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THE STABLE GRAPH: THE METRIC SPACE SCALING LIMIT OF A CRITICAL RANDOM GRAPH WITH I.I.D. POWER-LAW DEGREES

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We prove a metric space scaling limit for a critical random graph with independent and identically distributed degrees having power-law tail behaviour with exponent $\alpha + 1$, where $\alpha \in (1, 2)$. The limiting components are constructed from random $\mathbb{R}$-trees encoded by the excursions above its running infimum of a process whose law is locally absolutely continuous with respect to that of a spectrally positive α-stable Lévy process. These spanning $\mathbb{R}$-trees are measure-changed α-stable trees. In each such $\mathbb{R}$-tree, we make a random number of vertex identifications, whose locations are determined by an auxiliary Poisson process. This generalises results, which were already known in the case where the degree distribution has a finite third moment (a model which lies in the same universality class as the Erdős–Rényi random graph) and where the role of the α-stable Lévy process is played by a Brownian motion.

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ABSENCE OF BACKWARD INFINITE PATHS FOR FIRST-PASSAGE PERCOLATION IN ARBITRARY DIMENSION

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In first-passage percolation (FPP), one places nonnegative random variables (weights) (t_e) on the edges of a graph and studies the induced weighted graph metric. We consider FPP on $\mathbb{Z}^d$ for $d \geq 2$ and analyze the geometric properties of geodesics, which are optimizing paths for the metric. Specifically, we address the question of existence of bigeodesics, which are doubly-infinite paths whose subpaths are geodesics. It is a famous conjecture originating from a question of Furstenberg and most strongly supported for $d = 2$ that, for continuously distributed i.i.d. weights, there a.s. are no bigeodesics. We provide the first progress on this question in general dimensions under no unproven assumptions. Our main result is that geodesic graphs, introduced in a previous paper of two of the authors, constructed in any deterministic direction a.s. do not contain doubly-infinite paths. As a consequence, one can construct random graphs of subsequential limits of point-to-hyperplane geodesics, which contain no bigeodesics. This gives evidence that bigeodesics, if they exist, cannot be constructed in a translation-invariant manner as limits of point-to-hyperplane geodesics.

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GEOMETRIC AND O-MINIMAL LITTLEWOOD–OFFORD PROBLEMS

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The classical Erdős–Littlewood–Offord theorem says that for nonzero vectors $a_1, \dots, a_n \in \mathbb{R}^d$, any $x \in \mathbb{R}^d$, and uniformly random $(\xi_1, \dots, \xi_n) \in \{-1, 1\}^n$, we have $\Pr(a_1\xi_1 + \cdots + a_n\xi_n = x) = O(n^{-1/2})$. In this paper, we show that $\Pr(a_1\xi_1 + \cdots + a_n\xi_n \in S) \le n^{-1/2+o(1)}$ whenever S is definable with respect to an o-minimal structure (e.g., this holds when S is any algebraic hypersurface), under the necessary condition that it does not contain a line segment. We also obtain an inverse theorem in this setting.

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EXPANSION IN SUPERCRITICAL RANDOM SUBGRAPHS OF THE HYPERCUBE AND ITS CONSEQUENCES

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It is well known that the behaviour of a random subgraph of a d-dimensional hypercube, where we include each edge independently with probability p, undergoes a phase transition when p is around $\frac{1}{d}$. More precisely, standard arguments show that just below this value of p all components of this graph have order $O(d)$ with probability tending to one as $d \to \infty$ (whp for short), whereas Ajtai, Komlós and Szemerédi (*Combinatorica* **2** (1982) 1–7) showed that just above this value, in the *supercritical regime*, whp there is a unique "giant" component of order $\Theta(2^d)$. We show that whp the vertex expansion of the giant component is inverse polynomial in d. As a consequence, we obtain polynomial in d bounds on the diameter of the giant component and the mixing time of the lazy random walk on the giant component, answering questions of Bollobás, Kohayakawa and Łuczak (*Random Structures and Algorithms* **5** (1994) 627–648) and of Pete (*Electron. Commun. Probab.* **13** (2008) 377–392). Furthermore, our results imply lower bounds on the circumference and Hadwiger number of a random subgraph of the hypercube in this regime of p, which are tight up to polynomial factors in d.

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METASTABLE BEHAVIOR OF WEAKLY MIXING MARKOV CHAINS: THE CASE OF REVERSIBLE, CRITICAL ZERO-RANGE PROCESSES

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We present a general method to derive the metastable behavior of weakly mixing Markov chains. This approach is based on properties of the resolvent equations and can be applied to metastable dynamics, which do not satisfy the mixing conditions required in (*J. Stat. Phys.* **140** (2010) 1065–1114; *J. Stat. Phys.* **149** (2012) 598–618) or in Landim, Marcondes and Seo (2020).

As an application, we study the metastable behavior of critical zero-range processes. Let $r : S \times S \to \mathbb{R}_+$ be the jump rates of an irreducible random walk on a finite set S, reversible with respect to the uniform measure. For $\alpha > 0$, let $g : \mathbb{N} \to \mathbb{R}_+$ be given by $g(0) = 0$, $g(1) = 1$, $g(k) = [k/(k-1)]^\alpha$, $k \geq 2$. Consider a zero-range process on S in which a particle jumps from a site x, occupied by k particles, to a site y at rate $g(k)r(x,y)$. For $\alpha \geq 1$, in the stationary state, as the total number of particles, represented by N, tends to infinity, all particles but a negligible number accumulate at one single site. This phenomenon is called condensation. Since condensation occurs if and only if $\alpha \geq 1$, we call the case $\alpha = 1$ critical. By applying the general method established in the first part of the article to the critical case, we show that the site, which concentrates almost all particles, evolves in the time-scale $N^2 \log N$ as a random walk on S whose transition rates are proportional to the capacities of the underlying random walk.

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EXISTENCE OF AN UNBOUNDED NODAL HYPERSURFACE FOR SMOOTH GAUSSIAN FIELDS IN DIMENSION $d \geq 3$

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For the Bargmann–Fock field on $\mathbb{R}^d$ with $d \geq 3$, we prove that the critical level $\ell_c(d)$ of the percolation model formed by the excursion sets $\{f \geq \ell\}$ is strictly positive. This implies that for every ℓ sufficiently close to 0 (in particular for the nodal hypersurfaces corresponding to the case $\ell = 0$), $\{f = \ell\}$ contains an unbounded connected component that visits "most" of the ambient space. Our findings actually hold for a more general class of positively correlated smooth Gaussian fields with rapid decay of correlations. The results of this paper show that the behavior of nodal hypersurfaces of these Gaussian fields in $\mathbb{R}^d$ for $d \geq 3$ is very different from the behavior of nodal lines of their 2-dimensional analogues.

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POISSON STATISTICS AND LOCALIZATION AT THE SPECTRAL EDGE OF SPARSE ERDŐS–RÉNYI GRAPHS

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We consider the adjacency matrix A of the Erdős–Rényi graph on N vertices with edge probability d/N. For $(\log \log N)^4 \ll d \lesssim \log N$, we prove that the eigenvalues near the spectral edge form asymptotically a Poisson point process and the associated eigenvectors are exponentially localized. As a corollary, at the critical scale $d \asymp \log N$, the limiting distribution of the largest nontrivial eigenvalue does not match with any previously known distribution. Together with (*Comm. Math. Phys.* **388** (2021) 507–579), our result establishes the coexistence of a fully delocalized phase and a fully localized phase in the spectrum of A. The proof relies on a three-scale rigidity argument, which characterizes the fluctuations of the eigenvalues in terms of the fluctuations of sizes of spheres of radius 1 and 2 around vertices of large degree.

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FREE ENERGY OF A DILUTED SPIN GLASS MODEL WITH QUADRATIC HAMILTONIAN

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We study a diluted mean-field spin glass model with a quadratic Hamiltonian. Our main result establishes the limiting free energy in terms of an integral of a family of random variables that are the weak limits of the quenched variances of the spins in the system with varying edge connectivity. The key ingredient in our argument is played by the identification of these random variables as the unique solution to a recursive distributional equation. Our results in particular provide the first example of the diluted Shcherbina–Tirozzi model, whose limiting free energy can be derived at any inverse temperature and external field.

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