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AN ELLIPTIC HARNACK INEQUALITY FOR DIFFERENCE EQUATIONS WITH RANDOM BALANCED COEFFICIENTS

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We prove an elliptic Harnack inequality at large scale on the lattice $\mathbb{Z}^d$ for nonnegative solutions of a difference equation with balanced i.i.d. coefficients which are not necessarily elliptic. We also identify the optimal constant in the Harnack inequality. Our proof relies on a quantitative homogenization result of the corresponding invariance principle to Brownian motion and on percolation estimates. As a corollary of our main theorem, we derive an almost optimal Hölder estimate.

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A SMOLUCHOWSKI–KRAMERS APPROXIMATION FOR AN INFINITE DIMENSIONAL SYSTEM WITH STATE-DEPENDENT DAMPING

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We study the validity of a Smoluchowski–Kramers approximation for a class of wave equations in a bounded domain of $\mathbb{R}^n$ subject to a state-dependent damping and perturbed by a multiplicative noise. We prove that in the small mass limit the solution converges to the solution of a stochastic quasilinear parabolic equation where a noise-induced extra drift is created.

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SIMPLE CONFORMAL LOOP ENSEMBLES ON LIOUVILLE QUANTUM GRAVITY

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We show that, when one draws a simple conformal loop ensemble (CLE_κ for $\kappa \in (8/3, 4)$) on an independent $\sqrt{\kappa}$-Liouville quantum gravity (LQG) surface and explores the CLE in a natural Markovian way, the quantum surfaces (e.g., corresponding to the interior of the CLE loops) that are cut out form a Poisson point process of quantum disks. This construction allows us to make direct links between CLE on LQG, $(4/\kappa)$-stable processes, and labeled branching trees. The ratio between positive and negative jump intensities of these processes turns out to be $-\cos(4\pi/\kappa)$ which can be interpreted as a "density" of CLE loops in the CLE on LQG setting. Positive jumps correspond to the discovery of a CLE loop (where the LQG length of the loop is given by the jump size) and negative jumps correspond to the moments where the discovery process splits the remaining to be discovered domain into two pieces.

Some consequences of this result are the following: (i) It provides a construction of a CLE on LQG as a patchwork/welding of quantum disks. (ii) It allows us to construct the "natural quantum measure" that lives in a CLE carpet. (iii) It enables us to derive some new properties and formulas for SLE processes and CLE themselves (without LQG) such as the exact distribution of the trunk of the general $\mathrm{SLE}_\kappa(\kappa - 6)$ processes.

The present work deals directly with structures in the continuum and makes no reference to discrete models, but our calculations match those for scaling limits of $O(N)$ models on planar maps with large faces and CLE on LQG. Indeed, our Lévy-tree descriptions are exactly the ones that appear in the study of the large-scale limit of peeling of discrete decorated planar maps, such as in recent work of Bertoin, Budd, Curien and Kortchemski.

The case of nonsimple CLEs on LQG is studied in another paper.

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CONVERGENCE IN LAW FOR COMPLEX GAUSSIAN MULTIPLICATIVE CHAOS IN PHASE III

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Gaussian multiplicative chaos (GMC) is informally defined as a random measure $e^{\gamma X}\,\mathrm{d}x$ where X is Gaussian field on $\mathbb{R}^d$ (or an open subset of it) whose correlation function is of the form $K(x,y) = \log \frac{1}{|y-x|} + L(x,y)$, where L is a continuous function of x and y and $\gamma = \alpha + i\beta$ is a complex parameter. In the present paper we consider the case $\gamma \in \mathcal{P}'_{\mathrm{III}}$, where

$$\mathcal{P}'_{\mathrm{III}} := \{\alpha + i\beta : \alpha, \gamma \in \mathbb{R}, |\alpha| < \sqrt{d/2}, \alpha^2 + \beta^2 \geq d\}.$$

We prove that if X is replaced by an approximation X_ε obtained by convolution with a smooth kernel, then the random distribution $e^{\gamma X_\varepsilon}\,\mathrm{d}x$, when properly rescaled, has an explicit nontrivial limit in law when ε goes to zero. This limit does not depend on the specific convolution kernel which is used to define X_ε and can be described as a complex Gaussian white noise with a random intensity given by a real GMC associated with parameter 2α.

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NORMAL FLUCTUATION IN QUANTUM ERGODICITY FOR WIGNER MATRICES

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We consider the quadratic form of a general high-rank deterministic matrix on the eigenvectors of an $N \times N$ Wigner matrix and prove that it has Gaussian fluctuation for each bulk eigenvector in the large N limit. The proof is a combination of the energy method for the Dyson Brownian motion inspired by Marcinek and Yau (2021) and our recent multiresolvent local laws (*Comm. Math. Phys.* **388** (2021) 1005–1048).

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GEODESIC STARS IN RANDOM GEOMETRY

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A point of a metric space is called a geodesic star with m arms if it is the endpoint of m disjoint geodesics. For every $m \in \{1, 2, 3, 4\}$, we prove that the set of all geodesic stars with m arms in the Brownian sphere has dimension $5 - m$. This complements recent results of Miller and Qian, who proved that this dimension is smaller than or equal to $5 - m$.

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LOCAL LIMITS OF BIPARTITE MAPS WITH PRESCRIBED FACE DEGREES IN HIGH GENUS

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We study the local limits of uniform high genus bipartite maps with prescribed face degrees. We prove the convergence toward a family of infinite maps of the plane, the **q**-IBPMs, which exhibit both a spatial Markov property and a hyperbolic behaviour. Therefore, we observe a similar local behaviour for a wide class of models of random high genus maps which can be seen as a result of universality. Our results cover all the regimes where the expected degree of the root face remains finite in the limit. This follows a work by the same authors on high genus triangulations.

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CRITICAL PREWETTING IN THE 2D ISING MODEL

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In this paper, we develop a detailed analysis of critical prewetting in the context of the two-dimensional Ising model. Namely, we consider a two-dimensional nearest-neighbor Ising model in a $2N \times N$ rectangular box with a boundary condition inducing the coexistence of the $+$ phase in the bulk and a layer of $-$ phase along the bottom wall. The presence of an external magnetic field of intensity $h = \lambda/N$ (for some fixed $\lambda > 0$) makes the layer of $-$ phase unstable. For any $\beta > \beta_c$, we prove that, under a diffusing scaling by $N^{-2/3}$ horizontally and $N^{-1/3}$ vertically, the interface separating the layer of unstable phase from the bulk phase weakly converges to an explicit Ferrari–Spohn diffusion.

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SHARP CONCENTRATION FOR THE LARGEST AND SMALLEST FRAGMENT IN A *k*-REGULAR SELF-SIMILAR FRAGMENTATION

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We study the asymptotics of the k-regular self-similar fragmentation process. For $\alpha > 0$ and an integer $k \geq 2$, this is the Markov process $(I_t)_{t\geq 0}$ in which each I_t is a union of open subsets of $[0, 1)$, and independently each subinterval of I_t of size u breaks into k equally sized pieces at rate u^α. Let k^{-m_t} and k^{-M_t} be the respective sizes of the largest and smallest fragments in I_t. By relating $(I_t)_{t\geq 0}$ to a branching random walk, we find that there exist explicit deterministic functions $g(t)$ and $h(t)$ such that $|m_t - g(t)| \leq 1$ and $|M_t - h(t)| \leq 1$ for all sufficiently large t. Furthermore, for each n, we study the final time at which fragments of size k^{-n} exist. In particular, by relating our branching random walk to a certain point process, we show that, after suitable rescaling, the laws of these times converge to a Gumbel distribution as $n \to \infty$.

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A FORWARD-BACKWARD SDE FROM THE 2D NONLINEAR STOCHASTIC HEAT EQUATION

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We consider a nonlinear stochastic heat equation in spatial dimension $d = 2$, forced by a white-in-time multiplicative Gaussian noise with spatial correlation length $\varepsilon > 0$ but divided by a factor of $\sqrt{\log \varepsilon^{-1}}$. We impose a condition on the Lipschitz constant of the nonlinearity so that the problem is in the "weak noise" regime. We show that, as $\varepsilon \downarrow 0$, the one-point distribution of the solution converges, with the limit characterized in terms of the solution to a forward-backward stochastic differential equation (FBSDE). We also characterize the limiting multipoint statistics of the solution, when the points are chosen on appropriate scales, in similar terms. Our approach is new even for the linear case, in which the FBSDE can be solved explicitly and we recover results of Caravenna, Sun, and Zygouras (*Ann. Appl. Probab.* **27** (2017) 3050–3112).

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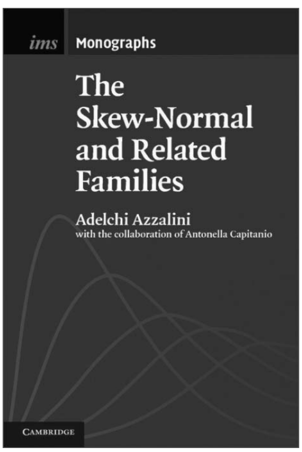

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