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DISCRETE SELF-SIMILAR AND ERGODIC MARKOV CHAINS

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The first aim of this paper is to introduce a class of Markov chains on $\mathbb{Z}_+$ which are discrete self-similar in the sense that their semigroups satisfy an invariance property expressed in terms of a discrete random dilation operator. After showing that this latter property requires the chains to be upward skip-free, we first establish a gateway intertwining relation between the semigroup of such chains and the one of spectrally negative self-similar Markov processes on $\mathbb{R}_+$. As a by-product, we prove that each of these Markov chains, after an appropriate scaling, converge in the Skorohod metric to the associated self-similar Markov process. By a linear perturbation of the generator of these Markov chains, we obtain a class of ergodic Markov chains which are nonreversible. By means of intertwining relations and their strengthened interweaving versions, we derive several deep analytical properties of such ergodic chains, including the description of the spectrum, the spectral expansion of their semigroups and the study of their convergence to equilibrium in the Φ-entropy sense as well as their hypercontractivity property.

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BROWNIAN BEES IN THE INFINITE SWARM LIMIT

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The *Brownian bees* model is a branching particle system with spatial selection. It is a system of N particles which move as independent Brownian motions in $\mathbb{R}^d$ and independently branch at rate 1, and, crucially, at each branching event, the particle which is the furthest away from the origin is removed to keep the population size constant. In the present work we prove that, as $N \to \infty$, the behaviour of the particle system is well approximated by the solution of a free boundary problem (which is the subject of a companion paper (*Trans. Amer. Math. Soc.* **374** (2021) 6269–6329)), the *hydrodynamic limit* of the system. We then show that for this model the so-called *selection principle* holds; that is, that as $N \to \infty$, the equilibrium density of the particle system converges to the steady-state solution of the free boundary problem.

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MEAN FIELD GAMES MASTER EQUATIONS WITH NONSEPARABLE HAMILTONIANS AND DISPLACEMENT MONOTONICITY

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In this manuscript we propose a structural condition on nonseparable Hamiltonians, which we term *displacement monotonicity* condition, to study second-order mean field games master equations. A rate of dissipation of a bilinear form is brought to bear a global (in time) well-posedness theory, based on a priori uniform Lipschitz estimates on the solution in the measure variable. Displacement monotonicity being sometimes in dichotomy with the widely used Lasry–Lions monotonicity condition, the novelties of this work persist even when restricted to separable Hamiltonians.

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THE FREE UNIFORM SPANNING FOREST IS DISCONNECTED IN SOME VIRTUALLY FREE GROUPS, DEPENDING ON THE GENERATOR SET

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We prove the rather counterintuitive result that there exist finite transitive graphs H and integers k such that the Free Uniform Spanning Forest in the direct product of the k-regular tree and H has infinitely many trees almost surely.

This shows that the number of trees in the FUSF is not a quasi-isometry invariant. Moreover, we give two different Cayley graphs of the same virtually free group such that the FUSF has infinitely many trees in one, but is connected in the other, answering a question of Lyons and Peres (*Probability on Trees and Networks* (2016) Cambridge Univ. Press) in the negative.

A version of our argument gives an example of a nonunimodular transitive graph where $\mathsf{WUSF} \neq \mathsf{FUSF}$, but some of the FUSF trees are light with respect to Haar measure. This disproves a conjecture of Tang (*Electron. J. Probab.* **26** (2021) Paper No. 141).

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CUTOFF FOR THE ASYMMETRIC RIFFLE SHUFFLE

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In the Gilbert–Shannon–Reeds shuffle, a deck of N cards is cut into two approximately equal parts which are riffled together uniformly at random. Bayer and Diaconis (*Ann. Appl. Probab.* **2** 294–313) famously showed that this Markov chain undergoes cutoff in total variation after $\frac{3\log(N)}{2\log(2)}$ shuffles. We establish cutoff for the more general *asymmetric* riffle shuffles in which one cuts the deck into differently sized parts. The value of the cutoff point confirms a conjecture of Lalley from 2000 (*Ann. Appl. Probab.* **10** 1302–1321). Some appealing consequences are that asymmetry always slows mixing and that total variation mixing is strictly faster than separation and L^∞ mixing.

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OPTIMAL REGULARITY IN TIME AND SPACE FOR STOCHASTIC POROUS MEDIUM EQUATIONS

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We prove optimal regularity estimates in Sobolev spaces in time and space for solutions to stochastic porous medium equations. The noise term considered here is multiplicative, white in time and coloured in space. The coefficients are assumed to be Hölder continuous, and the cases of smooth coefficients of, at most, linear growth as well as $\sqrt{u}$ are covered by our assumptions. The regularity obtained is consistent with the optimal regularity derived for the deterministic porous medium equation in (*J. Eur. Math. Soc.* **23** (2021) 425–465, *Anal. PDE* **13** (2020) 2441–2480) and the presence of the temporal white noise. The proof relies on a significant adaptation of velocity averaging techniques from their usual L^1 context to the natural L^2 setting of the stochastic case. We introduce a new mixed kinetic/mild representation of solutions to quasilinear SPDE and use L^2 based a priori bounds to treat the stochastic term.

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SLICING ℓ_p-BALLS RELOADED: STABILITY, PLANAR SECTIONS IN ℓ_1

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We show that the two-dimensional minimum-volume central section of the n-dimensional cross-polytope is attained by the regular $2n$-gon. We establish stability-type results for hyperplane sections of ℓ_p-balls in all the cases where the extremisers are known. Our methods are mainly probabilistic, exploring connections between negative moments of projections of random vectors uniformly distributed on convex bodies and volume of their sections.

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YAGLOM LIMIT FOR CRITICAL NONLOCAL BRANCHING MARKOV PROCESSES

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We consider the classical Yaglom limit theorem for a branching Markov process $X = (X_t, t \geq 0)$, with nonlocal branching mechanism in the setting that the mean semigroup is critical, that is, its leading eigenvalue is zero. In particular, we show that there exists a constant $c(f)$ such that

$$\mathrm{Law}\left(\frac{\langle f, X_t\rangle}{t} | \langle 1, X_t\rangle > 0\right) \to \mathbf{e}_{c(f)}, \quad t \to \infty,$$

where $\mathbf{e}_{c(f)}$ is an exponential random variable with rate $c(f)$ and the convergence is in distribution. As part of the proof, we also show that the probability of survival decays inversely proportionally to time. Although Yaglom limit theorems have recently been handled in the setting of branching Brownian motion in a bounded domain and superprocesses, (*Probab. Theory Related Fields* **173** (2019) 999–1062; *Electron. Commun. Probab.* **23** (2018) 42), these results do not allow for nonlocal branching which complicates the analysis. Our approach and the main novelty of this work is based around a precise result for the scaled asymptotics for the kth martingale moments of X (rather than the Yaglom limit itself). We then illustrate our results in the setting of neutron transport for which the nonlocality is essential, complementing recent developments in this domain (*Ann. Appl. Probab.* **30** (2020) 2573–2612; *Ann. Appl. Probab.* **30** (2020) 2815–2845; *SIAM J. Appl. Math.* **81** (2021) 982–1001; Cox et al. (2021); *J. Stat. Phys.* **176** (2019) 425–455).

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ASYMPTOTIC EXPANSIONS FOR A CLASS OF FREDHOLM PFAFFIANS AND INTERACTING PARTICLE SYSTEMS

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Motivated by the phenomenon of duality for interacting particle systems, we introduce two classes of Pfaffian kernels describing a number of Pfaffian point processes in the "bulk" and at the "edge." Using the probabilistic method due to Mark Kac, we prove two Szegő-type asymptotic expansion theorems for the corresponding Fredholm Pfaffians. The idea of the proof is to introduce an effective random walk with transition density determined by the Pfaffian kernel, express the logarithm of the Fredholm Pfaffian through expectations with respect to the random walk, and analyse the expectations using general results on random walks. We demonstrate the utility of the theorems by calculating asymptotics for the empty interval and noncrossing probabilities for a number of examples of Pfaffian point processes: coalescing/annihilating Brownian motions, massive coalescing Brownian motions, real zeros of Gaussian power series and Kac polynomials, and real eigenvalues for the real Ginibre ensemble.

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$\sqrt{\log t}$-SUPERDIFFUSIVITY FOR A BROWNIAN PARTICLE IN THE CURL OF THE 2D GFF

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The present work is devoted to the study of the large time behaviour of a critical Brownian diffusion in two dimensions, whose drift is divergence-free, ergodic and given by the curl of the 2-dimensional Gaussian free field. We prove the conjecture, made in (*J. Stat. Phys.* **147** (2012) 113–131), according to which the diffusion coefficient $D(t)$ diverges as $\sqrt{\log t}$ for $t \to \infty$. Starting from the fundamental work by Alder and Wainwright (*Phys. Rev. Lett.* **18** (1967) 988–990), logarithmically superdiffusive behaviour has been predicted to occur for a wide variety of out-of-equilibrium systems in the critical spatial dimension $d = 2$. Examples include the diffusion of a tracer particle in a fluid, self-repelling polymers and random walks, Brownian particles in divergence-free random environments and, more recently, the 2-dimensional critical Anisotropic KPZ equation. Even if in all of these cases it is expected that $D(t) \sim \sqrt{\log t}$, to the best of the authors' knowledge, this is the first instance in which such precise asymptotics is rigorously established.

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