

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

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THE ANNALS OF PROBABILITY

Vol. 49, No. 5, pp. 2141–2670 September 2021

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The Annals of Probability [ISSN 0091-1798 (print); ISSN 2168-894X (online)], Volume 49, Number 5, September 2021. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

UNIVERSALITY FOR CRITICAL KCM: FINITE NUMBER OF STABLE DIRECTIONS

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In this paper, we consider kinetically constrained models (KCM) on \mathbb{Z}^2 with general update families \mathcal{U} . For \mathcal{U} belonging to the so-called “critical class,” our focus is on the divergence of the infection time of the origin for the equilibrium process as the density of the facilitating sites vanishes. In a recent paper (*Probab. Theory Related Fields* **178** (2020) 289–326), Marêché and two of the present authors proved that if \mathcal{U} has an infinite number of “stable directions,” then on a doubly logarithmic scale the above divergence is twice the one in the corresponding \mathcal{U} -bootstrap percolation.

Here, we prove instead that, contrary to previous conjectures (*Comm. Math. Phys.* **369** (2019) 761–809), in the complementary case the two divergences are the same. In particular, we establish the full universality partition for critical \mathcal{U} . The main novel contribution is the identification of the leading mechanism governing the motion of infected critical droplets. It consists of a peculiar hierarchical combination of mesoscopic East-like motions.

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MSC2020 subject classifications. Primary 60K35; secondary 82C22, 60J27, 60C05.

Key words and phrases. Kinetically constrained models, bootstrap percolation, universality, Glauber dynamics, Poincaré inequality.

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SYMMETRIES OF STOCHASTIC COLORED VERTEX MODELS

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We discover a new property of the stochastic colored six-vertex model called *flip-invariance*. We use it to show that for a given collection of observables of the model, any transformation that preserves the distribution of each individual observable also preserves their joint distribution. This generalizes recent shift-invariance results of Borodin–Gorin–Wheeler. As limiting cases, we obtain similar statements for the Brownian last passage percolation, the Kardar–Parisi–Zhang equation, the Airy sheet and directed polymers. Our proof relies on an equivalence between the stochastic colored six-vertex model and the Yang–Baxter basis of the Hecke algebra. We conclude by discussing the relationship of the model with Kazhdan–Lusztig polynomials and positroid varieties in the Grassmannian.

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MSC2020 subject classifications. Primary 82C22; secondary 60K35, 14M15, 05E99.

Key words and phrases. Six-vertex model, flip-invariance, Hecke algebra, last passage percolation, KPZ equation, Airy sheet, directed polymers, Kazhdan–Lusztig polynomials, positroid varieties.

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THE HEIGHT OF MALLOWS TREES

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Random binary search trees are obtained by recursively inserting the elements $\sigma(1), \sigma(2), \dots, \sigma(n)$ of a uniformly random permutation σ of $[n] = \{1, \dots, n\}$ into a binary search tree data structure. Devroye (*J. Assoc. Comput. Mach.* **33** (1986) 489–498) proved that the height of such trees is asymptotically of order $c^* \log n$, where $c^* = 4.311\dots$ is the unique solution of $c \log((2e)/c) = 1$ with $c \geq 2$. In this paper, we study the structure of binary search trees $T_{n,q}$ built from Mallows permutations. A Mallows(q) permutation is a random permutation of $[n] = \{1, \dots, n\}$ whose probability is proportional to $q^{\text{Inv}(\sigma)}$, where $\text{Inv}(\sigma) = |\{i < j : \sigma(i) > \sigma(j)\}|$. This model generalizes random binary search trees, since Mallows(q) permutations with $q = 1$ are uniformly distributed. The laws of $T_{n,q}$ and $T_{n,q^{-1}}$ are related by a simple symmetry (switching the roles of the left and right children), so it suffices to restrict our attention to $q \leq 1$.

We show that, for $q \in [0, 1]$, the height of $T_{n,q}$ is asymptotically $(1 + o(1))(c^* \log n + n(1 - q))$ in probability. This yields three regimes of behaviour for the height of $T_{n,q}$, depending on whether $n(1 - q)/\log n$ tends to zero, tends to infinity or remains bounded away from zero and infinity. In particular, when $n(1 - q)/\log n$ tends to zero, the height of $T_{n,q}$ is asymptotically of order $c^* \log n$, like it is for random binary search trees. Finally, when $n(1 - q)/\log n$ tends to infinity, we prove stronger tail bounds and distributional limit theorems for the height of $T_{n,q}$.

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MSC2020 subject classifications. Primary 60B15, 60C05; secondary 05A05, 60F05, 60F15, 60K35, 82B23, 82B26.

Key words and phrases. Mallows trees, Mallows permutations, binary search trees, random trees, random permutations, heights of trees.

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FROZEN PERCOLATION ON THE BINARY TREE IS NONENDOGENOUS

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In frozen percolation, i.i.d. uniformly distributed activation times are assigned to the edges of a graph. At its assigned time an edge opens provided neither of its end vertices is part of an infinite open cluster; in the opposite case it freezes. Aldous (*Math. Proc. Cambridge Philos. Soc.* **128** (2000) 465–477) showed that such a process can be constructed on the infinite 3-regular tree and asked whether the event that a given edge freezes is a measurable function of the activation times assigned to all edges. We give a negative answer to this question, or, using an equivalent formulation and terminology introduced by Aldous and Bandyopadhyay (*Ann. Appl. Probab.* **15** (2005) 1047–1110), we show that the recursive tree process associated with frozen percolation on the oriented binary tree is nonendogenous. An essential role in our proofs is played by a frozen percolation process on a continuous-time binary Galton–Watson tree that has nice scale invariant properties.

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MSC2020 subject classifications. Primary 82C27; secondary 60K35, 82C26, 60J80.

Key words and phrases. Frozen percolation, self-organised criticality, recursive distributional equation, recursive tree process, endogeny, near-critical percolation, branching process.

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METASTABILITY AND EXIT PROBLEMS FOR SYSTEMS OF STOCHASTIC REACTION–DIFFUSION EQUATIONS

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In this paper we develop a metastability theory for a class of stochastic reaction–diffusion equations exposed to small multiplicative noise. We consider the case where the unperturbed reaction–diffusion equation features multiple asymptotically stable equilibria. When the system is exposed to small stochastic perturbations, it is likely to stay near one equilibrium for a long period of time but will eventually transition to the neighborhood of another equilibrium. We are interested in studying the exit time from the full domain of attraction (in a function space) surrounding an equilibrium and, therefore, do not assume that the domain of attraction features uniform attraction to the equilibrium. This means that the boundary of the domain of attraction is allowed to contain saddles and limit cycles. Our method of proof is purely infinite dimensional, that is, we do not go through finite dimensional approximations. In addition, we address the multiplicative noise case, and we do not impose gradient type of assumptions on the nonlinearity. We prove large deviations logarithmic asymptotics for the exit time and for the exit shape, also characterizing the most probable set of shapes of solutions at the time of exit from the domain of attraction.

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MSC2020 subject classifications. 60F10, 60H15, 35R60, 60G40.

Key words and phrases. Stochastic partial differential equations, stochastic reaction–diffusion equation, metastability, small noise, large deviations, exit time, exit place.

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ON STOCHASTIC EQUATIONS WITH DRIFT IN L_d

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For the Itô stochastic equations in \mathbb{R}^d with drift in L_d , several results are discussed, such as the existence of weak solutions, the existence of the corresponding Markov process, the Aleksandrov type estimates of their Green's functions, which yield their summability to the power of $d/(d-1)$, the Fabes–Stroock type estimates, which show that Green's functions are summable to a higher degree, the Fanghua Lin type estimates, which are one of the main tools in the W_p^2 -theory of fully nonlinear elliptic equations, the fact that Green's functions are in the class A_∞ of Muckenhoupt and a few other results.

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SHARP THRESHOLD FOR THE ISING PERCEPTRON MODEL

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Consider the discrete cube $\{-1, 1\}^N$ and a random collection of half spaces which includes each half space $H(x) := \{y \in \{-1, 1\}^N : x \cdot y \geq \kappa \sqrt{N}\}$ for $x \in \{-1, 1\}^N$ independently with probability p . Is the intersection of these half spaces empty? This is called the Ising perceptron model under Bernoulli disorder. We prove that this event has a sharp threshold, that is, the probability that the intersection is empty increases quickly from ϵ to $1 - \epsilon$ when p increases only by a factor of $1 + o(1)$ as $N \rightarrow \infty$.

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A GEOMETRIC REPRESENTATION OF FRAGMENTATION PROCESSES ON STABLE TREES

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We provide a new geometric representation of a family of fragmentation processes by nested laminations which are compact subsets of the unit disk made of noncrossing chords. We specifically consider a fragmentation, obtained by cutting a random stable tree at random points, which split the tree into smaller subtrees. When coding each of these cutpoints by a chord in the unit disk, we separate the disk into smaller connected components, corresponding to the smaller subtrees of the initial tree. This geometric point of view allows us in particular to highlight a new relation between the Aldous–Pitman fragmentation of the Brownian continuum random tree and minimal factorizations of the n -cycle, that is, factorizations of the permutation $(1\ 2\ \cdots\ n)$ into a product of $(n - 1)$ transpositions, proving this way a conjecture of Féray and Kortchemski. We discuss various properties of these new lamination-valued processes, and we notably show that they can be coded by explicit Lévy processes.

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MSC2020 subject classifications. Primary 60C05, 60F17; secondary 05A05.

Key words and phrases. Random trees, fragmentation processes, stable tree, scaling limit, lamination of the disk, permutation, minimal factorization, Lévy process.

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CHARACTERIZATION OF BROWNIAN GIBBSIAN LINE ENSEMBLES

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In this paper we show that a Brownian Gibbsian line ensemble is completely characterized by the finite-dimensional marginals of its top curve, that is, the finite-dimensional sets of the top curve form a separating class. A particular consequence of our result is that the parabolic Airy line ensemble is the unique Brownian Gibbsian line ensemble, whose top curve is the parabolic Airy₂ process.

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MSC2020 subject classifications. Primary 82C22; secondary 60J65.

Key words and phrases. Airy process, Airy line ensemble, Gibbs measures.

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CHASE-ESCAPE WITH DEATH ON TREES

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Chase-escape is a competitive growth process in which red particles spread to adjacent uncolored sites, while blue particles overtake adjacent red particles. We introduce the variant in which red particles die and describe the phase diagram for the resulting process on infinite d -ary trees. A novel connection to weighted Catalan numbers makes it possible to characterize the critical behavior.

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SMALL BALL PROBABILITIES AND A SUPPORT THEOREM FOR THE STOCHASTIC HEAT EQUATION

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We consider the following stochastic partial differential equation on $t \geq 0$, $x \in [0, J]$, $J \geq 1$, where we consider $[0, J]$ to be the circle with end points identified,

$$\partial_t \mathbf{u}(t, x) = \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \mathbf{g}(t, x, \mathbf{u}) + \sigma(t, x, \mathbf{u}) \dot{\mathbf{W}}(t, x),$$

$\dot{\mathbf{W}}(t, x)$ is 2-parameter d -dimensional vector valued white noise and σ is function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ to space of symmetric $d \times d$ matrices which is Lipschitz in \mathbf{u} . We assume that σ is uniformly elliptic and that \mathbf{g} is uniformly bounded. Assuming that $\mathbf{u}(0, x) \equiv \mathbf{0}$, we prove small ball probabilities for the solution u . We also prove a support theorem for solutions, when $\mathbf{u}(0, x)$ is not necessarily zero.

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MSC2020 subject classifications. Primary 60H15; secondary 60G17, 60G60.

Key words and phrases. Heat equation, white noise, stochastic partial differential equations, small ball, support.

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POLARITY OF ALMOST ALL POINTS FOR SYSTEMS OF NONLINEAR STOCHASTIC HEAT EQUATIONS IN THE CRITICAL DIMENSION

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We study vector-valued solutions $u(t, x) \in \mathbb{R}^d$ to systems of nonlinear stochastic heat equations with multiplicative noise,

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \sigma(u(t, x)) \dot{W}(t, x).$$

Here, $t \geq 0$, $x \in \mathbb{R}$ and $\dot{W}(t, x)$ is an \mathbb{R}^d -valued space–time white noise. We say that a point $z \in \mathbb{R}^d$ is polar if

$$P\{u(t, x) = z \text{ for some } t > 0 \text{ and } x \in \mathbb{R}\} = 0.$$

We show that, in the critical dimension $d = 6$, almost all points in \mathbb{R}^d are polar.

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MSC2020 subject classifications. Primary 60G15; secondary 60J45, 60G60.

Key words and phrases. Hitting probabilities, polarity of points, critical dimension, nonlinear stochastic partial differential equations.

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MOMENT ESTIMATES FOR SOME RENORMALIZED PARABOLIC ANDERSON MODELS

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The theory of regularity structures enables the definition of the following parabolic Anderson model in a very rough environment: $\partial_t u_t(x) = \frac{1}{2} \Delta u_t(x) + u_t(x) \dot{W}_t(x)$, for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$, where $\dot{W}_t(x)$ is a Gaussian noise whose space time covariance function is singular. In this rough context we shall give some information about the moments of $u_t(x)$ when the stochastic heat equation is interpreted in the Skorohod as well as the Stratonovich sense. Of special interest is the critical case, for which one observes a blowup of moments for large times.

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MSC2020 subject classifications. 60L30, 60L50, 60F10, 60K37.

Key words and phrases. Parabolic Anderson model, regularity structures, Skorohod equation, Stratonovich equation, moment estimate, critical time.

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TO FIXATE OR NOT TO FIXATE IN TWO-TYPE ANNIHILATING BRANCHING RANDOM WALKS

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We study a model of competition between two types evolving as branching random walks on \mathbb{Z}^d . The two types are represented by red and blue balls, respectively, with the rule that balls of different colour annihilate upon contact. We consider initial configurations in which the sites of \mathbb{Z}^d contain one ball each which are independently coloured red with probability p and blue otherwise. We address the question of *fixation*, referring to the sites and eventually settling for a given colour or not. Under a mild moment condition on the branching rule, we prove that the process will fixate almost surely for $p \neq 1/2$ and that every site will change colour infinitely often almost surely for the balanced initial condition $p = 1/2$.

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MSC2020 subject classifications. 60K35, 60J80, 82C22, 82C27.

Key words and phrases. Branching random walk, competing growth, nonequilibrium dynamics.

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SECOND ERRATA TO “DISTANCE COVARIANCE IN METRIC SPACES”

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There is a slight gap and error in Remark 3.4 of *Ann. Probab.* **41**, no. 5 (2013), 3284–3305, that was not noticed before the first errata were published (*Ann. Probab.* **46**, no. 4 (2018), 2400–2405). We take this opportunity to provide some additional updates as well.

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MSC2020 subject classifications. Primary 62H20, 62G20, 51K99; secondary 62H15, 30L05, 30L15.

Key words and phrases. Negative type, hypothesis testing, independence, distance correlation, Brownian covariance.

The Annals of Probability

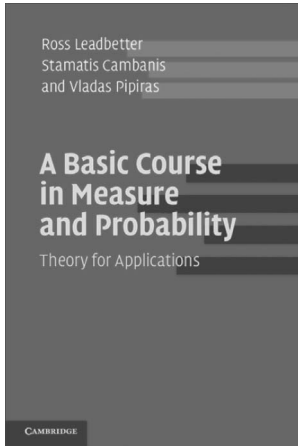
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Ross Leadbetter, Stamatis Cambanis, and
Vlaslas Pipiras

Originating from the authors' own graduate course at the University of North Carolina, this material has been thoroughly tried and tested over many years, making the book perfect for a two-term course or for self-study. It provides a concise introduction that covers all of the measure theory and probability most useful for statisticians, including Lebesgue integration, limit theorems in probability, martingales, and some theory of stochastic processes. Readers can test their understanding of the material through the 300 exercises provided.

The book is especially useful for graduate students in statistics and related fields of application (biostatistics, econometrics, finance, meteorology, machine learning, and so on) who want to shore up their mathematical foundation. The authors establish common ground for students of varied interests which will serve as a firm 'take-off point' for them as they specialize in areas that exploit mathematical machinery.

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