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MULTIPOINT DISTRIBUTION OF TASEP

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Recently Johansson and Rahman obtained the limiting multitime distribution for the discrete polynuclear growth model (Johansson and Rahman (2019)), which is equivalent to a discrete TASEP model with step initial condition. In this paper, we obtain a finite time multipoint distribution formula of continuous TASEP with general initial conditions in the space-time plane. We evaluate the limit of this distribution function when the times go to infinity at the same speed for both step and flat initial conditions. These limiting distributions are expected to be universal for all the models in the Kardar–Parisi–Zhang universality class.

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BROWNIAN LOOPS AND THE CENTRAL CHARGE OF A LIOUVILLE RANDOM SURFACE

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We explore the geometric meaning of the so-called *zeta-regularized* determinant of the Laplace–Beltrami operator on a compact surface, with or without boundary. We relate the $(-c/2)$th power of the determinant of the Laplacian to the appropriately regularized partition function of a Brownian loop soup of intensity c on the surface. This means that, in a certain sense, *decorating* a random surface by a Brownian loop soup of intensity c corresponds to *weighting* the law of the surface by the $(-c/2)$th power of the determinant of the Laplacian.

Next, we introduce a method of regularizing a Liouville quantum gravity (LQG) surface (with some matter central charge parameter $\mathbf{c}$) to produce a smooth surface. And we show that weighting the law of this random surface by the $(-\mathbf{c}'/2)$th power of the Laplacian determinant has precisely the effect of changing the matter central charge from $\mathbf{c}$ to $\mathbf{c}+\mathbf{c}'$. Taken together with the earlier results, this provides a way of interpreting an LQG surface of matter central charge $\mathbf{c}$ as a pure LQG surface *decorated* by a Brownian loop soup of intensity $\mathbf{c}$.

Building on this idea, we present several open problems about random planar maps and their continuum analogs. Although the original construction of LQG is well defined only for $\mathbf{c} \le 1$, some of the constructions and questions also make sense when $\mathbf{c} > 1$.

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SCALING AND LOCAL LIMITS OF BAXTER PERMUTATIONS AND BIPOLAR ORIENTATIONS THROUGH COALESCENT-WALK PROCESSES

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Baxter permutations, plane bipolar orientations, and a specific family of walks in the nonnegative quadrant, called *tandem walks*, are well-known to be related to each other through several bijections. We introduce a further new family of discrete objects, called *coalescent-walk processes* and we relate it to the three families mentioned above.

We prove joint Benjamini–Schramm convergence (both in the annealed and quenched sense) for uniform objects in the four families. Furthermore, we explicitly construct a new random measure on the unit square, called the *Baxter permuton* and we show that it is the scaling limit (in the permuton sense) of uniform Baxter permutations. In addition, we relate the limiting objects of the four families to each other, both in the local and scaling limit case.

The scaling limit result is based on the convergence of the trajectories of the coalescent-walk process to the *coalescing flow*—in the terminology of Le Jan and Raimond (*Ann. Probab.* **32** (2004) 1247–1315)—of a perturbed version of the Tanaka stochastic differential equation. Our scaling result entails joint convergence of the tandem walks of a plane bipolar orientation and its dual, giving an alternative answer to Conjecture 4.4 of Kenyon, Miller, Sheffield, Wilson (*Ann. Probab.* **47** (2019) 1240–1269) compared to the one of Gwynne, Holden, Sun ((2016), arXiv:1603.01194).

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FORESTS, CUMULANTS, MARTINGALES

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This work is concerned with forest and cumulant type expansions of general random variables on a filtered probability space. We establish a "broken exponential martingale" expansion that generalizes and unifies the exponentiation result of Alòs, Gatheral, and Radoičić (SSRN'17; *Quant. Finance* **20** (2020) 13–27) and the cumulant recursion formula of Lacoin, Rhodes, and Vargas (arXiv; (2019)). Specifically, we exhibit the two previous results as lower dimensional projections of the same generalized forest expansion, subsequently related by forest reordering. Our approach also leads to sharp integrability conditions for validity of the cumulant formula, as required by many of our examples, including iterated stochastic integrals, Lévy area, Bessel processes, KPZ with smooth noise, Wiener–Itô chaos, and "rough" stochastic (forward) variance models.

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A FLUCTUATION RESULT FOR THE DISPLACEMENT IN THE OPTIMAL MATCHING PROBLEM

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The aim of this paper is to justify in dimensions two and three the ansatz of Caracciolo et al. stating that the displacement in the optimal matching problem is essentially given by the solution to the linearized equation that is, the Poisson equation. Moreover, we prove that at all mesoscopic scales, this displacement is close in suitable negative Sobolev spaces to a curl-free Gaussian free field. For this, we combine a quantitative estimate on the difference between the displacement and the linearized object, which is based on the large-scale regularity theory recently developed in collaboration with F. Otto, together with a qualitative convergence result for the linearized problem.

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THE DISORDERED LATTICE FREE FIELD PINNING MODEL APPROACHING CRITICALITY

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We continue the study, initiated in (*J. Eur. Math. Soc. (JEMS)* **20** (2018) 199–257), of the localization transition of a lattice free field $\phi = (\phi(x))_{x \in \mathbb{Z}^d}$, $d \geq 3$, in presence of a quenched disordered substrate. The presence of the substrate affects the interface at the spatial sites in which the interface height is close to zero. This corresponds to the Hamiltonian

$$\sum_{x \in \mathbb{Z}^d} (\beta \omega_x + h) \delta_x,$$

where $\delta_x = \mathbf{1}_{[-1,1]}(\phi(x))$, and $(\omega_x)_{x \in \mathbb{Z}^d}$ is an i.i.d. centered field. A transition takes place when the average pinning potential h goes past a threshold $h_c(\beta)$: from a delocalized phase $h < h_c(\beta)$, where the field is macroscopically repelled by the substrate, to a localized one $h > h_c(\beta)$ where the field sticks to the substrate. In (*J. Eur. Math. Soc. (JEMS)* **20** (2018) 199–257), the critical value of h is identified and it coincides, up to the sign, with the log-Laplace transform of $\omega = \omega_x$, that is $-h_c(\beta) = \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}]$. Here, we obtain the sharp critical behavior of the free energy approaching criticality:

$$\lim_{u \searrow 0} \frac{\mathrm{d}(\beta, h_c(\beta) + u)}{u^2} = \frac{1}{2 \operatorname{Var}(e^{\beta\omega - \lambda(\beta)})}.$$

Moreover, we give a precise description of the trajectories of the field in the same regime: to leading order as $h \searrow h_c(\beta)$ the absolute value of the field is $\sqrt{2\sigma_d^2 |\log(h - h_c(\beta))|}$ except on a vanishing fraction of sites (σ_d^2 is the single site variance of the free field).

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SURFACE TRANSITION IN THE COLLAPSED PHASE OF A SELF-INTERACTING WALK ADSORBED ALONG A HARD WALL

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The present paper is dedicated to the 2-dimensional Interacting Partially Directed Self Avoiding Walk constrained to remain in the upper-half plane and interacting with the horizontal axis. This model has originally been introduced to investigate the behavior of a homopolymer dipped in a poor solvent and adsorbed along a horizontal hard wall. It is known to undergo a *collapse* transition between an *extended* phase, inside which typical configurations of the polymer have a large horizontal extension (comparable to their total size), and a *collapsed* phase inside which the polymer looks like a globule.

It is conjectured in the physics literature (see, e.g., (*Phys. A, Stat. Mech. Appl.* **318** (2002) 171) or (*Phys. Rev. E* **65** (2002) 056124)) that inside the collapsed phase, a *surface* transition occurs between an *adsorbed-collapsed* regime where the bottommost layer of the globule is pinned at the hard wall, and a *desorbed-collapsed* regime where the globule wanders away from the wall. In the present paper, we consider a simplified "single-bead" version of the model, for which we establish rigorously the existence of the surface transition and exhibit its associated critical curve. To that aim, we display some sharp asymptotics of the partition function of this simplified model within the collapsed phase.

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STATIONARY DISTRIBUTIONS FOR THE VOTER MODEL IN $d \geq 3$ ARE FACTORS OF IID

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For the Voter Model on $\mathbb{Z}^d$, $d \geq 3$, we show that the (extremal) stationary distributions are isomorphic to Bernoulli shifts, and answer an open question asked by Steif and Tykesson in (*ALEA Lat. Am. J. Probab. Math. Stat.* **16** (2019) 899–955). The proof gives explicit constructions of the stationary distributions as factors of IID processes on $\mathbb{Z}^d$.

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DOMAINS OF ATTRACTION OF INVARIANT DISTRIBUTIONS OF THE INFINITE ATLAS MODEL

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The infinite Atlas model describes a countable system of competing Brownian particles where the lowest particle gets a unit upward drift and the rest evolve as standard Brownian motions. The stochastic process of gaps between the particles in the infinite Atlas model does not have a unique stationary distribution and in fact for every $a \geq 0$, $\pi_a := \bigotimes_{i=1}^{\infty} \text{Exp}(2 + ia)$ is a stationary measure for the gap process. We say that an initial distribution of gaps is in the weak domain of attraction of the stationary measure π_a if the time averaged laws of the stochastic process of the gaps, when initialized using that distribution, converge to π_a weakly in the large time limit. We provide general sufficient conditions on the initial gap distribution of the Atlas particles for it to lie in the weak domain of attraction of π_a for each $a \geq 0$. The cases $a = 0$ and $a > 0$ are qualitatively different as is seen from the analysis and the sufficient conditions that we provide. Proofs are based on the analysis of synchronous couplings, namely, couplings of the ranked particle systems started from different initial configurations, but driven using the same set of Brownian motions.

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HIDDEN SYMMETRIES AND LIMIT LAWS IN THE EXTREME ORDER STATISTICS OF THE LAPLACE RANDOM WALK

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This paper is concerned with the limit laws of the extreme order statistics derived from a symmetric Laplace walk. We provide two different descriptions of the point process of the limiting extreme order statistics: a branching representation and a squared Bessel representation. These complementary descriptions expose various hidden symmetries in branching processes and Brownian motion which lie behind some striking formulas found by Schehr and Majumdar (*Phys. Rev. Lett.* **108** (2012) 040601). In particular, the Bessel process of dimension $4 = 2 + 2$ appears in the descriptions as a path decomposition of Brownian motion at a local minimum and the Ray–Knight description of Brownian local times near the minimum.

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ERRATUM TO "CHARACTERIZATION OF POSITIVELY CORRELATED SQUARED GAUSSIAN PROCESSES"

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Articles