*
[3]*Department of Statistics, University of Oxford, dominicjyeo@gmail.com*

We study the distribution of ages in the mean field forest fire model introduced by Ráth and Tóth. This model is an evolving random graph whose dynamics combine Erdős–Rényi edge-addition with a Poisson rain of *lightning strikes*. All edges in a connected component are deleted when any of its vertices is struck by lightning. We consider the asymptotic regime of lightning rates for which the model displays self-organized criticality. The *age* of a vertex increases at unit rate, but it is reset to zero at each burning time. We show that the empirical age distribution converges as a process to a deterministic solution of an autonomous measure-valued differential equation. The main technique is to observe that, conditioned on the vertex ages, the graph is an inhomogeneous random graph in the sense of Bollobás, Janson and Riordan. We then study the evolution of the ages via the multitype Galton–Watson trees that arise as the limit in law of the component of an identified vertex at any fixed time. These trees are critical from the gelation time onwards.

REFERENCES

[1] ALDOUS, D. J. (2000). The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.* **128** 465–477. MR1744108 https://doi.org/10.1017/S0305004199004326

[2] ALEKSANDROV, A. B., JANSON, S., PELLER, V. V. and ROCHBERG, R. (2002). An interesting class of operators with unusual Schatten–von Neumann behavior. In *Function Spaces, Interpolation Theory and Related Topics* (*Lund*, 2000) (M. Cwikel et al., eds.) 61–149. de Gruyter, Berlin. MR1943283

[3] BAK, P., TANG, C. and WIESENFELD, K. (1988). Self-organized criticality. *Phys. Rev. A* (3) **38** 364–374. MR0949160 https://doi.org/10.1103/PhysRevA.38.364

[4] BENJAMINI, I., LYONS, R. and SCHRAMM, O. (2015). Unimodular random trees. *Ergodic Theory Dynam. Systems* **35** 359–373. MR3316916 https://doi.org/10.1017/etds.2013.56

[5] BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396 https://doi.org/10.1002/rsa.20168

[6] BUFFET, E. and PULÉ, J. V. (1990). On Lushnikov's model of gelation. *J. Stat. Phys.* **58** 1041–1058. MR1049055 https://doi.org/10.1007/BF01026562

[7] BUFFET, E. and PULÉ, J. V. (1991). Polymers and random graphs. *J. Stat. Phys.* **64** 87–110. MR1117648 https://doi.org/10.1007/BF01057869

[8] CERF, R. and GORNY, M. (2014). A Curie–Weiss model of self-organized criticality. *Ann. Probab.* **44** 444–478.

[9] CRANE, E. (2018). Steady state clusters and the Ráth–Tóth mean field forest fire model. Available at arXiv:1809.03462 [math.PR].

[10] CRANE, E. (2020). Well-posedness of the mean field forest fire age evolution equation. Available at arXiv:2007.05807 [math.AP].

[11] CRANE, E., FREEMAN, N. and TÓTH, B. (2015). Cluster growth in the dynamical Erdős–Rényi process with forest fires. *Electron. J. Probab.* **20** 101. MR3407218 https://doi.org/10.1214/EJP.v20-4035

[12] DROSSEL, B. and SCHWABL, F. (1992). Self-organized critical forest fires model. *Phys. Rev. Lett.* **69** 1629–1632.

MSC2020 subject classifications. 05C80, 60J80, 46N55, 35Q82.
Key words and phrases. Inhomogeneous random graph, multitype branching process, self-organized criticality, Perron–Frobenius theory, differential equations.

[13] DÜRRE, M. (2006). Existence of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. J. Probab.* **11** 513–539. MR2242654 https://doi.org/10.1214/EJP.v11-333

[14] DÜRRE, M. (2006). Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. Commun. Probab.* **11** 304–315. MR2266720 https://doi.org/10.1214/ECP.v11-1229

[15] DÜRRE, M. (2009). Self-organized Critical Phenomena; Forest Fire and Sandpile Model Ph.D. thesis Ludwig-Maximilians-Universität München.

[16] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magy. Tud. Akad. Mat. Kut. Intéz. Közl.* **5** 17–61. MR0125031

[17] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II. Second Edition*. Wiley, New York. MR0270403

[18] FOURNIER, N. and LAURENÇOT, P. (2009). Marcus–Lushnikov processes, Smoluchowski's and Flory's models. *Stochastic Process. Appl.* **119** 167–189. MR2485023 https://doi.org/10.1016/j.spa.2008.02.003

[19] GRAF, R. (2014). A forest-fire model on the upper half-plane. *Electron. J. Probab.* **19** 8. MR3164761 https://doi.org/10.1214/EJP.v19-2625

[20] JÁRAI, A. A. (2018). Sandpile models. *Probab. Surv.* **15** 243–306. MR3857602 https://doi.org/10.1214/14-PS228

[21] KISS, D., MANOLESCU, I. and SIDORAVICIUS, V. (2015). Planar lattices do not recover from forest fires. *Ann. Probab.* **43** 3216–3238. MR3433580 https://doi.org/10.1214/14-AOP958

[22] MERLE, M. and NORMAND, R. (2014). Self-organized criticality in a discrete model for Smoluchowski's equation. Available at arXiv:1410.8338 [math.PR].

[23] MERLE, M. and NORMAND, R. (2015). Self-organized criticality in a discrete model for Smoluchowski's equation with limited aggregations. Available at arXiv:1509.00934 [math.PR].

[24] MODE, C. J. (1971). *Multitype Branching Processes. Theory and Applications*. Elsevier, New York. MR0279901

[25] NORMAND, R. and ZAMBOTTI, L. (2011). Uniqueness of post-gelation solutions of a class of coagulation equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** 189–215. MR2784069 https://doi.org/10.1016/j.anihpc.2010.10.005

[26] RÁTH, B. (2009). Mean field frozen percolation. *J. Stat. Phys.* **137** 459–499. MR2564286 https://doi.org/10.1007/s10955-009-9863-5

[27] RÁTH, B. and TÓTH, B. (2009). Erdős–Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab.* **14** 1290–1327. MR2511285 https://doi.org/10.1214/EJP.v14-653

[28] REED, M. and SIMON, B. (1980). *Methods of Modern Mathematical Physics. I: Functional Analysis*, 2nd ed. Academic Press, New York. MR0751959

[29] SWART, J. M. (2013). Duality and Intertwining of Markov Chains. ALEA in Europe School, CIRM, Luminy (Marseille). Available at: http://staff.utia.cas.cz/swart/dualtwine.pdf.

[30] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[31] VAN DEN BERG, J. and BROUWER, R. (2004). Self-destructive percolation. *Random Structures Algorithms* **24** 480–501. MR2060632 https://doi.org/10.1002/rsa.20003

[32] VAN DER HOFSTAD, R. *Random Graphs and Complex Networks. Vol. 2. Cambridge Series in Statistical and Probabilistic Mathematics*. To appear. Draft available at http://www.win.tue.nl/~rhofstad/.

[33] YEO, D. (2016). Self-organised criticality in random graph processes DPhil thesis, University of Oxford. Available at https://ora.ox.ac.uk.

[34] YEO, D. (2018). Frozen percolation on inhomogeneous random graphs. Available at arXiv:1810.02750 [math.PR].

The Annals of Probability
2021, Vol. 49, No. 4, 2076–2140
https://doi.org/10.1214/21-AOP1505

LOCAL AND GLOBAL GEOMETRY OF THE 2D ISING INTERFACE IN CRITICAL PREWETTING

BY SHIRSHENDU GANGULY[*] AND REZA GHEISSARI[†]

Department of Statistics, University of California, Berkeley, [*]*sganguly@berkeley.edu;* [†]*gheissari@berkeley.edu*

Consider the Ising model at low temperatures and positive external field λ on an $N \times N$ box with *Dobrushin boundary conditions* that are plus on the north, east and west boundaries and minus on the south boundary. If $\lambda = 0$, the interface separating the plus and minus phases is diffusive, having $O(\sqrt{N})$ height fluctuations, and the model is *fully wetted*. Under an order one field, the interface fluctuations are $O(1)$, and the interface is only partially wetted, being pinned to its southern boundary. We study the *critical prewetting regime* of $\lambda_N \downarrow 0$, where the height fluctuations are expected to scale as $\lambda^{-1/3}$ and the rescaled interface is predicted to converge to the Ferrari–Spohn diffusion. Velenik (*Probab. Theory Related Fields* **129** (2004) 83–112) identified the order of the area under the interface up to logarithmic corrections. Since then, more refined features of such interfaces have only been identified in simpler models of random walks under area tilts.

In this paper we resolve several conjectures of Velenik regarding the refined features of the Ising interface in the critical prewetting regime. Our main result is a sharp bound on the one-point height fluctuation, proving $e^{-\Theta(x^{3/2})}$ upper tails reminiscent of the Tracy–Widom distribution, capturing a tradeoff between the locally Brownian oscillations and the global field effect. We further prove a concentration estimate for the number of points above which the interface attains a large height. These are used to deduce various geometric properties of the interface, including the order and tails of the area it confines and the polylogarithmic prefactor governing its maximum height fluctuation. Our arguments combine classical inputs from the random-line representation of the Ising interface with novel local resampling and coupling schemes.

REFERENCES

[1] ABRAHAM, D. B. and SMITH, E. R. (1986). An exactly solved model with a wetting transition. *J. Stat. Phys.* **43** 621–643. MR0845731 https://doi.org/10.1007/BF01020656

[2] ALEXANDER, K. S. (2001). Cube-root boundary fluctuations for droplets in random cluster models. *Comm. Math. Phys.* **224** 733–781. MR1871907 https://doi.org/10.1007/s220-001-8022-2

[3] BRICMONT, J., EL MELLOUKI, A. and FRÖHLICH, J. (1986). Random surfaces in statistical mechanics: Roughening, rounding, wetting,.... *J. Stat. Phys.* **42** 743–798. MR0833220 https://doi.org/10.1007/BF01010444

[4] CAMPANINO, M., IOFFE, D. and VELENIK, Y. (2003). Ornstein–Zernike theory for finite range Ising models above T_c. *Probab. Theory Related Fields* **125** 305–349. MR1964456 https://doi.org/10.1007/s00440-002-0229-z

[5] CAMPANINO, M., IOFFE, D. and VELENIK, Y. (2008). Fluctuation theory of connectivities for subcritical random cluster models. *Ann. Probab.* **36** 1287–1321. MR2435850 https://doi.org/10.1214/07-AOP359

[6] CAPUTO, P., LUBETZKY, E., MARTINELLI, F., SLY, A. and TONINELLI, F. L. (2016). Scaling limit and cube-root fluctuations in SOS surfaces above a wall. *J. Eur. Math. Soc.* (*JEMS*) **18** 931–995. MR3500829 https://doi.org/10.4171/JEMS/606

[7] CHAYES, J. T., CHAYES, L. and SCHONMANN, R. H. (1987). Exponential decay of connectivities in the two-dimensional Ising model. *J. Stat. Phys.* **49** 433–445. MR0926191 https://doi.org/10.1007/BF01009344

MSC2020 subject classifications. Primary 60K35, 82B24; secondary 82B20, 82B41.
Key words and phrases. Ising model, wetting transition, entropic repulsion, interface, cube-root fluctuations.

[8] CORWIN, I. (2012). The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl*. **1** 1130001. MR2930377 https://doi.org/10.1142/S2010326311300014
[9] CORWIN, I. (2016). Kardar–Parisi–Zhang universality. *Notices Amer. Math. Soc*. **63** 230–239. MR3445162 https://doi.org/10.1090/noti1334
[10] COURONNÉ, O. and MESSIKH, R. J. (2004). Surface order large deviations for 2D FK-percolation and Potts models. *Stochastic Process. Appl*. **113** 81–99. MR2078538 https://doi.org/10.1016/j.spa.2004.03.010
[11] DOBRUSHIN, R. and HRYNIV, O. (1997). Fluctuations of the phase boundary in the 2D Ising ferromagnet. *Comm. Math. Phys*. **189** 395–445. MR1480026 https://doi.org/10.1007/s002200050209
[12] DOBRUSHIN, R., KOTECKÝ, R. and SHLOSMAN, S. (1992). *Wulff Construction. Translations of Mathematical Monographs* **104**. Amer. Math. Soc., Providence, RI. MR1181197 https://doi.org/10.1090/mmono/104
[13] FERRARI, P. L. and SPOHN, H. (2005). Constrained Brownian motion: Fluctuations away from circular and parabolic barriers. *Ann. Probab.* **33** 1302–1325. MR2150190 https://doi.org/10.1214/009117905000000125
[14] FISHER, M. E. (1984). Walks, walls, wetting, and melting. *J. Stat. Phys*. **34** 667–729. MR0751710 https://doi.org/10.1007/BF01009436
[15] GHEISSARI, R. and LUBETZKY, E. (2018). Mixing times of critical two-dimensional Potts models. *Comm. Pure Appl. Math*. **71** 994–1046. MR3794520 https://doi.org/10.1002/cpa.21718
[16] GHEISSARI, R. and LUBETZKY, E. (2018). The effect of boundary conditions on mixing of 2D Potts models at discontinuous phase transitions. *Electron. J. Probab.* **23** 57. MR3814251 https://doi.org/10.1214/18-ejp180
[17] GREENBERG, L. and IOFFE, D. (2005). On an invariance principle for phase separation lines. *Ann. Inst. Henri Poincaré Probab. Stat*. **41** 871–885. MR2165255 https://doi.org/10.1016/j.anihpb.2005.05.001
[18] GRIMMETT, G. (2006). *The Random-Cluster Model. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **333**. Springer, Berlin. MR2243761 https://doi.org/10.1007/978-3-540-32891-9
[19] HAMMOND, A. (2012). Phase separation in random cluster models I: Uniform upper bounds on local deviation. *Comm. Math. Phys*. **310** 455–509. MR2890306 https://doi.org/10.1007/s00220-011-1370-2
[20] HAMMOND, A. (2012). Phase separation in random cluster models II: The droplet at equilibrium, and local deviation lower bounds. *Ann. Probab.* **40** 921–978. MR2962083 https://doi.org/10.1214/11-AOP646
[21] HRYNIV, O. (1998). On local behaviour of the phase separation line in the 2D Ising model. *Probab. Theory Related Fields* **110** 91–107. MR1602044 https://doi.org/10.1007/s004400050146
[22] HRYNIV, O. and VELENIK, Y. (2004). Universality of critical behaviour in a class of recurrent random walks. *Probab. Theory Related Fields* **130** 222–258. MR2093763 https://doi.org/10.1007/s00440-004-0353-z
[23] IOFFE, D., OTT, S., VELENIK, Y. and WACHTEL, V. (2020). Invariance principle for a Potts interface along a wall. *J. Stat. Phys*. **180** 832–861. MR4131016 https://doi.org/10.1007/s10955-020-02546-8
[24] IOFFE, D. and SCHONMANN, R. H. (1998). Dobrushin–Kotecký–Shlosman theorem up to the critical temperature. *Comm. Math. Phys*. **199** 117–167. MR1660207 https://doi.org/10.1007/s002200050497
[25] IOFFE, D., SHLOSMAN, S. and VELENIK, Y. (2015). An invariance principle to Ferrari–Spohn diffusions. *Comm. Math. Phys*. **336** 905–932. MR3322390 https://doi.org/10.1007/s00220-014-2277-5
[26] IOFFE, D. and VELENIK, Y. (2018). Low-temperature interfaces: Prewetting, layering, faceting and Ferrari–Spohn diffusions. *Markov Process. Related Fields* **24** 487–537. MR3821253
[27] IOFFE, D., VELENIK, Y. and WACHTEL, V. (2018). Dyson Ferrari–Spohn diffusions and ordered walks under area tilts. *Probab. Theory Related Fields* **170** 11–47. MR3748320 https://doi.org/10.1007/s00440-016-0751-z
[28] KARDAR, M., PARISI, G. and ZHANG, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett*. **56** 889–892.
[29] LUBETZKY, E., MARTINELLI, F., SLY, A. and TONINELLI, F. L. (2013). Quasi-polynomial mixing of the 2D stochastic Ising model with "plus" boundary up to criticality. *J. Eur. Math. Soc*. (*JEMS*) **15** 339–386. MR3017041 https://doi.org/10.4171/JEMS/363
[30] MARTINELLI, F. and TONINELLI, F. L. (2010). On the mixing time of the 2D stochastic Ising model with "plus" boundary conditions at low temperature. *Comm. Math. Phys*. **296** 175–213. MR2606632 https://doi.org/10.1007/s00220-009-0963-5
[31] MARTIROSJAN, D. G. (1975). Uniqueness of Gibbs limit distributions for the perturbed Ising model. *Teoret. Mat. Fiz*. **22** 335–342. MR0489616
[32] MCCOY, B. M. and WU, T. T. (2014). *The Two-Dimensional Ising Model*, 2nd ed. Dover, Mineola, NY. MR3838431
[33] PFISTER, C.-E. and VELENIK, Y. (1997). Large deviations and continuum limit in the 2D Ising model. *Probab. Theory Related Fields* **109** 435–506. MR1483597 https://doi.org/10.1007/s004400050139

[34] PFISTER, C.-E. and VELENIK, Y. (1999). Interface, surface tension and reentrant pinning transition in the 2D Ising model. *Comm*. *Math*. *Phys*. **204** 269–312. MR1704276 https://doi.org/10.1007/s002200050646

[35] PISZTORA, A. (1996). Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Related Fields* **104** 427–466. MR1384040 https://doi.org/10.1007/BF01198161

[36] PRÄHOFER, M. and SPOHN, H. (2002). Scale invariance of the PNG droplet and the Airy process **108** 1071–1106. MR1933446 https://doi.org/10.1023/A:1019791415147

[37] SCHONMANN, R. H. (1987). Second order large deviation estimates for ferromagnetic systems in the phase coexistence region. *Comm*. *Math*. *Phys*. **112** 409–422. MR0908546

[38] SCHONMANN, R. H. and SHLOSMAN, S. B. (1995). Complete analyticity for 2D Ising completed. *Comm*. *Math*. *Phys*. **170** 453–482. MR1334405

[39] SCHONMANN, R. H. and SHLOSMAN, S. B. (1996). Constrained variational problem with applications to the Ising model. *J*. *Stat*. *Phys*. **83** 867–905. MR1392417 https://doi.org/10.1007/BF02179548

[40] VELENIK, Y. (1997). Phase separation as a large deviations problem a microscopic derivation of surface thermodynamics for some 2D spin systems. 312.

[41] VELENIK, Y. (2004). Entropic repulsion of an interface in an external field. *Probab. Theory Related Fields* **129** 83–112. MR2052864 https://doi.org/10.1007/s00440-003-0328-5

[42] VELENIK, Y. (2006). Localization and delocalization of random interfaces. *Probab. Surv.* **3** 112–169. MR2216964 https://doi.org/10.1214/154957806000000050

The Annals of Probability

Vol. 49 September 2021 No. 5

Articles