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# ON THE RADIUS OF GAUSSIAN FREE FIELD EXCURSION CLUSTERS

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We consider the Gaussian free field $\varphi$ on $\mathbb{Z}^d$, for $d \geq 3$, and give sharp bounds on the probability that the radius of a finite cluster in the excursion set $\{\varphi \geq h\}$ exceeds a large value $N$ for any height $h \neq h_*$, where $h_*$ refers to the corresponding percolation critical parameter. In dimension 3, we prove that this probability is subexponential in $N$ and decays as $\exp\{-\frac{\pi}{6}(h-h_*)^2\frac{N}{\log N}\}$ as $N \to \infty$ to principal exponential order. When $d \geq 4$, we prove that these tails decay exponentially in $N$. Our results extend to other quantities of interest, such as truncated two-point functions and the two-arms probability for annuli crossings at scale $N$.

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# RECURRENCE OF THE UNIFORM INFINITE HALF-PLANE MAP VIA DUALITY OF RESISTANCES

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We study the simple random walk on the Uniform Infinite Half-Plane Map, which is the local limit of critical Boltzmann planar maps with a large and simple boundary. We prove that the simple random walk is recurrent, and that the resistance between the root and the boundary of the hull of radius $r$ is at least of order $\log r$. This resistance bound is expected to be sharp, and is better than those following from previous proofs of recurrence for nonbounded-degree planar maps models. Our main tools are the self-duality of uniform planar maps, a classical lemma about duality of resistances and some peeling estimates. The proof shares some ideas with Russo–Seymour–Welsh theory in percolation.

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# INTEGRATION BY PARTS AND THE KPZ TWO-POINT FUNCTION

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In this article, we consider the KPZ fixed point starting from a two-sided Brownian motion with an arbitrary diffusion coefficient. We apply the integration by parts formula from Malliavin calculus to establish a key relation between the two-point (correlation) function of the spatial derivative process and the location of the maximum of an Airy process plus Brownian motion minus a parabola. Integration by parts also allows us to deduce the density of this location in terms of the second derivative of the variance of the KPZ fixed point. In the stationary regime, we find the same density related to limit fluctuations of a second-class particle. We further develop an adaptation of Stein's method that implies asymptotic independence of the spatial derivative process from the initial data.

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# THE TIME CONSTANT FOR BERNOULLI PERCOLATION IS LIPSCHITZ CONTINUOUS STRICTLY ABOVE $p_c$

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*Dedicated to the memory of Vladas Sidoravicius*[*]

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We consider the standard model of i.i.d. first passage percolation on $\mathbb{Z}^d$ given a distribution $G$ on $[0, +\infty]$ ($+\infty$ is allowed). When $G([0, +\infty)) > p_c(d)$, it is known that the time constant $\mu_G$ exists. We are interested in the regularity properties of the map $G \mapsto \mu_G$. We first study the specific case of distributions of the form $G_p = p\delta_1 + (1 - p)\delta_\infty$ for $p > p_c(d)$. In this case, the travel time between two points is equal to the length of the shortest path between the two points in a bond percolation of parameter $p$. We show that the function $p \mapsto \mu_{G_p}$ is Lipschitz continuous on every interval $[p_0, 1]$, where $p_0 > p_c(d)$.

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# MEAN FIELD BEHAVIOR DURING THE BIG BANG REGIME FOR COALESCING RANDOM WALKS

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In this paper, we consider coalescing random walks on a general connected graph $\mathcal{G} = (V, E)$. We set up a unified framework to study the leading order of the decay rate of $P_t$, the expectation of the fraction of occupied sites at time $t$, particularly for the 'Big Bang' regime where $t \ll t_{\text{coal}} := \mathbb{E}(\inf\{s :$ There is only one particle at time $s\})$. Our results show that $P_t$ satisfies certain 'mean-field behaviors' if the graphs satisfy certain 'transience-like' conditions.

We apply this framework to two families of graphs: (1) graphs given by the configuration model whose degree distribution is supported in the interval $[3, \bar{d}]$ for some $\bar{d} \geq 3$, and (2) finite and infinite transitive graphs. In the first case, we show that for $1 \ll t \ll |V|$, $P_t$ decays in the order of $t^{-1}$, and $(tP_t)^{-1}$ is approximately the probability that two particles starting from the root of the corresponding unimodular Galton–Watson tree never collide after one of them leaves the root. The number $(tP_t)^{-1}$ is also roughly $|V|/(2t_{\text{meet}})$, where $t_{\text{meet}}$ is the mean meeting time of two independent walkers. By taking the local weak limit, we prove convergence of $tP_t$ as $t \to \infty$ for the corresponding unimodular Galton–Watson tree. For the second family of graphs, we consider a growing sequence of finite transitive graphs $\mathcal{G}_n = (V_n, E_n)$, satisfying that $t_{\text{meet}} = O(|V_n|)$ and the inverse of the spectral gap $t_{\text{rel}}$ is $o(|V_n|)$. We show that $t_{\text{rel}} \ll t \ll t_{\text{coal}}$, $(tP_t)^{-1}$ is approximately the probability that two random walks never meet before time $t$, and it is also roughly $|V|/(2t_{\text{meet}})$. In addition, we define a natural 'uniform transience' condition, and show that in the transitive setup it implies the above estimates of $tP_t$ for all $1 \ll t \ll t_{\text{coal}}$. Estimates of $tP_t$ are also obtained for all infinite transient transitive unimodular graphs, in particular, all transient transitive amenable graphs.

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# QUANTITATIVE HOMOGENIZATION OF INTERACTING PARTICLE SYSTEMS

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For a class of interacting particle systems in continuous space, we show that finite-volume approximations of the bulk diffusion matrix converge at an algebraic rate. The models we consider are reversible with respect to the Poisson measures with constant density, and are of nongradient type. Our approach is inspired by recent progress in the quantitative homogenization of elliptic equations. Along the way, we develop suitable modifications of the Caccioppoli and multiscale Poincaré inequalities, which are of independent interest.

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# THREE-HALVES VARIATION OF GEODESICS IN THE DIRECTED LANDSCAPE

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We show that geodesics in the directed landscape have $3/2$-variation and that weight functions along the geodesics have cubic variation.

We show that the geodesic and its landscape environment around an interior point have a small-scale limit. This limit is given in terms of the directed landscape with Brownian–Bessel boundary conditions. The environments around different interior points are asymptotically independent.

We give tail bounds with optimal exponents for geodesic and weight function increments.

As an application of our results, we show that geodesics are not Hölder-$2/3$ and that weight functions are not Hölder-$1/3$, although these objects are known to be Hölder with all lower exponents.

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# THE MULTINOMIAL TILING MODEL

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Given a graph $\mathcal{G}$ and collection of subgraphs $T$ (called tiles), we consider covering $\mathcal{G}$ with copies of tiles in $T$ so that each vertex $v \in \mathcal{G}$ is covered with a predetermined multiplicity. The *multinomial tiling model* is a natural probability measure on such configurations (it is the uniform measure on standard tilings of the corresponding "blow-up" of $\mathcal{G}$).

In the limit of large multiplicities, we compute the asymptotic growth rate of the number of multinomial tilings. We show that the individual tile densities tend to a Gaussian field defined by an associated discrete Laplacian. We also find an exact discrete Coulomb gas limit when we vary the multiplicities.

For tilings of $\mathbb{Z}^d$ with translates of a single tile and a small density of defects, we study a crystallization phenomenon when the defect density tends to zero, and give examples of naturally occurring quasicrystals in this framework.

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# ISOPERIMETRIC INEQUALITIES IN THE BROWNIAN PLANE

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We consider the model of the Brownian plane, which is a pointed non-compact random metric space with the topology of the complex plane. The Brownian plane can be obtained as the scaling limit in distribution of the uniform infinite planar triangulation or the uniform infinite planar quadrangulation and is conjectured to be the universal scaling limit of many others random planar lattices. We establish sharp bounds on the probability of having a short cycle separating the ball of radius $r$ centered at the distinguished point from infinity. Then we prove a strong version of the spatial Markov property of the Brownian plane. Combining our study of short cycles with this strong spatial Markov property we obtain sharp isoperimetric bounds for the Brownian plane.

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# STABLE SHREDDED SPHERES AND CAUSAL RANDOM MAPS WITH LARGE FACES

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We introduce a new familiy of random compact metric spaces $\mathcal{S}_\alpha$ for $\alpha \in (1, 2)$, which we call *stable shredded spheres*. They are constructed from excursions of $\alpha$-stable Lévy processes on [0, 1] possessing no negative jumps. Informally, viewing the graph of the Lévy excursion in the plane, each jump of the process is "cut open" and replaced by a circle, and then all points on the graph at equal height, which are not separated by a jump, are identified. We show that the shredded spheres arise as scaling limits of models of causal random planar maps with large faces introduced by Di Francesco and Guitter. We also establish that their Hausdorff dimension is almost surely equal to $\alpha$. Point identification in the shredded spheres is intimately connected to the presence of decrease points in stable spectrally positive Lévy processes, as studied by Bertoin in the 1990s.

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