ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 50, No. 2—March 2022

THE ANNALS OF PROBABILITY Vol. 50, No. 2, pp. 419–833 March 2022

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

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The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 50, Number 2, March 2022. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Probability
2022, Vol. 50, No. 2, 419–454
https://doi.org/10.1214/21-AOP1536

THE TWO-DIMENSIONAL CONTINUUM RANDOM FIELD ISING MODEL

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In this paper, we construct the two-dimensional continuum random field Ising model via scaling limits of a random field perturbation of the critical two-dimensional Ising model with diminishing disorder strength. Furthermore, we show that almost surely with respect to the continuum random field given by a white noise, the law of the magnetisation field is singular with respect to that of the two-dimensional continuum pure Ising model constructed by Camia, Garban and Newman (*Ann. Probab.* **43** (2015) 528–571).

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MSC2020 subject classifications. Primary 82B44; secondary 82B20, 82B27, 60G60, 60K35.
Key words and phrases. Random field Ising model, continuum scaling limit, magnetisation field.

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The Annals of Probability
2022, Vol. 50, No. 2, 455–508
https://doi.org/10.1214/21-AOP1537

MAXIMUM AND COUPLING OF THE SINE-GORDON FIELD

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For $0 < \beta < 6\pi$, we prove that the distribution of the centred maximum of the ε-regularised continuum sine-Gordon field on the two-dimensional torus converges to a randomly shifted Gumbel distribution as $\varepsilon \to 0$. Our proof relies on a strong coupling at all scales of the sine-Gordon field with the Gaussian free field, of independent interest, and extensions of existing methods for the maximum of the lattice Gaussian free field.

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MSC2020 subject classifications. Primary 60G60; secondary 82B41.
Key words and phrases. Maximum, sine-Gordon field, Gaussian free field, coupling.

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The Annals of Probability
2022, Vol. 50, No. 2, 509–558
https://doi.org/10.1214/21-AOP1538

EXTREMAL DISTANCE AND CONFORMAL RADIUS OF A CLE_4 LOOP

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Consider CLE_4 in the unit disk, and let ℓ be the loop of the CLE_4 surrounding the origin. Schramm, Sheffield and Wilson determined the law of the conformal radius seen from the origin of the domain surrounded by ℓ. We complement their result by determining the law of the extremal distance between ℓ and the boundary of the unit disk. More surprisingly, we also compute the joint law of these conformal radius and extremal distance. This law involves first and last hitting times of a one-dimensional Brownian motion. Similar techniques also allow us to determine joint laws of some extremal distances in a critical Brownian loop-soup cluster.

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MSC2020 subject classifications. 31A15, 60G15, 60G60, 60J65, 60J67, 81T40.
Key words and phrases. Conformal loop ensemble, Gaussian free field, isomorphism theorems, local set, loop soup, metric graph, Schramm–Loewner evolution.

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The Annals of Probability
2022, Vol. 50, No. 2, 559–590
https://doi.org/10.1214/21-AOP1540

ON INHOMOGENEOUS POLYNUCLEAR GROWTH

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This article studies the inhomogeneous geometric polynuclear growth model; the distribution of which is related to Schur functions. We explain a method to derive its distribution functions in both space-like and time-like directions, focusing on the two-time distribution. Asymptotics of the two-time distribution in the KPZ-scaling limit is then considered, extending to two times several single-time distributions in the KPZ universality class.

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MSC2020 subject classifications. Primary 60F05, 60K35, 82C23, 82C24; secondary 05E10, 15A15, 30E20.
Key words and phrases. KPZ universality, last passage percolation, polynuclear growth, two-time distribution.

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The Annals of Probability
2022, Vol. 50, No. 2, 591–648
https://doi.org/10.1214/21-AOP1541

THE DIRICHLET–FERGUSON DIFFUSION ON THE SPACE OF PROBABILITY MEASURES OVER A CLOSED RIEMANNIAN MANIFOLD

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We construct a recurrent diffusion process with values in the space of probability measures over an arbitrary closed Riemannian manifold of dimension $d \geq 2$. The process is associated with the Dirichlet form defined by integration of the Wasserstein gradient w.r.t. the Dirichlet–Ferguson measure, and is the counterpart on multidimensional base spaces to the modified massive Arratia flow over the unit interval described in V. Konarovskyi and M.-K. von Renesse (*Comm. Pure Appl. Math.* **72** (2019) 764–800). Together with two different constructions of the process, we discuss its ergodicity, invariant sets, finite-dimensional approximations, and Varadhan short-time asymptotics.

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MSC2020 subject classifications. Primary 60J46, 60J60; secondary 47D07, 60G57.

Key words and phrases. Dirichlet–Ferguson measure, Dirichlet–Ferguson diffusion, Wasserstein diffusion, modified massive Arratia flow.

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The Annals of Probability
2022, Vol. 50, No. 2, 649–687
https://doi.org/10.1214/21-AOP1542

PRECISE LOCAL ESTIMATES FOR DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION: HYPOELLIPTIC CASE

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This article is concerned with stochastic differential equations driven by a d dimensional fractional Brownian motion with Hurst parameter $H > 1/4$ and understood in the rough paths sense. Whenever the coefficients of the equation satisfy a uniform hypoellipticity condition, we establish a sharp local estimate on the associated control distance function and a sharp local lower estimate on the density of the solution. Our methodology relies heavily on the rough paths structure of the equation.

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MSC2020 subject classifications. 60H10, 60H07, 60G15.
Key words and phrases. Rough paths, Malliavin calculus, fractional Brownian motion.

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The Annals of Probability
2022, Vol. 50, No. 2, 688–734
https://doi.org/10.1214/21-AOP1543

HYPERCONTRACTIVITY AND LOWER DEVIATION ESTIMATES IN NORMED SPACES

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We consider the problem of estimating small ball probabilities $\mathbb{P}\{f(G) \leq \delta \mathbb{E} f(G)\}$ for subadditive, positively homogeneous functions f with respect to the Gaussian measure. We establish estimates that depend on global parameters of the underlying function, which take into account analytic and statistical measures, such as the variance and the L^1-norms of its partial derivatives. This leads to dimension-dependent bounds for small ball and lower small deviation estimates for seminorms when the linear structure is appropriately chosen to optimize the aforementioned parameters. Our bounds are best possible up to numerical constants. In all regimes, $\|G\|_\infty = \max_{i\leq n} |g_i|$ arises as an extremal case in this study. The proofs exploit the convexity and hypercontractivity properties of the Gaussian measure.

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MSC2020 subject classifications. Primary 46B09; secondary 52A21.
Key words and phrases. Talagrand's $L^1 - L^2$ bound, Alon–Milman theorem, Ornstein–Uhlenbeck semigroup, Gaussian convexity, hypercontractvity, superconcentration.

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The Annals of Probability
2022, Vol. 50, No. 2, 735–770
https://doi.org/10.1214/21-AOP1545

CONCENTRATION INEQUALITIES FOR LOG-CONCAVE DISTRIBUTIONS WITH APPLICATIONS TO RANDOM SURFACE FLUCTUATIONS

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We derive two concentration inequalities for linear functions of log-concave distributions: an enhanced version of the classical Brascamp–Lieb concentration inequality and an inequality quantifying log-concavity of marginals in a manner suitable for obtaining variance and tail probability bounds.

These inequalities are applied to the statistical mechanics problem of estimating the fluctuations of random surfaces of the $\nabla\varphi$ type. The classical Brascamp–Lieb inequality bounds the fluctuations whenever the interaction potential is uniformly convex. We extend these bounds to the case of convex potentials whose second derivative vanishes only on a zero measure set, when the underlying graph is a d-dimensional discrete torus. The result applies, in particular, to potentials of the form $U(x) = |x|^p$ with $p > 1$ and answers a question discussed by Brascamp–Lieb–Lebowitz (In *Statistical Mechanics* (1975) 379–390, Springer). Additionally, new tail probability bounds are obtained for the family of potentials $U(x) = |x|^p + x^2$, $p > 2$. This result answers a question mentioned by Deuschel and Giacomin (*Stochastic Process. Appl.* **89** (2000) 333–354).

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MSC2020 subject classifications. Primary 60E15, 60K35, 82C24; secondary 82B20.

Key words and phrases. Brascamp–Lieb concentration inequality, log-concave distributions, random surfaces of the $\nabla\varphi$ type, effective interface models, localization, tail probability bounds.

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The Annals of Probability
2022, Vol. 50, No. 2, 771–790
https://doi.org/10.1214/21-AOP1547

ON MULTIPLE SLE FOR THE FK–ISING MODEL

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We prove the convergence of multiple interfaces in the critical planar $q = 2$ random cluster model and provide an explicit description of the scaling limit. Remarkably, the expression for the partition function of the resulting multiple $\mathrm{SLE}_{16/3}$ coincides with the bulk spin correlation in the critical Ising model in the half-plane, after formally replacing a position of each spin and its complex conjugate with a pair of points on the real line. As a corollary we recover Belavin–Polyakov–Zamolodchikov equations for the spin correlations.

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MSC2020 subject classifications. 60J67, 82B20.
Key words and phrases. Schramm–Lowner evolution, Ising model, random cluster models.

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The Annals of Probability
2022, Vol. 50, No. 2, 791–833
https://doi.org/10.1214/21-AOP1548

MCKEAN–VLASOV OPTIMAL CONTROL: THE DYNAMIC PROGRAMMING PRINCIPLE

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We study the McKean–Vlasov optimal control problem with common noise which allow the law of the control process to appear in the state dynamics under various formulations: strong and weak ones, Markovian or non-Markovian. By interpreting the controls as probability measures on an appropriate canonical space with two filtrations, we then develop the classical measurable selection, conditioning and concatenation arguments in this new context, and establish the dynamic programming principle under general conditions.

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MSC2020 subject classifications. Primary 49L20, 93E20; secondary 60K35, 60H30.
Key words and phrases. McKean–Vlasov optimal control, dynamic programming principle, measurable selection.

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The Annals of Probability

Vol. 50 **May 2022** **No. 3**

Articles