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CONFORMAL GROWTH RATES AND SPECTRAL GEOMETRY ON DISTRIBUTIONAL LIMITS OF GRAPHS

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For a unimodular random graph (G, ρ), we consider deformations of its intrinsic path metric by a (random) weighting of its vertices. This leads to the notion of the *conformal growth exponent of* (G, ρ), which is the best asymptotic degree of volume growth of balls that can be achieved by such a reweighting. Under moment conditions on the degree of the root, we show that the conformal growth exponent of a unimodular random graph bounds its almost sure spectral dimension. This has interesting consequences for many low-dimensional models.

The consequences in dimension two are particularly strong. It establishes that models like the uniform infinite planar triangulation (UIPT) and quadrangulation (UIPQ) almost surely have spectral dimension at most two. It also establishes a conjecture of Benjamini and Schramm (*Electron. J. Probab.* **6** (2001) no. 23) by extending their recurrence theorem from planar graphs to arbitrary families of H-minor-free graphs. More generally, it strengthens the work of Gurel-Gurevich and Nachmias (*Ann. of Math. (2)* **177** (2013) 761–781) who established recurrence for distributional limits of planar graphs when the degree of the root has exponential tails.

We further present a general method for proving subdiffusivity of the random walk on a large class of models, including UIPT and UIPQ, using only the volume growth profile of balls in the intrinsic metric.

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LIOUVILLE QUANTUM GRAVITY AND THE BROWNIAN MAP II: GEODESICS AND CONTINUITY OF THE EMBEDDING

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We endow the $\sqrt{8/3}$-Liouville quantum gravity sphere with a metric space structure and show that the resulting metric measure space agrees in law with the Brownian map. Recall that a Liouville quantum gravity sphere is a priori naturally parameterized by the Euclidean sphere $\mathbf{S}^2$. Previous work in this series used quantum Loewner evolution (QLE) to construct a metric $d_{\mathcal{Q}}$ on a countable dense subset of $\mathbf{S}^2$. Here, we show that $d_{\mathcal{Q}}$ a.s. extends uniquely and continuously to a metric $\overline{d}_{\mathcal{Q}}$ on all of $\mathbf{S}^2$. Letting d denote the Euclidean metric on $\mathbf{S}^2$, we show that the identity map between $(\mathbf{S}^2, d)$ and $(\mathbf{S}^2, \overline{d}_{\mathcal{Q}})$ is a.s. Hölder continuous in both directions. We establish several other properties of $(\mathbf{S}^2, \overline{d}_{\mathcal{Q}})$, culminating in the fact that (as a random metric measure space) it agrees in law with the Brownian map. We establish analogous results for the Brownian disk and plane. Our proofs involve new estimates on the size and shape of QLE balls and related quantum surfaces, as well as a careful analysis of $(\mathbf{S}^2, \overline{d}_{\mathcal{Q}})$ geodesics.

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LIMITS OF SPARSE CONFIGURATION MODELS AND BEYOND: GRAPHEXES AND MULTIGRAPHEXES

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We investigate structural properties of large, sparse random graphs through the lens of *sampling convergence* (Borgs et al. (*Ann. Probab.* **47** (2019) 2754–2800). Sampling convergence generalizes left convergence to sparse graphs, and describes the limit in terms of a *graphex*. We introduce a notion of sampling convergence for sequences of multigraphs, and establish the graphex limit for the configuration model, a preferential attachment model, the generalized random graph and a bipartite variant of the configuration model. The results for the configuration model, preferential attachment model and bipartite configuration model provide necessary and sufficient conditions for these random graph models to converge. The limit for the configuration model and the preferential attachment model is an augmented version of an exchangeable random graph model introduced by Caron and Fox (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** (2017) 1295–1366).

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PERIODIC HOMOGENIZATION OF NONSYMMETRIC LÉVY-TYPE PROCESSES

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In this paper we study homogenization problem for strong Markov processes on $\mathbb{R}^d$ having infinitesimal generators

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\le 1\}}\big) k(x,z)\Pi(dz) + \langle b(x), \nabla f(x)\rangle, \quad f \in C_b^2(\mathbb{R}^d)$$

in periodic media, where Π is a nonnegative measure on $\mathbb{R}^d$ that does not charge the origin 0, satisfies $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\Pi(dz) < \infty$ and can be singular with respect to the Lebesgue measure on $\mathbb{R}^d$. Under a proper scaling we show the scaled processes converge weakly to Lévy processes on $\mathbb{R}^d$. The results are a counterpart of the celebrated work (*Asymptotic Analysis for Periodic Structures* (1978) North-Holland; *Ann. Probab.* **13** (1985) 385–396) in the jump-diffusion setting. In particular, we completely characterize the homogenized limiting processes when $b(x)$ is a bounded continuous multivariate 1-periodic $\mathbb{R}^d$-valued function, $k(x,z)$ is a nonnegative bounded continuous function that is multivariate 1-periodic in both x and z variables and, in spherical coordinate $z = (r,\theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$,

$$\mathbb{1}_{\{|z|>1\}}\Pi(dz) = \mathbb{1}_{\{r>1\}}\varrho_0(d\theta)\frac{dr}{r^{1+\alpha}}$$

with $\alpha \in (0,\infty)$ and ϱ_0 being any finite measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Different phenomena occur depending on the values of α; there are five cases: $\alpha \in (0,1)$, $\alpha = 1$, $\alpha \in (1,2)$, $\alpha = 2$ and $\alpha \in (2,\infty)$.

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OPTIMIZATION OF MEAN-FIELD SPIN GLASSES

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Mean-field spin glasses are families of random energy functions (Hamiltonians) on high-dimensional product spaces. In this paper, we consider the case of Ising mixed p-spin models,; namely, Hamiltonians $H_N : \Sigma_N \to \mathbb{R}$ on the Hamming hypercube $\Sigma_N = \{\pm 1\}^N$, which are defined by the property that $\{H_N(\boldsymbol{\sigma})\}_{\boldsymbol{\sigma}\in\Sigma_N}$ is a centered Gaussian process with covariance $\mathbb{E}\{H_N(\boldsymbol{\sigma}_1)H_N(\boldsymbol{\sigma}_2)\}$ depending only on the scalar product $\langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\rangle$.

The asymptotic value of the optimum $\max_{\boldsymbol{\sigma}\in\Sigma_N} H_N(\boldsymbol{\sigma})$ was characterized in terms of a variational principle known as the Parisi formula, first proved by Talagrand and, in a more general setting, by Panchenko. The structure of superlevel sets is extremely rich and has been studied by a number of authors. Here, we ask whether a near optimal configuration $\boldsymbol{\sigma}$ can be computed in polynomial time.

We develop a message passing algorithm whose complexity per-iteration is of the same order as the complexity of evaluating the gradient of H_N, and characterize the typical energy value it achieves. When the p-spin model H_N satisfies a certain no-overlap gap assumption, for any $\varepsilon > 0$, the algorithm outputs $\boldsymbol{\sigma} \in \Sigma_N$ such that $H_N(\boldsymbol{\sigma}) \ge (1-\varepsilon)\max_{\boldsymbol{\sigma}'} H_N(\boldsymbol{\sigma}')$, with high probability. The number of iterations is bounded in N and depends uniquely on ε. More generally, regardless of whether the no-overlap gap assumption holds, the energy achieved is given by an extended variational principle, which generalizes the Parisi formula.

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MULTIVARIATE NORMAL APPROXIMATION FOR TRACES OF RANDOM UNITARY MATRICES

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In this article we obtain a superexponential rate of convergence in total variation between the traces of the first m powers of a $n \times n$ random unitary matrices and a $2m$-dimensional Gaussian random variable. This generalizes previous results in the scalar case to the multivariate setting, and we also give the precise dependence on the dimensions m and n in the estimates with explicit constants. We are especially interested in the regime where m grows with n and our main result basically states that if $m \ll \sqrt{n}$, then the rate of convergence in the Gaussian approximation is $\Gamma(\frac{n}{m}+1)^{-1}$ times a correction. We also show that the Gaussian approximation remains valid for all $m \ll n^{2/3}$ without a fast rate of convergence.

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ON THE REAL DAVIES' CONJECTURE

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We show that every matrix $A \in \mathbb{R}^{n\times n}$ is, at least, $\delta\|A\|$-close to a *real* matrix $A+E \in \mathbb{R}^{n\times n}$ whose eigenvectors have condition number, at most, $\tilde{O}_n(\delta^{-1})$. In fact, we prove that, with high probability, taking E to be a sufficiently small multiple of an i.i.d. *real* sub-Gaussian matrix of bounded density suffices. This essentially confirms a speculation of Davies and of Banks, Kulkarni, Mukherjee and Srivastava, who recently proved such a result for i.i.d. *complex Gaussian* matrices.

Along the way we also prove nonasymptotic estimates on the minimum possible distance between any two eigenvalues of a random matrix whose entries have arbitrary means; this part of our paper may be of independent interest.

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SCALING LIMITS OF THE THREE-DIMENSIONAL UNIFORM SPANNING TREE AND ASSOCIATED RANDOM WALK

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We show that the law of the three-dimensional uniform spanning tree (UST) is tight under rescaling in a space whose elements are measured, rooted real trees, continuously embedded into Euclidean space. We also establish that the relevant laws actually converge along a particular scaling sequence. The techniques that we use to establish these results are further applied to obtain various properties of the intrinsic metric and measure of any limiting space, including showing that the Hausdorff dimension of such is given by $3/\beta$, where $\beta \approx 1.624\dots$ is the growth exponent of three-dimensional loop-erased random walk. Additionally, we study the random walk on the three-dimensional uniform spanning tree, deriving its walk dimension (with respect to both the intrinsic and Euclidean metric) and its spectral dimension, demonstrating the tightness of its annealed law under rescaling, and deducing heat kernel estimates for any diffusion that arises as a scaling limit.

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MOMENTS OF THE RIEMANN ZETA FUNCTION ON SHORT INTERVALS OF THE CRITICAL LINE

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We show that as $T \to \infty$, for all $t \in [T, 2T]$ outside of a set of measure $\mathrm{o}(T)$,

$$\int_{-\log^\theta T}^{\log^\theta T} \left|\zeta\left(\frac{1}{2} + \mathrm{i}t + \mathrm{i}h\right)\right|^\beta \mathrm{d}h = (\log T)^{f_\theta(\beta)+\mathrm{o}(1)},$$

for some explicit exponent $f_\theta(\beta)$, where $\theta > -1$ and $\beta > 0$. This proves an extended version of a conjecture of Fyodorov and Keating (*Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** (2014) 20120503, 32). In particular, it shows that, for all $\theta > -1$, the moments exhibit a phase transition at a critical exponent $\beta_c(\theta)$, below which $f_\theta(\beta)$ is quadratic and above which $f_\theta(\beta)$ is linear. The form of the exponent f_θ also differs between mesoscopic intervals ($-1 < \theta < 0$) and macroscopic intervals ($\theta > 0$), a phenomenon that stems from an approximate tree structure for the correlations of zeta. We also prove that, for all $t \in [T, 2T]$ outside a set of measure $\mathrm{o}(T)$,

$$\max_{|h| \le \log^\theta T} \left|\zeta\left(\frac{1}{2} + \mathrm{i}t + \mathrm{i}h\right)\right| = (\log T)^{m(\theta)+\mathrm{o}(1)},$$

for some explicit $m(\theta)$. This generalizes earlier results of Najnudel (*Probab. Theory Related Fields* **172** (2018) 387–452) and Arguin et al. (*Comm. Pure Appl. Math.* **72** (2019) 500–535) for $\theta = 0$. The proofs are unconditional, except for the upper bounds when $\theta > 3$, where the Riemann hypothesis is assumed.

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ON STRONG SOLUTIONS OF ITÔ'S EQUATIONS WITH $\sigma \in W^1_d$ AND $b \in L_d$

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We consider Itô uniformly nondegenerate equations with time independent coefficients, the diffusion coefficient in $W^1_{d,\mathrm{loc}}$ and the drift in L_d. We prove the unique strong solvability for any starting point and prove that, as a function of the starting point, the solutions are Hölder continuous with any exponent < 1. We also prove that if we are given a sequence of coefficients converging in an appropriate sense to the original ones, then the solutions of approximating equations converge to the solution of the original one.

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CONVERGENCE OF THE RANDOM ABELIAN SANDPILE

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We prove that Abelian sandpiles with random initial states converge almost surely to unique scaling limits. The proof follows the Armstrong–Smart program for stochastic homogenization of uniformly elliptic equations.

Using simple random walk estimates, we prove an analogous result for the divisible sandpile and identify its scaling limit as exactly that of the averaged divisible sandpile. As a corollary, this gives a new quantitative proof of known results on the stabilizability of Abelian sandpiles.

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