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ROUGH SEMIMARTINGALES AND *p*-VARIATION ESTIMATES FOR MARTINGALE TRANSFORMS

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We establish a new scale of p-variation estimates for martingale paraproducts, martingale transforms and Itô integrals, of relevance in rough paths theory, stochastic and harmonic analysis. As an application, we introduce rough semimartingales, a common generalization of classical semimartingales and (controlled) rough paths and their integration theory.

REFERENCES

[1] BAIN, A. and CRISAN, D. (2009). *Fundamentals of Stochastic Filtering*. *Stochastic Modelling and Applied Probability* **60**. Springer, New York. MR2454694 https://doi.org/10.1007/978-0-387-76896-0

[2] BICHTELER, K. (1981). Stochastic integration and L^p-theory of semimartingales. *Ann. Probab.* **9** 49–89. MR0606798

[3] BOURGAIN, J. (1989). Pointwise ergodic theorems for arithmetic sets. *Publ. Math. Inst. Hautes Études Sci.* **69** 5–45. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein. MR1019960

[4] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications*. *I*. *Probability Theory and Stochastic Modelling* **83**. Springer, Cham. Mean field FBSDEs, control, and games. MR3752669

[5] CHEVYREV, I. and FRIZ, P. K. (2019). Canonical RDEs and general semimartingales as rough paths. *Ann. Probab.* **47** 420–463. MR3909973 https://doi.org/10.1214/18-AOP1264

[6] CHEVYREV, I., FRIZ, P. K., KOREPANOV, A., MELBOURNE, I. and ZHANG, H. (2019). Multiscale systems, homogenization, and rough paths. In *Probability and Analysis in Interacting Physical Systems*. *Springer Proc. Math. Stat.* **283** 17–48. Springer, Cham. MR3968507 https://doi.org/10.1007/978-3-030-15338-0_2

[7] COGHI, M. and FLANDOLI, F. (2016). Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26** 1407–1442. MR3513594 https://doi.org/10.1214/15-AAP1120

[8] COQUET, F., JAKUBOWSKI, A., MÉMIN, J. and SŁOMIŃSKI, L. (2006). Natural decomposition of processes and weak Dirichlet processes. In *In Memoriam Paul-André Meyer*: *Séminaire de Probabilités XXXIX*. *Lecture Notes in Math.* **1874** 81–116. Springer, Berlin. MR2276891 https://doi.org/10.1007/978-3-540-35513-7_8

[9] COUTIN, L. and LEJAY, A. (2005). Semi-martingales and rough paths theory. *Electron. J. Probab.* **10** 761–785. MR2164030 https://doi.org/10.1214/EJP.v10-162

[10] COVIELLO, R. and RUSSO, F. (2007). Nonsemimartingales: Stochastic differential equations and weak Dirichlet processes. *Ann. Probab.* **35** 255–308. MR2303950 https://doi.org/10.1214/009117906000000566

[11] CRISAN, D., DIEHL, J., FRIZ, P. K. and OBERHAUSER, H. (2013). Robust filtering: Correlated noise and multidimensional observation. *Ann. Appl. Probab.* **23** 2139–2160. MR3134732 https://doi.org/10.1214/12-AAP896

[12] DAVIE, A. M. (2008). Differential equations driven by rough paths: An approach via discrete approximation. *Appl. Math. Res. Express. AMRX* **2008** 1–40.

[13] DAVIS, B. (1970). On the integrability of the martingale square function. *Israel J. Math.* **8** 187–190. MR0268966 https://doi.org/10.1007/BF02771313

[14] DAVIS, M. H. A. (2011). Pathwise nonlinear filtering with correlated noise. In *The Oxford Handbook of Nonlinear Filtering* 403–424. Oxford Univ. Press, Oxford. MR2884603

[15] DEUSCHEL, J.-D., ORENSHTEIN, T. and PERKOWSKI, N. (2021). Additive functionals as rough paths. *Ann. Probab.* **49** 1450–1479. MR4255150 https://doi.org/10.1214/20-aop1488

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[16] DIEHL, J. and FRIZ, P. (2012). Backward stochastic differential equations with rough drivers. *Ann. Probab.* **40** 1715–1758. MR2978136 https://doi.org/10.1214/11-AOP660

[17] DIEHL, J., FRIZ, P. K. and STANNAT, W. (2017). Stochastic partial differential equations: A rough paths view on weak solutions via Feynman–Kac. *Ann. Fac. Sci. Toulouse Math.* (6) **26** 911–947. MR3746646 https://doi.org/10.5802/afst.1556

[18] DIEHL, J., OBERHAUSER, H. and RIEDEL, S. (2015). A Lévy area between Brownian motion and rough paths with applications to robust nonlinear filtering and rough partial differential equations. *Stochastic Process. Appl.* **125** 161–181. MR3274695 https://doi.org/10.1016/j.spa.2014.08.005

[19] DO, Y., MUSCALU, C. and THIELE, C. (2012). Variational estimates for paraproducts. *Rev. Mat. Iberoam.* **28** 857–878. MR2949622 https://doi.org/10.4171/RMI/694

[20] DO, Y., MUSCALU, C. and THIELE, C. (2017). Variational estimates for the bilinear iterated Fourier integral. *J. Funct. Anal.* **272** 2176–2233. MR3596720 https://doi.org/10.1016/j.jfa.2016.09.010

[21] ERRAMI, M. and RUSSO, F. (2003). n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.* **104** 259–299. MR1961622 https://doi.org/10.1016/S0304-4149(02)00238-7

[22] FÖLLMER, H. (1981). Dirichlet processes. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980). *Lecture Notes in Math.* **851** 476–478. Springer, Berlin. MR0621001

[23] FRIZ, P. and VICTOIR, N. (2008). The Burkholder–Davis–Gundy inequality for enhanced martingales. In *Séminaire de Probabilités XLI. Lecture Notes in Math.* **1934** 421–438. Springer, Berlin. MR2483743 https://doi.org/10.1007/978-3-540-77913-1_20

[24] FRIZ, P. and VICTOIR, N. (2010). Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 369–413. MR2667703 https://doi.org/10.1214/09-AIHP202

[25] FRIZ, P. K. and HAIRER, M. *A Course on Rough Paths*, 2nd ed. *Universitext*. Springer, Berlin.

[26] FRIZ, P. K., HOCQUET, A. and LÊ, K. (2021). Rough stochastic differential equations. Preprint.

[27] FRIZ, P. K. and SHEKHAR, A. (2017). General rough integration, Lévy rough paths and a Lévy–Kintchine-type formula. *Ann. Probab.* **45** 2707–2765. MR3693973 https://doi.org/10.1214/16-AOP1123

[28] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths. Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. Theory and applications. MR2604669 https://doi.org/10.1017/CBO9780511845079

[29] FRIZ, P. K. and ZHANG, H. (2018). Differential equations driven by rough paths with jumps. *J. Differential Equations* **264** 6226–6301. MR3770049 https://doi.org/10.1016/j.jde.2018.01.031

[30] GAINES, J. G. and LYONS, T. J. (1997). Variable step size control in the numerical solution of stochastic differential equations. *SIAM J. Appl. Math.* **57** 1455–1484. MR1470933 https://doi.org/10.1137/S0036139995286515

[31] GUBINELLI, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. MR2091358 https://doi.org/10.1016/j.jfa.2004.01.002

[32] GUBINELLI, M. (2010). Ramification of rough paths. *J. Differential Equations* **248** 693–721. MR2578445 https://doi.org/10.1016/j.jde.2009.11.015

[33] GUERRA, J. and NUALART, D. (2008). Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion. *Stoch. Anal. Appl.* **26** 1053–1075. MR2440915 https://doi.org/10.1080/07362990802286483

[34] HAIRER, M. and KELLY, D. (2015). Geometric versus non-geometric rough paths. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 207–251. MR3300969 https://doi.org/10.1214/13-AIHP564

[35] HAIRER, M. and LI, X.-M. (2020). Averaging dynamics driven by fractional Brownian motion. *Ann. Probab.* **48** 1826–1860. MR4124526 https://doi.org/10.1214/19-AOP1408

[36] HYTÖNEN, T., VAN NEERVEN, J., VERAAR, M. and WEIS, L. *Martingales and Littlewood-Paley Theory* **I**. Springer, Berlin.

[37] JAIN, N. C. and MONRAD, D. (1983). Gaussian measures in B_p. *Ann. Probab.* **11** 46–57. MR0682800

[38] JONES, R. L., SEEGER, A. and WRIGHT, J. (2008). Strong variational and jump inequalities in harmonic analysis. *Trans. Amer. Math. Soc.* **360** 6711–6742. MR2434308 https://doi.org/10.1090/S0002-9947-08-04538-8

[39] KLOEDEN, P. E. and NEUENKIRCH, A. (2007). The pathwise convergence of approximation schemes for stochastic differential equations. *LMS J. Comput. Math.* **10** 235–253. MR2320830 https://doi.org/10.1112/S1461157000001388

[40] KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solution of Stochastic Differential Equations. Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374 https://doi.org/10.1007/978-3-662-12616-5

[41] KOVAČ, V. and ZORIN-KRANICH, P. (2019). Variational estimates for martingale paraproducts. *Electron. Commun. Probab.* **24** 48. MR4003122 https://doi.org/10.1214/19-ecp257

[42] LÊ, K. (2020). A stochastic sewing lemma and applications. *Electron. J. Probab.* **25** 38. MR4089788 https://doi.org/10.1214/20-ejp442

[43] LÉPINGLE, D. (1976). La variation d'ordre p des semi-martingales. *Z. Wahrsch. Verw. Gebiete* **36** 295–316. MR0420837 https://doi.org/10.1007/BF00532696

[44] LIONS, P.-L. and SOUGANIDIS, P. E. (1998). Fully nonlinear stochastic partial differential equations: Nonsmooth equations and applications. *C. R. Acad. Sci. Paris Sér. I Math.* **327** 735–741. MR1659958 https://doi.org/10.1016/S0764-4442(98)80161-4

[45] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527 https://doi.org/10.4171/RMI/240

[46] MANSTAVIČIUS, M. (2004). p-variation of strong Markov processes. *Ann. Probab.* **32** 2053–2066. MR2073185 https://doi.org/10.1214/009117904000000423

[47] MIREK, M., STEIN, E. M. and ZORIN-KRANICH, P. (2020). Jump inequalities via real interpolation. *Math. Ann.* **376** 797–819. MR4055178 https://doi.org/10.1007/s00208-019-01889-2

[48] MUSCALU, C. (2014). Calderón commutators and the Cauchy integral on Lipschitz curves revisited II. The Cauchy integral and its generalizations. *Rev. Mat. Iberoam.* **30** 1089–1122. MR3255002 https://doi.org/10.4171/RMI/808

[49] MUSCALU, C., TAO, T. and THIELE, C. (2002). Uniform estimates on paraproducts. *J. Anal. Math.* **87** 369–384. Dedicated to the memory of Thomas H. Wolff.

[50] OSĘKOWSKI, A. (2017). A Fefferman–Stein inequality for the martingale square and maximal functions. *Statist. Probab. Lett.* **129** 81–85. MR3688518 https://doi.org/10.1016/j.spl.2017.05.008

[51] PISIER, G. and XU, Q. H. (1988). The strong p-variation of martingales and orthogonal series. *Probab. Theory Related Fields* **77** 497–514. MR0933985 https://doi.org/10.1007/BF00959613

[52] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. Second edition. Version 2.1, Corrected third printing. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[53] TAYLOR, S. J. (1972). Exact asymptotic estimates of Brownian path variation. *Duke Math. J.* **39** 219–241. MR0295434

[54] WILLIAMS, D. R. E. (2001). Path-wise solutions of stochastic differential equations driven by Lévy processes. *Rev. Mat. Iberoam.* **17** 295–329. MR1891200 https://doi.org/10.4171/RMI/296

[55] YOUNG, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** 251–282. MR1555421 https://doi.org/10.1007/BF02401743

[56] ZORIN-KRANICH, P. (2020). Weighted Lépingle inequality. *Bernoulli* **26** 2311–2318. MR4091110 https://doi.org/10.3150/20-BEJ1194

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LOGARITHMIC HEAT KERNEL ESTIMATES WITHOUT CURVATURE RESTRICTIONS

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The main results of the article are short time estimates and asymptotic estimates for the first two order derivatives of the logarithmic heat kernel of a complete Riemannian manifold. We remove all curvature restrictions and also develop several techniques.

A basic tool developed here is intrinsic stochastic variations with prescribed second order covariant differentials, allowing to obtain a path integration representation for the second order derivatives of the heat semigroup P_t on a complete Riemannian manifold, again without any assumptions on the curvature. The novelty is the introduction of an ϵ^2 term in the variation allowing greater control. We also construct a family of cut-off stochastic processes adapted to an exhaustion by compact subsets with smooth boundaries, each process is constructed path by path and differentiable in time. Furthermore, the differentials have locally uniformly bounded moments with respect to the Brownian motion measures, allowing to bypass the lack of continuity of the exit time of the Brownian motions on its initial position.

REFERENCES

[1] AIDA, S. (2000). Logarithmic derivatives of heat kernels and logarithmic Sobolev inequalities with unbounded diffusion coefficients on loop spaces. *J. Funct. Anal.* **174** 430–477. MR1768982 https://doi.org/10.1006/jfan.2000.3592

[2] AIDA, S. (2004). Precise Gaussian estimates of heat kernels on asymptotically flat Riemannian manifolds with poles. In *Recent Developments in Stochastic Analysis and Related Topics* 1–19. World Sci. Publ., Hackensack, NJ. MR2200501

[3] AIDA, S. (2015). Asymptotics of spectral gaps on loop spaces over a class of Riemannian manifolds. *J. Funct. Anal.* **269** 3714–3764. MR3418070 https://doi.org/10.1016/j.jfa.2015.09.023

[4] AIRAULT, H. and MALLIAVIN, P. (1992). Integration on loop groups. II. Heat equation for the Wiener measure. *J. Funct. Anal.* **104** 71–109. MR1152460 https://doi.org/10.1016/0022-1236(92)90091-V

[5] AIRAULT, H., MALLIAVIN, P. and REN, J. (1999). Smoothness of stopping times of diffusion processes. *J. Math. Pures Appl.* (9) **78** 1069–1091. MR1732053 https://doi.org/10.1016/S0021-7824(99)00026-4

[6] ARNAUDON, M., PLANK, H. and THALMAIER, A. (2003). A Bismut type formula for the Hessian of heat semigroups. *C. R. Math. Acad. Sci. Paris* **336** 661–666. MR1988128 https://doi.org/10.1016/S1631-073X(03)00123-7

[7] AZENCOTT, R. (1980). Grandes déviations et applications. In *Eighth Saint Flour Probability Summer School*—1978 (*Saint Flour*, 1978). *Lecture Notes in Math.* **774** 1–176. Springer, Berlin. MR0590626

[8] AZENCOTT, R., ed. (1981). Géodésiques et diffusions en temps petit, sḿinaire de probabilitś universitde ́Paris VII, Asterisque 84–85, Ch. 5 by L. Elie and Ch. IX by C. Bellaiche. Math. Soc. France.

[9] BALL, K., BARTHE, F., BEDNORZ, W., OLESZKIEWICZ, K. and WOLFF, P. (2013). L^1-smoothing for the Ornstein–Uhlenbeck semigroup. *Mathematika* **59** 160–168. MR3028177 https://doi.org/10.1112/S0025579312001064

[10] BEN AROUS, G. (1988). Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. Éc. Norm. Supér.* (4) **21** 307–331. MR0974408

[11] BEN AROUS, G. and LÉANDRE, R. (1991). Décroissance exponentielle du noyau de la chaleur sur la diagonale. I. *Probab. Theory Related Fields* **90** 175–202. MR1128069 https://doi.org/10.1007/BF01192161

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[12] BISMUT, J.-M. (1984). *Large Deviations and the Malliavin Calculus*. *Progress in Mathematics* **45**. Birkhäuser, Inc., Boston, MA. MR0755001

[13] CHEEGER, J., GROMOV, M. and TAYLOR, M. (1982). Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.* **17** 15–53. MR0658471

[14] CHEEGER, J. and YAU, S. T. (1981). A lower bound for the heat kernel. *Comm. Pure Appl. Math.* **34** 465–480. MR0615626 https://doi.org/10.1002/cpa.3160340404

[15] CHEN, X., LI, X.-M. and WU, B. Stochastic analysis on loop space over general Riemannian manifold. Preprint.

[16] CHEN, X., LI, X.-M. and WU, B. (2010). A Poincaré inequality on loop spaces. *J. Funct. Anal.* **259** 1421–1442. MR2659766 https://doi.org/10.1016/j.jfa.2010.05.006

[17] CHEN, X., LI, X.-M. and WU, B. (2011). A concrete estimate for the weak Poincaré inequality on loop space. *Probab. Theory Related Fields* **151** 559–590. MR2851693 https://doi.org/10.1007/s00440-010-0308-5

[18] CHEN, X. and WU, B. (2014). Functional inequality on path space over a non-compact Riemannian manifold. *J. Funct. Anal.* **266** 6753–6779. MR3198853 https://doi.org/10.1016/j.jfa.2014.03.017

[19] CHENG, S. Y., LI, P. and YAU, S. T. (1981). On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.* **103** 1021–1063. MR0630777 https://doi.org/10.2307/2374257

[20] DAVIES, E. B. (1989). *Heat Kernels and Spectral Theory*. *Cambridge Tracts in Mathematics* **92**. Cambridge Univ. Press, Cambridge. MR0990239 https://doi.org/10.1017/CBO9780511566158

[21] DRIVER, B. K. (1992). A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. *J. Funct. Anal.* **110** 272–376. MR1194990 https://doi.org/10.1016/0022-1236(92)90035-H

[22] DRIVER, B. K. (1994). A Cameron–Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold. *Trans. Amer. Math. Soc.* **342** 375–395. MR1154540 https://doi.org/10.2307/2154699

[23] EBERLE, A. (2001). Local Poincaré inequalities on loop spaces. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 1023–1028. MR1872466 https://doi.org/10.1016/S0764-4442(01)02174-7

[24] EBERLE, A. (2002). Absence of spectral gaps on a class of loop spaces. *J. Math. Pures Appl.* (9) **81** 915–955. MR1946909 https://doi.org/10.1016/S0021-7824(02)01260-6

[25] EELLS, J. and ELWORTHY, K. D. (1971). On Fredholm manifolds. In *Actes du Congrès International des Mathématiciens* (*Nice*, 1970), *Tome* 2 215–219. Gauthier-Villars, Paris. MR0433487

[26] ELDAN, R. and LEE, J. R. (2015). Talagrand's convolution conjecture on Gaussian space. In 2015 *IEEE* 56*th Annual Symposium on Foundations of Computer Science—FOCS* 2015 1395–1408. IEEE Computer Soc., Los Alamitos, CA. MR3473377

[27] ELWORTHY, K. D. (1978). Stochastic dynamical systems and their flows. In *Stochastic Analysis* (*Proc. Internat. Conf., Northwestern Univ., Evanston, Ill.*, 1978) 79–95. Academic Press, New York. MR0517235

[28] ELWORTHY, K. D. (1982). *Stochastic Differential Equations on Manifolds*. *London Mathematical Society Lecture Note Series* **70**. Cambridge Univ. Press, Cambridge. MR0675100

[29] ELWORTHY, K. D., LE JAN, Y. and LI, X.-M. (1999). *On the Geometry of Diffusion Operators and Stochastic Flows*. *Lecture Notes in Math.* **1720**. Springer, Berlin. MR1735806 https://doi.org/10.1007/BFb0103064

[30] ELWORTHY, K. D. and LI, X.-M. (1994). Formulae for the derivatives of heat semigroups. *J. Funct. Anal.* **125** 252–286. MR1297021 https://doi.org/10.1006/jfan.1994.1124

[31] ELWORTHY, K. D. and LI, X.-M. (1994). Differentiation of heat semigroups and applications. In *Probability Theory and Mathematical Statistics* (*Vilnius*, 1993) 239–251. TEV, Vilnius. MR1649579

[32] ELWORTHY, K. D. and LI, X.-M. (1996). A class of integration by parts formulae in stochastic analysis. I. In *Itô's Stochastic Calculus and Probability Theory* 15–30. Springer, Tokyo. MR1439515

[33] ENGOULATOV, A. (2006). A universal bound on the gradient of logarithm of the heat kernel for manifolds with bounded Ricci curvature. *J. Funct. Anal.* **238** 518–529. MR2253731 https://doi.org/10.1016/j.jfa.2006.02.013

[34] FANG, S. and MALLIAVIN, P. (1993). Stochastic analysis on the path space of a Riemannian manifold. I. Markovian stochastic calculus. *J. Funct. Anal.* **118** 249–274. MR1245604 https://doi.org/10.1006/jfan.1993.1145

[35] GONG, F., RÖCKNER, M. and WU, L. (2001). Poincaré inequality for weighted first order Sobolev spaces on loop spaces. *J. Funct. Anal.* **185** 527–563. MR1856276 https://doi.org/10.1006/jfan.2001.3775

[36] GONG, F.-Z. and MA, Z.-M. (1998). The log-Sobolev inequality on loop space over a compact Riemannian manifold. *J. Funct. Anal.* **157** 599–623. MR1638269 https://doi.org/10.1006/jfan.1998.3263

[37] GOZLAN, N., LI, X.-M., MADIMAN, M., ROBERTO, C. and SAMSON, P. M. (2021). Log-Hessian and deviation bounds for Markov semi-groups, and regularization effect in $\mathbb{L}^1$. *Potential Anal.*. Available at arXiv:1907.10896. https://doi.org/10.1007/s11118-021-09934-z
[38] GREENE, R. E. and WU, H. (1979). *Function Theory on Manifolds Which Possess a Pole*. *Lecture Notes in Math*. **699**. Springer, Berlin. MR0521983
[39] GROSS, L. (1975). Logarithmic Sobolev inequalities. *Amer. J. Math*. **97** 1061–1083. MR0420249 https://doi.org/10.2307/2373688
[40] GROSS, L. (1991). Logarithmic Sobolev inequalities on loop groups. *J. Funct. Anal*. **102** 268–313. MR1140628 https://doi.org/10.1016/0022-1236(91)90123-M
[41] HAMILTON, R. S. (1993). A matrix Harnack estimate for the heat equation. *Comm. Anal. Geom*. **1** 113–126. MR1230276 https://doi.org/10.4310/CAG.1993.v1.n1.a6
[42] HSU, E. P. (1999). Estimates of derivatives of the heat kernel on a compact Riemannian manifold. *Proc. Amer. Math. Soc*. **127** 3739–3744. MR1618694 https://doi.org/10.1090/S0002-9939-99-04967-9
[43] HSU, E. P. (2002). *Stochastic Analysis on Manifolds*. *Graduate Studies in Mathematics* **38**. Amer. Math. Soc., Providence, RI. MR1882015 https://doi.org/10.1090/gsm/038
[44] HSU, E. P. and OUYANG, C. (2009). Quasi-invariance of the Wiener measure on the path space over a complete Riemannian manifold. *J. Funct. Anal*. **257** 1379–1395. MR2541273 https://doi.org/10.1016/j.jfa.2009.05.017
[45] HSU, P. (1990). Heat kernel on noncomplete manifolds. *Indiana Univ. Math. J*. **39** 431–442. MR1089046 https://doi.org/10.1512/iumj.1990.39.39023
[46] KOTSCHWAR, B. L. (2007). Hamilton's gradient estimate for the heat kernel on complete manifolds. *Proc. Amer. Math. Soc*. **135** 3013–3019. MR2317980 https://doi.org/10.1090/S0002-9939-07-08837-5
[47] LEHEC, J. (2016). Regularization in L_1 for the Ornstein–Uhlenbeck semigroup. *Ann. Fac. Sci. Toulouse Math*. (6) **25** 191–204. MR3485296 https://doi.org/10.5802/afst.1492
[48] LI, P. and YAU, S.-T. (1986). On the parabolic kernel of the Schrödinger operator. *Acta Math*. **156** 153–201. MR0834612 https://doi.org/10.1007/BF02399203
[49] LI, X.-D. (2016). Hamilton's Harnack inequality and the W-entropy formula on complete Riemannian manifolds. *Stochastic Process. Appl*. **126** 1264–1283. MR3461198 https://doi.org/10.1016/j.spa.2015.11.002
[50] LI, X.-M. (1992). Stochastic differential equations on noncompact manifolds. Thesis, Univ. Warwick.
[51] LI, X.-M. (1994). Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Related Fields* **100** 485–511. MR1305784 https://doi.org/10.1007/BF01268991
[52] LI, X.-M. (2018). Generalised Brownian bridges: Examples. *Markov Process. Related Fields* **24** 151–163. MR3793947
[53] LI, X.-M. (2018). Doubly damped stochastic parallel translations and Hessian formulas. In *Stochastic Partial Differential Equations and Related Fields*. *Springer Proc. Math. Stat*. **229** 345–357. Springer, Cham. MR3828180 https://doi.org/10.1007/978-3-319-74929-7_2
[54] LI, X.-M. (2021). Hessian formulas and estimates for parabolic Schrödinger operators. *J. Stoch. Anal*. **2** Art. 7, 53 pp. MR4304478
[55] LI, X.-M. and SCHEUTZOW, M. (2011). Lack of strong completeness for stochastic flows. *Ann. Probab*. **39** 1407–1421. MR2857244 https://doi.org/10.1214/10-AOP585
[56] LI, X.-M. and THOMPSON, J. (2018). First order Feynman–Kac formula. *Stochastic Process. Appl*. **128** 3006–3029. MR3834851 https://doi.org/10.1016/j.spa.2017.10.010
[57] MALLIAVIN, P. (1978). *Géométrie Différentielle Stochastique*. *Séminaire de Mathématiques Supérieures* [*Seminar on Higher Mathematics*] **64**. Presses de l'Université de Montréal, Montreal, Que. Notes prepared by Danièle Dehen and Dominique Michel. MR0540035
[58] MALLIAVIN, P. (1999). The canonic diffusion above the diffeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math*. **329** 325–329. MR1713340 https://doi.org/10.1016/S0764-4442(00)88575-4
[59] MALLIAVIN, P. and STROOCK, D. W. (1996). Short time behavior of the heat kernel and its logarithmic derivatives. *J. Differential Geom*. **44** 550–570. MR1431005
[60] MOLČANOV, S. A. (1975). Diffusion processes, and Riemannian geometry. *Uspekhi Mat. Nauk* **30** 3–59. MR0413289
[61] NEEL, R. (2007). The small-time asymptotics of the heat kernel at the cut locus. *Comm. Anal. Geom*. **15** 845–890. MR2395259
[62] NORRIS, J. R. (1993). Path integral formulae for heat kernels and their derivatives. *Probab. Theory Related Fields* **94** 525–541. MR1201558 https://doi.org/10.1007/BF01192562
[63] SETTI, A. G. (1993). Eigenvalue estimates for the Laplacian with lower order terms on a compact Riemannian manifold. In *Differential Geometry*: *Riemannian Geometry* (*Los Angeles*, *CA*, 1990). *Proc. Sympos. Pure Math*. **54** 521–527. Amer. Math. Soc., Providence, RI. MR1216643

[64] SHEU, S. J. (1991). Some estimates of the transition density of a nondegenerate diffusion Markov process. *Ann. Probab.* **19** 538–561. MR1106275

[65] SOUPLET, P. and ZHANG, Q. S. (2006). Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds. *Bull. Lond. Math. Soc.* **38** 1045–1053. MR2285258 https://doi.org/10.1112/S0024609306018947

[66] STROOCK, D. W. (1996). An estimate on the Hessian of the heat kernel. In *Itô's Stochastic Calculus and Probability Theory* 355–371. Springer, Tokyo. MR1439536

[67] THALMAIER, A. (1997). On the differentiation of heat semigroups and Poisson integrals. *Stoch. Stoch. Rep.* **61** 297–321. MR1488139 https://doi.org/10.1080/17442509708834123

[68] THALMAIER, A. and WANG, F.-Y. (1998). Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *J. Funct. Anal.* **155** 109–124. MR1622800 https://doi.org/10.1006/jfan.1997.3220

[69] THOMPSON, J. (2019). Derivatives of Feynman–Kac semigroups. *J. Theoret. Probab.* **32** 950–973. MR3959634 https://doi.org/10.1007/s10959-018-0824-2

[70] VARADHAN, S. R. S. (1966). Asymptotic probabilities and differential equations. *Comm. Pure Appl. Math.* **19** 261–286. MR0203230 https://doi.org/10.1002/cpa.3160190303

[71] VARADHAN, S. R. S. (1967). Diffusion processes in a small time interval. *Comm. Pure Appl. Math.* **20** 659–685. MR0217881 https://doi.org/10.1002/cpa.3160200404

[72] VAROPOULOS, N. T. (1989). Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique. *Bull. Sci. Math.* **113** 253–277. MR1016211

[73] VAROPOULOS, N. T. (1990). Small time Gaussian estimates of heat diffusion kernels. II. The theory of large deviations. *J. Funct. Anal.* **93** 1–33. MR1070036 https://doi.org/10.1016/0022-1236(90)90136-9

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UNIVERSALITY OF CUTOFF FOR EXCLUSION WITH RESERVOIRS

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We consider the exclusion process with reservoirs on arbitrary networks. We characterize the spectral gap, mixing time, and mixing window of the process, in terms of certain simple spectral statistics of the underlying network. Among other consequences we establish a nonconservative analogue of Aldous's spectral gap conjecture, and we show that cutoff occurs if and only if the product condition is satisfied. We illustrate this by providing explicit cutoffs on discrete lattices of arbitrary dimensions and boundary conditions which substantially generalize recent one-dimensional results. We also obtain cutoff phenomena in relative entropy, Hilbert norm, separation distance, and supremum norm. Our proof exploits negative dependence in a novel, simple way to reduce the understanding of the whole process to that of single-site marginals. We believe that this approach will find other applications.

REFERENCES

[1] ALDOUS, D. and FILL, J. A. (2002). Reversible Markov Chains and Random Walks on Graphs. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] ALON, G. and KOZMA, G. (2020). Comparing with octopi. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2672–2685. MR4164852 https://doi.org/10.1214/20-AIHP1054

[3] BASU, R., HERMON, J. and PERES, Y. (2017). Characterization of cutoff for reversible Markov chains. *Ann. Probab.* **45** 1448–1487. MR3650406 https://doi.org/10.1214/16-AOP1090

[4] BERTINI, L., DE SOLE, A., GABRIELLI, D., JONA-LASINIO, G. and LANDIM, C. (2003). Large deviations for the boundary driven symmetric simple exclusion process. *Math. Phys. Anal. Geom.* **6** 231–267. MR1997915 https://doi.org/10.1023/A:1024967818899

[5] BORCEA, J., BRÄNDÉN, P. and LIGGETT, T. M. (2009). Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.* **22** 521–567. MR2476782 https://doi.org/10.1090/S0894-0347-08-00618-8

[6] BRISTIEL, A. and CAPUTO, P. (2021). Entropy inequalities for random walks and permutations.

[7] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. MR2429910

[8] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2020). Mixing time of the adjacent walk on the simplex. *Ann. Probab.* **48** 2449–2493. MR4152648 https://doi.org/10.1214/20-AOP1428

[9] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2022). Spectral gap and cutoff phenomenon for the Gibbs sampler of $\nabla\varphi$ interfaces with convex potential. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 794–826. MR4421608 https://doi.org/10.1214/21-aihp1174

[10] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4

[11] CHEN, G.-Y. and SALOFF-COSTE, L. (2008). The cutoff phenomenon for ergodic Markov processes. *Electron. J. Probab.* **13** 26–78. MR2375599 https://doi.org/10.1214/EJP.v13-474

[12] DIACONIS, P. (1996). The cutoff phenomenon in finite Markov chains. *Proc. Natl. Acad. Sci. USA* **93** 1659–1664. MR1374011 https://doi.org/10.1073/pnas.93.4.1659

[13] DING, J., LUBETZKY, E. and PERES, Y. (2010). Total variation cutoff in birth-and-death chains. *Probab. Theory Related Fields* **146** 61–85. MR2550359 https://doi.org/10.1007/s00440-008-0185-3

[14] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[15] GANTERT, N., NESTORIDI, E. and SCHMID, D. (2021). Mixing times for the simple exclusion process with open boundaries.

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[16] GONÇALVES, P., JARA, M., MARINHO, R. and MENEZES, O. (2021). Sharp Convergence to Equilibrium for the SSEP with Reservoirs.

[17] HERMON, J. and PYMAR, R. (2020). The exclusion process mixes (almost) faster than independent particles. *Ann. Probab.* **48** 3077–3123. MR4164461 https://doi.org/10.1214/20-AOP1455

[18] HERMON, J. and SALEZ, J. (2019). A version of Aldous' spectral-gap conjecture for the zero range process. *Ann. Appl. Probab.* **29** 2217–2229. MR3984254 https://doi.org/10.1214/18-AAP1449

[19] LACOIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 https://doi.org/10.1214/15-AOP1004

[20] LACOIN, H. (2016). The cutoff profile for the simple exclusion process on the circle. *Ann. Probab.* **44** 3399–3430. MR3551201 https://doi.org/10.1214/15-AOP1053

[21] LACOIN, H. (2017). The simple exclusion process on the circle has a diffusive cutoff window. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1402–1437. MR3689972 https://doi.org/10.1214/16-AIHP759

[22] LANDIM, C., MILANÉS, A. and OLLA, S. (2008). Stationary and nonequilibrium fluctuations in boundary driven exclusion processes. *Markov Process. Related Fields* **14** 165–184. MR2437527

[23] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904 https://doi.org/10.1090/mbk/107

[24] LIGGETT, T. M. (2005). *Interacting Particle Systems*. *Classics in Mathematics*. Springer, Berlin. MR2108619 https://doi.org/10.1007/b138374

[25] LUBETZKY, E. and SLY, A. (2013). Cutoff for the Ising model on the lattice. *Invent. Math.* **191** 719–755. MR3020173 https://doi.org/10.1007/s00222-012-0404-5

[26] LUBETZKY, E. and SLY, A. (2014). Cutoff for general spin systems with arbitrary boundary conditions. *Comm. Pure Appl. Math.* **67** 982–1027. MR3193965 https://doi.org/10.1002/cpa.21489

[27] LUBETZKY, E. and SLY, A. (2015). An exposition to information percolation for the Ising model. *Ann. Fac. Sci. Toulouse Math.* (6) **24** 745–761. MR3434254 https://doi.org/10.5802/afst.1462

[28] LUBETZKY, E. and SLY, A. (2016). Information percolation and cutoff for the stochastic Ising model. *J. Amer. Math. Soc.* **29** 729–774. MR3486171 https://doi.org/10.1090/jams/841

[29] MONTENEGRO, R. and TETALI, P. (2006). Mathematical aspects of mixing times in Markov chains. *Found. Trends Theor. Comput. Sci.* **1** x+121. MR2341319 https://doi.org/10.1561/0400000003

[30] MORRIS, B. (2006). The mixing time for simple exclusion. *Ann. Appl. Probab.* **16** 615–635. MR2244427 https://doi.org/10.1214/105051605000000728

[31] OLIVEIRA, R. I. (2013). Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk. *Ann. Probab.* **41** 871–913. MR3077529 https://doi.org/10.1214/11-AOP714

[32] PERES, Y. American Institute of Mathematics (AIM) research workshop "Sharp Thresholds for Mixing Times" (Palo Alto, December 2004). Summary available at http://www.aimath.org/WWN/mixingtimes.

[33] QUATTROPANI, M. and SAU, F. (2021). Mixing of the Averaging process and its discrete dual on finite-dimensional geometries.

[34] SALEZ, J. (2021). Cutoff for non-negatively curved Markov chains.

[35] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290 (1970). MR0268959 https://doi.org/10.1016/0001-8708(70)90034-4

[36] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. MR2023023 https://doi.org/10.1214/aoap/1075828054

[37] YAU, H.-T. (1991). Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys.* **22** 63–80. MR1121850 https://doi.org/10.1007/BF00400379

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BERRY–ESSEEN TYPE BOUNDS FOR THE LEFT RANDOM WALK ON $\mathrm{GL}_d(\mathbb{R})$ UNDER POLYNOMIAL MOMENT CONDITIONS

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Let $A_n = \varepsilon_n \cdots \varepsilon_1$, where $(\varepsilon_n)_{n\geq 1}$ is a sequence of independent random matrices, taking values in $\mathrm{GL}_d(\mathbb{R})$, $d \geq 2$, with common distribution μ. In this paper, under standard assumptions on μ (strong irreducibility and proximality) we prove Berry–Esseen type theorems for $\log(\|A_n\|)$ when μ has a polynomial moment. More precisely, we get the rate $((\log n)/n)^{q/2-1}$, when μ has a moment of order $q \in]2, 3]$ and the rate $1/\sqrt{n}$ when μ has a moment of order 4, which significantly improves earlier results in this setting.

REFERENCES

[1] BENOIST, Y. and QUINT, J.-F. (2016). Central limit theorem for linear groups. *Ann. Probab.* **44** 1308–1340. MR3474473 https://doi.org/10.1214/15-AOP1002

[2] BOLTHAUSEN, E. (1982). Exact convergence rates in some martingale central limit theorems. *Ann. Probab.* **10** 672–688. MR0659537

[3] BOUGEROL, P. and LACROIX, J. (1985). *Products of Random Matrices with Applications to Schrödinger Operators. Progress in Probability and Statistics* **8**. Birkhäuser, Inc., Boston, MA. MR0886674 https://doi.org/10.1007/978-1-4684-9172-2

[4] BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19–42. MR0365692 https://doi.org/10.1214/aop/1176997023

[5] CUNY, C., DEDECKER, J. and JAN, C. (2017). Limit theorems for the left random walk on $\mathrm{GL}_d(\mathbb{R})$. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1839–1865. MR3729637 https://doi.org/10.1214/16-AIHP773

[6] CUNY, C., DEDECKER, J. and MERLEVÈDE, F. (2018). On the Komlós, Major and Tusnády strong approximation for some classes of random iterates. *Stochastic Process. Appl.* **128** 1347–1385. MR3769665 https://doi.org/10.1016/j.spa.2017.07.011

[7] CUNY, C., DEDECKER, J., MERLEVÈDE, F. and PELIGRAD, M. (2022). Berry–Esseen type bounds for the matrix coefficients and the spectral radius of the left random walk on $\mathrm{GL}_d(\mathbb{R})$. *C. R. Math. Acad. Sci. Paris* **360** 475–482. MR4449853 https://doi.org/10.5802/crmath.312

[8] DEDECKER, J. and DOUKHAN, P. (2003). A new covariance inequality and applications. *Stochastic Process. Appl.* **106** 63–80. MR1983043 https://doi.org/10.1016/S0304-4149(03)00040-1

[9] DEDECKER, J., MERLEVÈDE, F. and RIO, E. (2009). Rates of convergence for minimal distances in the central limit theorem under projective criteria. *Electron. J. Probab.* **14** 978–1011. MR2506123 https://doi.org/10.1214/EJP.v14-648

[10] DINH, T.-C., KAUFMANN, L. and WU, H. Random walks on $\mathrm{SL}_2(\mathbb{C})$: Spectral gap and local limit theorems. Available at https://arxiv.org/pdf/2106.04019.pdf.

[11] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. MR0270403

[12] FERNANDO, K. and PÈNE, F. (2022). Expansions in the local and the central limit theorems for dynamical systems. *Comm. Math. Phys.* **389** 273–347. MR4365142 https://doi.org/10.1007/s00220-021-04255-z

[13] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Stat.* **31** 457–469. MR0121828 https://doi.org/10.1214/aoms/1177705909

[14] GUIVARC'H, Y. and RAUGI, A. (1985). Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence. *Z. Wahrsch. Verw. Gebiete* **69** 187–242. MR0779457 https://doi.org/10.1007/BF02450281

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[15] HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30. MR0144363

[16] JAN, C. (2001). Vitesse de convergence dans le TCL pour des processus associés à des systèmes dynamiques ou des produits de matrices aléatoires, Thèse de l'Université de Rennes 1 (2001), thesis number 01REN10073.

[17] JIRAK, M. (2016). Berry–Esseen theorems under weak dependence. *Ann. Probab.* **44** 2024–2063. MR3502600 https://doi.org/10.1214/15-AOP1017

[18] JIRAK, M. (2020). A Berry–Esseen bound with (almost) sharp dependence conditions. Available at arXiv:1606.01617.

[19] LE PAGE, É. (1982). Théorèmes limites pour les produits de matrices aléatoires. In *Probability Measures on Groups* (*Oberwolfach*, 1981). *Lecture Notes in Math.* **928** 258–303. Springer, Berlin. MR0669072

[20] MERLEVÈDE, F., PELIGRAD, M. and UTEV, S. (2019). *Functional Gaussian Approximation for Dependent Structures*. *Oxford Studies in Probability* **6**. Oxford Univ. Press, Oxford. MR3930596 https://doi.org/10.1093/oso/9780198826941.001.0001

[21] XIAO, H., GRAMA, I. and LIU, Q. (2021). Berry–Esseen bounds and moderate deviations for random walks on $\mathrm{GL}_d(\mathbb{R})$. *Stochastic Process. Appl.* **142** 293–318. MR4318368 https://doi.org/10.1016/j.spa.2021.08.005

[22] XIAO, H., GRAMA, I. and LIU, Q. (2022). Berry–Esseen bound and precise moderate deviations for products of random matrices. *J. Eur. Math. Soc.* (*JEMS*) **24** 2691–2750. MR4416588 https://doi.org/10.4171/jems/1142

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GLOBAL-IN-TIME PROBABILISTICALLY STRONG AND MARKOV SOLUTIONS TO STOCHASTIC 3D NAVIER–STOKES EQUATIONS: EXISTENCE AND NONUNIQUENESS

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We are concerned with the three-dimensional incompressible Navier–Stokes equations driven by an additive stochastic forcing of trace class. First, for every divergence free initial condition in L^2 we establish existence of infinitely many global-in-time probabilistically strong and analytically weak solutions, solving one of the open problems in the field. This result, in particular, implies nonuniqueness in law. Second, we prove nonuniqueness of the associated Markov processes in a suitably chosen class of analytically weak solutions satisfying a relaxed form of an energy inequality. Translated to the deterministic setting, we obtain nonuniqueness of the associated semiflows.

REFERENCES

[1] BASARIĆ, D. (2021). Semiflow selection for the compressible Navier–Stokes system. *J. Evol. Equ.* **21** 277–295. MR4238206 https://doi.org/10.1007/s00028-020-00578-x
[2] BREIT, D., FEIREISL, E. and HOFMANOVÁ, M. (2020). Dissipative solutions and semiflow selection for the complete Euler system. *Comm. Math. Phys.* **376** 1471–1497. MR4103973 https://doi.org/10.1007/s00220-019-03662-7
[3] BREIT, D., FEIREISL, E. and HOFMANOVÁ, M. (2020). Markov selection for the stochastic compressible Navier–Stokes system. *Ann. Appl. Probab.* **30** 2547–2572. MR4187121 https://doi.org/10.1214/20-AAP1566
[4] BREIT, D., FEIREISL, E. and HOFMANOVÁ, M. (2020). On solvability and ill-posedness of the compressible Euler system subject to stochastic forces. *Anal. PDE* **13** 371–402. MR4078230 https://doi.org/10.2140/apde.2020.13.371
[5] BREIT, D., FEIREISL, E. and HOFMANOVÁ, M. (2020). Solution semiflow to the isentropic Euler system. *Arch. Ration. Mech. Anal.* **235** 167–194. MR4062476 https://doi.org/10.1007/s00205-019-01420-6
[6] BUCKMASTER, T., COLOMBO, M. and VICOL, V. (2022). Wild solutions of the Navier–Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. *J. Eur. Math. Soc.* (*JEMS*) **24** 3333–3378. MR4422213 https://doi.org/10.4171/jems/1162
[7] BUCKMASTER, T., DE LELLIS, C., ISETT, P. and SZÉKELYHIDI, L. JR. (2015). Anomalous dissipation for 1/5-Hölder Euler flows. *Ann. of Math.* (2) **182** 127–172. MR3374958 https://doi.org/10.4007/annals.2015.182.1.3
[8] BUCKMASTER, T., DE LELLIS, C., SZÉKELYHIDI, L. JR. and VICOL, V. (2019). Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.* **72** 229–274. MR3896021 https://doi.org/10.1002/cpa.21781
[9] BUCKMASTER, T. and VICOL, V. (2019). Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.* **6** 173–263. MR4073888 https://doi.org/10.4171/emss/34
[10] BUCKMASTER, T. and VICOL, V. (2019). Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. of Math.* (2) **189** 101–144. MR3898708 https://doi.org/10.4007/annals.2019.189.1.3
[11] BUCKMASTER, T. and VICOL, V. (2021). Convex integration constructions in hydrodynamics. *Bull. Amer. Math. Soc.* (*N.S.*) **58** 1–44. MR4188806 https://doi.org/10.1090/bull/1713
[12] BURCZAK, J., MODENA, S. and SZÉKELYHIDI, L. (2021). Non uniqueness of power-law flows. *Comm. Math. Phys.* **388** 199–243. MR4328053 https://doi.org/10.1007/s00220-021-04231-7

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[13] CARDONA, J. E. and KAPITANSKI, L. (2020). Semiflow selection and Markov selection theorems. *Topol. Methods Nonlinear Anal.* **56** 197–227. MR4175077 https://doi.org/10.12775/tmna.2020.010

[14] CHERNYĬ, A. S. (2001). On strong and weak uniqueness for stochastic differential equations. *Teor. Veroyatn. Primen.* **46** 483–497. MR1978664 https://doi.org/10.1137/S0040585X97979093

[15] CHESKIDOV, A. and LUO, X. (2022). Sharp nonuniqueness for the Navier–Stokes equations. *Invent. Math.* **229** 987–1054. MR4462623 https://doi.org/10.1007/s00222-022-01116-x

[16] CHIODAROLI, E., FEIREISL, E. and FLANDOLI, F. (2021). Ill-posedness for the full Euler system driven by multiplicative white noise. *Indiana Univ. Math. J.* **70** 1267–1282. MR4318474 https://doi.org/10.1512/iumj.2021.70.8591

[17] DA PRATO, G. and DEBUSSCHE, A. (2003). Ergodicity for the 3D stochastic Navier–Stokes equations. *J. Math. Pures Appl.* (9) **82** 877–947. MR2005200 https://doi.org/10.1016/S0021-7824(03)00025-4

[18] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[19] DANERI, S. and SZÉKELYHIDI, L. JR. (2017). Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* **224** 471–514. MR3614753 https://doi.org/10.1007/s00205-017-1081-8

[20] DE LELLIS, C. and SZÉKELYHIDI, L. JR. (2009). The Euler equations as a differential inclusion. *Ann. of Math.* (2) **170** 1417–1436. MR2600877 https://doi.org/10.4007/annals.2009.170.1417

[21] DE LELLIS, C. and SZÉKELYHIDI, L. JR. (2010). On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* **195** 225–260. MR2564474 https://doi.org/10.1007/s00205-008-0201-x

[22] DE LELLIS, C. and SZÉKELYHIDI, L. JR. (2013). Dissipative continuous Euler flows. *Invent. Math.* **193** 377–407. MR3090182 https://doi.org/10.1007/s00222-012-0429-9

[23] DEBUSSCHE, A. and VOVELLE, J. (2015). Invariant measure of scalar first-order conservation laws with stochastic forcing. *Probab. Theory Related Fields* **163** 575–611. MR3418750 https://doi.org/10.1007/s00440-014-0599-z

[24] FLANDOLI, F. (2008). An introduction to 3D stochastic fluid dynamics. In *SPDE in Hydrodynamic*: *Recent Progress and Prospects*. *Lecture Notes in Math.* **1942** 51–150. Springer, Berlin. MR2459085 https://doi.org/10.1007/978-3-540-78493-7_2

[25] FLANDOLI, F., HOFMANOVÁ, M., LUO, D. and NILSSEN, T. (2022). Global well-posedness of the 3D Navier–Stokes equations perturbed by a deterministic vector field. *Ann. Appl. Probab.* **32** 2568–2586. MR4474514 https://doi.org/10.1214/21-aap1740

[26] FLANDOLI, F. and LUO, D. (2021). High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. *Probab. Theory Related Fields* **180** 309–363. MR4265023 https://doi.org/10.1007/s00440-021-01037-5

[27] FLANDOLI, F. and ROMITO, M. (2008). Markov selections for the 3D stochastic Navier–Stokes equations. *Probab. Theory Related Fields* **140** 407–458. MR2365480 https://doi.org/10.1007/s00440-007-0069-y

[28] GOLDYS, B., RÖCKNER, M. and ZHANG, X. (2009). Martingale solutions and Markov selections for stochastic partial differential equations. *Stochastic Process. Appl.* **119** 1725–1764. MR2513126 https://doi.org/10.1016/j.spa.2008.08.009

[29] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6, 75. MR3406823 https://doi.org/10.1017/fmp.2015.2

[30] HOFMANOVÁ, M., ZHU, R. and ZHU, X. (2019). Nonuniqueness in law of stochastic 3D Navier–Stokes equations. arXiv:1912.11841v2.

[31] HOFMANOVÁ, M., ZHU, R. and ZHU, X. (2021). Global existence and nonuniqueness for 3D Navier–Stokes equations with space–time white noise. arXiv:2112.14093.

[32] HOFMANOVÁ, M., ZHU, R. and ZHU, X. (2022). On ill- and well-posedness of dissipative martingale solutions to stochastic 3D Euler equations. *Comm. Pure Appl. Math.* **75** 2446–2510. MR4491876 https://doi.org/10.1002/cpa.22023

[33] ISETT, P. (2018). A proof of Onsager's conjecture. *Ann. of Math.* (2) **188** 871–963. MR3866888 https://doi.org/10.4007/annals.2018.188.3.4

[34] KRYLOV, N. V. (1973). The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. *Izv. Ross. Akad. Nauk Ser. Mat.* **37** 691–708. MR0339338

[35] KURTZ, T. G. (2007). The Yamada–Watanabe–Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.* **12** 951–965. MR2336594 https://doi.org/10.1214/EJP.v12-431

[36] REHMEIER, M. and SCHENKE, A. (2021). Nonuniqueness in law for stochastic hypodissipative Navier–Stokes equations. arXiv:2104.10798.

[37] RÖCKNER, M., ZHU, R. and ZHU, X. (2014). Local existence and non-explosion of solutions for stochastic fractional partial differential equations driven by multiplicative noise. *Stochastic Process. Appl.* **124** 1974–2002. MR3170231 https://doi.org/10.1016/j.spa.2014.01.010

[38] STROOCK, D. W. and VARADHAN, S. R. S. (2006). *Multidimensional Diffusion Processes. Classics in Mathematics*. Springer, Berlin. Reprint of the 1997 edition. MR2190038

[39] YAMAZAKI, K. (2021). Nonuniqueness in law of three-dimensional magnetohydrodynamics system forced by random noise. arXiv:2109.07015.

[40] YAMAZAKI, K. (2021). Nonuniqueness in law of three-dimensional Navier–Stokes equations diffused via a fractional Laplacian with power less than one half. arXiv:2104.10294.

[41] YAMAZAKI, K. (2022). Nonuniqueness in law for two-dimensional Navier–Stokes equations with diffusion weaker than a full Laplacian. *SIAM J. Math. Anal.* **54** 3997–4042. MR4448828 https://doi.org/10.1137/21M1451087

[42] YAMAZAKI, K. (2022). Remarks on the non-uniqueness in law of the Navier–Stokes equations up to the J.-L. Lions' exponent. *Stochastic Process. Appl.* **147** 226–269. MR4379382 https://doi.org/10.1016/j.spa.2022.01.016

[43] YAMAZAKI, K. (2022). Non-uniqueness in law for the Boussinesq system forced by random noise. *Calc. Var. Partial Differential Equations* **61** Paper No. 177, 65. MR4450128 https://doi.org/10.1007/s00526-022-02285-6

[44] ZHU, R. and ZHU, X. (2015). Three-dimensional Navier–Stokes equations driven by space–time white noise. *J. Differential Equations* **259** 4443–4508. MR3373412 https://doi.org/10.1016/j.jde.2015.06.002

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HIGH-DIMENSIONAL NEAR-CRITICAL PERCOLATION AND THE TORUS PLATEAU

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We consider percolation on $\mathbb{Z}^d$ and on the d-dimensional discrete torus, in dimensions $d \geq 11$ for the nearest-neighbour model and in dimensions $d > 6$ for spread-out models. For $\mathbb{Z}^d$ we employ a wide range of techniques and previous results to prove that there exist positive constants c and C such that the slightly subcritical two-point function and one-arm probabilities satisfy

$$\mathbb{P}_{p_c-\varepsilon}(0 \leftrightarrow x) \leq \frac{C}{\|x\|^{d-2}} e^{-c\varepsilon^{1/2}\|x\|},$$

$$\frac{c}{r^2} e^{-C\varepsilon^{1/2} r} \leq \mathbb{P}_{p_c-\varepsilon}\big(0 \leftrightarrow \partial[-r,r]^d\big) \leq \frac{C}{r^2} e^{-c\varepsilon^{1/2} r}.$$

Using this, we prove that throughout the critical window the torus two-point function has a "plateau," meaning that it decays for small x as $\|x\|^{-(d-2)}$ but for large x is essentially constant and of order $V^{-2/3}$ where V is the volume of the torus. The plateau for the two-point function leads immediately to a proof of the torus triangle condition, which is known to have many implications for the critical behaviour on the torus, and also leads to a proof that the critical values on the torus and on $\mathbb{Z}^d$ are separated by a multiple of $V^{-1/3}$. The torus triangle condition and the size of the separation of critical points have been proved previously, but our proofs are different and are direct consequences of the bound on the $\mathbb{Z}^d$ two-point function. In particular, we use results derived from the lace expansion on $\mathbb{Z}^d$, but in contrast to previous work on high-dimensional torus percolation, we do not need or use a separate torus lace expansion.

REFERENCES

[1] AIZENMAN, M. (1997). On the number of incipient spanning clusters. *Nuclear Phys. B* **485** 551–582. MR1431856 https://doi.org/10.1016/S0550-3213(96)00626-8

[2] AIZENMAN, M. and NEWMAN, C. M. (1984). Tree graph inequalities and critical behavior in percolation models. *J. Stat. Phys.* **36** 107–143. MR0762034 https://doi.org/10.1007/BF01015729

[3] BARSKY, D. J. and AIZENMAN, M. (1991). Percolation critical exponents under the triangle condition. *Ann. Probab.* **19** 1520–1536. MR1127713

[4] BENJAMINI, I. and SCHRAMM, O. (1996). Percolation beyond Z^d, many questions and a few answers. *Electron. Commun. Probab.* **1** 71–82. MR1423907 https://doi.org/10.1214/ECP.v1-978

[5] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2005). Random subgraphs of finite graphs. I. The scaling window under the triangle condition. *Random Structures Algorithms* **27** 137–184. MR2155704 https://doi.org/10.1002/rsa.20051

[6] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2005). Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. *Ann. Probab.* **33** 1886–1944. MR2165583 https://doi.org/10.1214/009117905000000260

[7] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2006). Random subgraphs of finite graphs. III. The phase transition for the n-cube. *Combinatorica* **26** 395–410. MR2260845 https://doi.org/10.1007/s00493-006-0022-1

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[8] BORGS, C., CHAYES, J. T., KESTEN, H. and SPENCER, J. (1999). Uniform boundedness of critical crossing probabilities implies hyperscaling. *Random Structures Algorithms* **15** 368–413.

[9] CAMIA, F., JIANG, J. and NEWMAN, C. M. (2021). The effect of free boundary conditions on the Ising model in high dimensions. *Probab. Theory Related Fields* **181** 311–328. MR4341075 https://doi.org/10.1007/s00440-021-01041-9

[10] CAMPANINO, M., CHAYES, J. T. and CHAYES, L. (1991). Gaussian fluctuations of connectivities in the subcritical regime of percolation. *Probab. Theory Related Fields* **88** 269–341. MR1100895 https://doi.org/10.1007/BF01418864

[11] CHATTERJEE, S. and HANSON, J. (2020). Restricted percolation critical exponents in high dimensions. *Comm. Pure Appl. Math.* **73** 2370–2429. MR4156622 https://doi.org/10.1002/cpa.21938

[12] CHATTERJEE, S., HANSON, J. and SOSOE, P. (2021). Subcritical connectivity and some exact tail exponents in high dimensional percolation. Preprint. Available at https://arxiv.org/pdf/2107.14347.

[13] CHAYES, J. T. and CHAYES, L. (1987). On the upper critical dimension of Bernoulli percolation. *Comm. Math. Phys.* **113** 27–48. MR0918403

[14] CHEN, L.-C. and SAKAI, A. (2015). Critical two-point functions for long-range statistical-mechanical models in high dimensions. *Ann. Probab.* **43** 639–681. MR3306002 https://doi.org/10.1214/13-AOP843

[15] DUMINIL-COPIN, H., RAOUFI, A. and TASSION, V. (2019). Sharp phase transition for the random-cluster and Potts models via decision trees. *Ann. of Math.* (2) **189** 75–99. MR3898174 https://doi.org/10.4007/annals.2019.189.1.2

[16] ESSAM, J. W., GAUNT, D. S. and GUTTMANN, A. J. (1978). Percolation theory at the critical dimension. *J. Phys. A: Math. Gen.* **11** 1983–1990.

[17] FITZNER, R. and VAN DER HOFSTAD, R. (2017). Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electron. J. Probab.* **22** Paper No. 43. MR3646069 https://doi.org/10.1214/17-EJP56

[18] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[19] HARA, T. (1990). Mean-field critical behaviour for correlation length for percolation in high dimensions. *Probab. Theory Related Fields* **86** 337–385. MR1069285 https://doi.org/10.1007/BF01208256

[20] HARA, T. (2008). Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.* **36** 530–593. MR2393990 https://doi.org/10.1214/009117907000000231

[21] HARA, T., VAN DER HOFSTAD, R. and SLADE, G. (2003). Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31** 349–408. MR1959796 https://doi.org/10.1214/aop/1046294314

[22] HARA, T. and SLADE, G. (1990). Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128** 333–391. MR1043524

[23] HARA, T. and SLADE, G. (1994). Mean-field behaviour and the lace expansion. In *Probability and Phase Transition* (*Cambridge*, 1993) (G. Grimmett, ed.). *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.* **420** 87–122. Kluwer Academic, Dordrecht. MR1283177

[24] HEYDENREICH, M. and VAN DER HOFSTAD, R. (2007). Random graph asymptotics on high-dimensional tori. *Comm. Math. Phys.* **270** 335–358. MR2276449 https://doi.org/10.1007/s00220-006-0152-8

[25] HEYDENREICH, M. and VAN DER HOFSTAD, R. (2011). Random graph asymptotics on high-dimensional tori II: Volume, diameter and mixing time. *Probab. Theory Related Fields* **149** 397–415. Correction: *Probab. Theory Related Fields*, *175*:1183–1185, (2019). MR2776620 https://doi.org/10.1007/s00440-009-0258-y

[26] HEYDENREICH, M. and VAN DER HOFSTAD, R. (2017). *Progress in High-Dimensional Percolation and Random Graphs*. *CRM Short Courses*. Springer, Cham. MR3729454

[27] HEYDENREICH, M., VAN DER HOFSTAD, R. and SAKAI, A. (2008). Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *J. Stat. Phys.* **132** 1001–1049. MR2430773 https://doi.org/10.1007/s10955-008-9580-5

[28] VAN DER HOFSTAD, R. and NACHMIAS, A. (2017). Hypercube percolation. *J. Eur. Math. Soc.* (*JEMS*) **19** 725–814. MR3612867 https://doi.org/10.4171/JEMS/679

[29] VAN DER HOFSTAD, R. and SAPOZHNIKOV, A. (2014). Cycle structure of percolation on high-dimensional tori. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 999–1027. MR3224297 https://doi.org/10.1214/13-AIHP565

[30] HULSHOF, T. and NACHMIAS, A. (2020). Slightly subcritical hypercube percolation. *Random Structures Algorithms* **56** 557–593. MR4060356 https://doi.org/10.1002/rsa.20853

[31] HUTCHCROFT, T. (2020). New critical exponent inequalities for percolation and the random cluster model. *Probab. Math. Phys.* **1** 147–165. MR4408005 https://doi.org/10.2140/pmp.2020.1.147

[32] HUTCHCROFT, T. (2020). Locality of the critical probability for transitive graphs of exponential growth. *Ann. Probab.* **48** 1352–1371. MR4112717 https://doi.org/10.1214/19-AOP1395

[33] HUTCHCROFT, T. (2022). Slightly supercritical percolation on nonamenable graphs I: The distribution of finite clusters. *Proc. Lond. Math. Soc.* Online first. Available at https://doi.org/10.1112/plms.12474.

[34] HUTCHCROFT, T. (2022). On the derivation of mean-field percolation critical exponents from the triangle condition. *J. Stat. Phys.* **189** Paper No. 6. MR4462652 https://doi.org/10.1007/s10955-022-02967-7

[35] KANNAN, R. and KRUEGER, C. K. (1996). *Advanced Analysis on the Real Line. Universitext*. Springer, New York. MR1390758

[36] KOZMA, G. and NACHMIAS, A. (2009). The Alexander–Orbach conjecture holds in high dimensions. *Invent. Math.* **178** 635–654. MR2551766 https://doi.org/10.1007/s00222-009-0208-4

[37] KOZMA, G. and NACHMIAS, A. (2011). Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.* **24** 375–409. MR2748397 https://doi.org/10.1090/S0894-0347-2010-00684-4

[38] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/CBO9780511750854

[39] MENSHIKOV, M. V. (1986). Coincidence of critical points in percolation problems. *Sov. Math., Dokl.* **33** 856–859.

[40] MICHTA, E. and SLADE, G. (2021). Weakly self-avoiding walk on a high-dimensional torus. Preprint. Available at https://arxiv.org/pdf/2107.14170.

[41] MICHTA, E. and SLADE, G. (2022). Asymptotic behaviour of the lattice Green function. *ALEA Lat. Am. J. Probab. Math. Stat.* **19** 957–981. MR4448687 https://doi.org/10.30757/alea.v19-38

[42] O'DONNELL, R., SAKS, M., SCHRAMM, O. and SERVEDIO, R. A. (2005). Every decision tree has an influential variable. In *46th Annual IEEE Symposium on Foundations of Computer Science* (*FOCS'05*) 31–39. IEEE, New York.

[43] PAPATHANAKOS, V. (2006). Finite-Size Effects in High-Dimensional Statistical Mechanical Systems: The Ising Model with Periodic Boundary Conditions. Ph.D. thesis, Princeton Univ.

[44] SLADE, G. (2006). *The Lace Expansion and Its Applications. Lecture Notes in Math.* **1879**. Springer, Berlin. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, Edited and with a foreword by Jean Picard. MR2239599

[45] SLADE, G. (2020). The near-critical two-point function and the torus plateau for weakly self-avoiding walk in high dimensions. Preprint. Available at https://arxiv.org/pdf/2008.00080.

[46] SMIRNOV, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 239–244. MR1851632 https://doi.org/10.1016/S0764-4442(01)01991-7

[47] SMIRNOV, S. and WERNER, W. (2001). Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8** 729–744. MR1879816 https://doi.org/10.4310/MRL.2001.v8.n6.a4

[48] TASAKI, H. (1987). Hyperscaling inequalities for percolation. *Comm. Math. Phys.* **113** 49–65. MR0918404

[49] TOULOUSE, G. (1974). Perspectives from the theory of phase transitions. *Nuovo Cimento* **23B** 234–240.

[50] VANNEUVILLE, H. (2022). Sharpness of Bernoulli percolation via couplings. Preprint. Available at https://arxiv.org/pdf/2201.08223.pdf.

[51] ZHOU, Z., GRIMM, J., DENG, Y. and GARONI, T. M. (2020). Random-length random walks and finite-size scaling on high-dimensional hypercubic lattices I: Periodic boundary conditions. Preprint. Available at https://arxiv.org/pdf/2008.00913.

[52] ZHOU, Z., GRIMM, J., FANG, S., DENG, Y. and GARONI, T. M. (2018). Random-length random walks and finite-size scaling in high dimensions. *Phys. Rev. Lett.* **121** 185701.

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THERMODYNAMIC AND SCALING LIMITS OF THE NON-GAUSSIAN MEMBRANE MODEL

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We characterize the behavior of a random discrete interface ϕ on $[-L, L]^d \cap \mathbb{Z}^d$ with energy $\sum V(\Delta\phi(x))$ as $L \to \infty$, where Δ is the discrete Laplacian and V is a uniformly convex, symmetric, and smooth potential. The interface ϕ is called the non-Gaussian membrane model. By analyzing the Helffer–Sjöstrand representation, associated to $\Delta\phi$, we provide a unified approach to continuous scaling limits of the rescaled and interpolated interface in dimensions $d = 2, 3$, Gaussian approximation in negative regularity spaces for all $d \geq 2$, and the infinite volume limit in $d \geq 5$.

REFERENCES

[1] ARMSTRONG, S. and DARIO, P. (2022). Quantitative hydrodynamic limits of the Langevin dynamics for gradient interface models. Preprint. Available at arXiv:2203.14926.
[2] ARMSTRONG, S. and WU, W. (2022). C^2 regularity of the surface tension for the $\nabla\phi$ interface model. *Comm. Pure Appl. Math.* **75** 349–421. MR4373172 https://doi.org/10.1002/cpa.22031
[3] BOBKOV, S. G. and LEDOUX, M. (2000). From Brunn–Minkowski to Brascamp–Lieb and to logarithmic Sobolev inequalities. *Geom. Funct. Anal.* **10** 1028–1052. MR1800062 https://doi.org/10.1007/PL00001645
[4] BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389. MR0450480 https://doi.org/10.1016/0022-1236(76)90004-5
[5] BUCHHOLZ, S., DEUSCHEL, J.-D., KURT, N. and SCHWEIGER, F. (2019). Probability to be positive for the membrane model in dimensions 2 and 3. *Electron. Commun. Probab.* **24** Paper No. 44, 14 pp. MR3978693 https://doi.org/10.1214/19-ECP245
[6] CARAVENNA, F. and DEUSCHEL, J.-D. (2009). Scaling limits of $(1 + 1)$-dimensional pinning models with Laplacian interaction. *Ann. Probab.* **37** 903–945. MR2537545 https://doi.org/10.1214/08-AOP424
[7] CHIARINI, A., CIPRIANI, A. and HAZRA, R. S. (2016). Extremes of some Gaussian random interfaces. *J. Stat. Phys.* **165** 521–544. MR3562423 https://doi.org/10.1007/s10955-016-1634-5
[8] CHIARINI, A. and NITZSCHNER, M. (2021). Phase transition for level-set percolation of the membrane model in dimensions $d \geq 5$. Preprint. Available at arXiv:2112.09116.
[9] CIPRIANI, A., DAN, B. and HAZRA, R. S. (2019). The scaling limit of the membrane model. *Ann. Probab.* **47** 3963–4001. MR4038046 https://doi.org/10.1214/19-aop1351
[10] CIPRIANI, A., DAN, B. and HAZRA, R. S. (2021). The scaling limit of the $(\nabla + \Delta)$-model. *J. Stat. Phys.* **182** Paper No. 39, 23 pp. MR4215327 https://doi.org/10.1007/s10955-021-02717-1
[11] FUNAKI, T. (2005). Stochastic interface models. In *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1869** 103–274. Springer, Berlin. MR2228384 https://doi.org/10.1007/11429579_2
[12] GAZZOLA, F., GRUNAU, H.-C. and SWEERS, G. (2010). *Polyharmonic Boundary Value Problems: Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains. Lecture Notes in Math.* **1991**. Springer, Berlin. MR2667016 https://doi.org/10.1007/978-3-642-12245-3
[13] GIACOMIN, G., OLLA, S. and SPOHN, H. (2001). Equilibrium fluctuations for $\nabla\phi$ interface model. *Ann. Probab.* **29** 1138–1172. MR1872740 https://doi.org/10.1214/aop/1015345600
[14] HELFFER, B. and SJÖSTRAND, J. (1994). On the correlation for Kac-like models in the convex case. *J. Stat. Phys.* **74** 349–409. MR1257821 https://doi.org/10.1007/BF02186817
[15] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8
[16] KURT, N. (2007). Entropic repulsion for a class of Gaussian interface models in high dimensions. *Stochastic Process. Appl.* **117** 23–34. MR2287101 https://doi.org/10.1016/j.spa.2006.05.011

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[17] KURT, N. (2008). Entropic repulsion for a Gaussian membrane model in the critical and supercritical dimensions. PhD thesis, Universität Zürich.

[18] KURT, N. (2009). Maximum and entropic repulsion for a Gaussian membrane model in the critical dimension. *Ann. Probab.* **37** 687–725. MR2510021 https://doi.org/10.1214/08-AOP417

[19] LAWLER, G. F. (1991). *Intersections of Random Walks. Probability and Its Applications*. Birkhäuser, Inc., Boston, MA. MR1117680

[20] MUIRHEAD, S. (2022). Percolation of strongly correlated Gaussian fields II. Sharpness of the phase transition. Preprint. Available at arXiv:2206.10724.

[21] MÜLLER, S. and SCHWEIGER, F. (2019). Estimates for the Green's function of the discrete bilaplacian in dimensions 2 and 3. *Vietnam J. Math.* **47** 133–181. MR3913854 https://doi.org/10.1007/s10013-018-0325-8

[22] NADDAF, A. and SPENCER, T. (1997). On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.* **183** 55–84. MR1461951 https://doi.org/10.1007/BF02509796

[23] SAKAGAWA, H. (2003). Entropic repulsion for a Gaussian lattice field with certain finite range interaction. *J. Math. Phys.* **44** 2939–2951. MR1982781 https://doi.org/10.1063/1.1581354

[24] SCHWEIGER, F. (2020). The maximum of the four-dimensional membrane model. *Ann. Probab.* **48** 714–741. MR4089492 https://doi.org/10.1214/19-AOP1372

[25] SCHWEIGER, F. (2021). Pinning for the critical and supercritical membrane model. *Probab. Math. Phys.* **2** 745–820. MR4408024 https://doi.org/10.2140/pmp.2021.2.745

[26] SJÖSTRAND, J. (1996). Correlation asymptotics and Witten Laplacians. *Algebra i Analiz* **8** 160–191. MR1392018

[27] THOMÉE, V. (1964). Elliptic difference operators and Dirichlet's problem. *Contrib. Differ. Equ.* **3** 301–324. MR0163444

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LOWER TAILS VIA RELATIVE ENTROPY

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We show that the naive mean-field approximation correctly predicts the leading term of the logarithmic lower tail probabilities for the number of copies of a given subgraph in $G(n, p)$ and of arithmetic progressions of a given length in random subsets of the integers in the entire range of densities where the mean-field approximation is viable.

Our main technical result provides sufficient conditions on the maximum degrees of a uniform hypergraph $\mathcal{H}$ that guarantee that the logarithmic lower tail probabilities for the number of edges, induced by a binomial random subset of the vertices of $\mathcal{H}$, can be well approximated by considering only product distributions. This may be interpreted as a weak, probabilistic version of the hypergraph container lemma that is applicable to all sparser-than-average (and not only independent) sets.

REFERENCES

[1] AUGERI, F. (2020). Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 2404–2448. MR4152647 https://doi.org/10.1214/20-AOP1427

[2] AVEZ, A. (1976). Harmonic functions on groups. In *Differential Geometry and Relativity*. *Mathematical Phys. and Appl. Math.* **3** 27–32. Reidel, Dordrecht. MR0507229

[3] BALOGH, J., MORRIS, R. and SAMOTIJ, W. (2015). Independent sets in hypergraphs. *J. Amer. Math. Soc.* **28** 669–709. MR3327533 https://doi.org/10.1090/S0894-0347-2014-00816-X

[4] BALOGH, J. and SAMOTIJ, W. (2020). An efficient container lemma. *Discrete Anal.* Paper No. 17, 56. MR4186910 https://doi.org/10.19086/da

[5] BASAK, A. and BASU, R. (2023). Upper tail large deviations of regular subgraph counts in Erdős–Rényi graphs in the full localized regime. *Comm. Pure Appl. Math.* **76** 3–72. Available at arXiv:1912.11410. https://doi.org/10.1002/cpa.22036

[6] BHATTACHARYA, B. B., GANGULY, S., LUBETZKY, E. and ZHAO, Y. (2017). Upper tails and independence polynomials in random graphs. *Adv. Math.* **319** 313–347. MR3695877 https://doi.org/10.1016/j.aim.2017.08.003

[7] CHATTERJEE, S. (2012). The missing log in large deviations for triangle counts. *Random Structures Algorithms* **40** 437–451. MR2925306 https://doi.org/10.1002/rsa.20381

[8] CHATTERJEE, S. and DEMBO, A. (2016). Nonlinear large deviations. *Adv. Math.* **299** 396–450. MR3519474 https://doi.org/10.1016/j.aim.2016.05.017

[9] CHATTERJEE, S. and VARADHAN, S. R. S. (2011). The large deviation principle for the Erdős–Rényi random graph. *European J. Combin.* **32** 1000–1017. MR2825532 https://doi.org/10.1016/j.ejc.2011.03.014

[10] CONLON, D. and GOWERS, W. T. (2016). Combinatorial theorems in sparse random sets. *Ann. of Math.* (2) **184** 367–454. MR3548529 https://doi.org/10.4007/annals.2016.184.2.2

[11] COOK, N. and DEMBO, A. (2020). Large deviations of subgraph counts for sparse Erdős–Rényi graphs. *Adv. Math.* **373** 107289, 53. MR4130460 https://doi.org/10.1016/j.aim.2020.107289

[12] COOK, N., DEMBO, A. and PHAM, H. T. Regularity method and large deviation principles for the Erdős–Rényi hypergraph. arXiv:2102.09100 [math.PR].

[13] COVER, T. M. and THOMAS, J. A. (2006). *Elements of Information Theory*, 2nd ed. Wiley Interscience, Hoboken, NJ. MR2239987

[14] CSISZÁR, I. (1984). Sanov property, generalized I-projection and a conditional limit theorem. *Ann. Probab.* **12** 768–793. MR0744233

MSC2020 subject classifications. 60F10.
Key words and phrases. Lower tail, subgraph counts, random graph.

[15] Csiszár, I. and Körner, J. (2011). *Information Theory*: *Coding Theorems for Discrete Memoryless Systems*, 2nd ed. Cambridge Univ. Press, Cambridge. MR2839250 https://doi.org/10.1017/CBO9780511921889

[16] DeMarco, B. and Kahn, J. (2012). Upper tails for triangles. *Random Structures Algorithms* **40** 452–459. MR2925307 https://doi.org/10.1002/rsa.20382

[17] Eldan, R. (2018). Gaussian-width gradient complexity, reverse log-Sobolev inequalities and nonlinear large deviations. *Geom*. *Funct*. *Anal*. **28** 1548–1596. MR3881829 https://doi.org/10.1007/s00039-018-0461-z

[18] Erdős, P., Kleitman, D. J. and Rothschild, B. L. (1976). Asymptotic enumeration of K_n-free graphs. In *Colloquio Internazionale Sulle Teorie Combinatorie* (*Rome*, 1973), *Tomo II*. *Atti dei Convegni Lincei* **17** 19–27. Accad. Naz. Lincei, Rome. MR0463020

[19] Frieze, A. and Kannan, R. (1999). Quick approximation to matrices and applications. *Combinatorica* **19** 175–220. MR1723039 https://doi.org/10.1007/s004930050052

[20] Harel, M., Mousset, F. and Samotij, W. (2022). Upper tails via high moments and entropic stability. *Duke Math*. *J*. **171** 2089–2192. MR4484206 https://doi.org/10.1215/00127094-2021-0067

[21] Jain, V., Risteski, A. and Koehler, F. (2019). Mean-field approximation, convex hierarchies, and the optimality of correlation rounding: A unified perspective. In *STOC'*19*—Proceedings of the* 51*st Annual ACM SIGACT Symposium on Theory of Computing* 1226–1236. ACM, New York. MR4003424 https://doi.org/10.1145/3313276.3316299

[22] Janson, S. (1990). Poisson approximation for large deviations. *Random Structures Algorithms* **1** 221–229. MR1138428 https://doi.org/10.1002/rsa.3240010209

[23] Janson, S., Łuczak, T. and Ruciński, A. (1990). An exponential bound for the probability of nonexistence of a specified subgraph in a random graph. In *Random Graphs '*87 (*Poznań*, 1987) 73–87. Wiley, Chichester. MR1094125

[24] Kohayakawa, Y., Łuczak, T. and Rödl, V. (1997). On K^4-free subgraphs of random graphs. *Combinatorica* **17** 173–213. MR1479298 https://doi.org/10.1007/BF01200906

[25] Kozma, G., Meyerovitch, T., Peled, R. and Samotij, W. What does a typical metric space look like? *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. To appear. Available at arXiv:2104.01689. https://doi.org/10.1214/22-AIHP1262

[26] Lubetzky, E. and Zhao, Y. (2015). On replica symmetry of large deviations in random graphs. *Random Structures Algorithms* **47** 109–146. MR3366814 https://doi.org/10.1002/rsa.20536

[27] Lubetzky, E. and Zhao, Y. (2017). On the variational problem for upper tails in sparse random graphs. *Random Structures Algorithms* **50** 420–436. MR3632418 https://doi.org/10.1002/rsa.20658

[28] Łuczak, T. (2000). On triangle-free random graphs. *Random Structures Algorithms* **16** 260–276. MR1749289 https://doi.org/10.1002/(SICI)1098-2418(200005)16:3<260::AID-RSA3>3.3.CO;2-H

[29] Manurangsi, P. and Raghavendra, P. (2017). A birthday repetition theorem and complexity of approximating dense CSPs. In 44*th International Colloquium on Automata*, *Languages*, *and Programming*. *LIPIcs*. *Leibniz Int*. *Proc*. *Inform*. **80** Art. No. 78, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR3685818

[30] Montanari, A. (2008). Estimating random variables from random sparse observations. *Eur. Trans*. *Telecommun*. **19** 385–403.

[31] Raghavendra, P. and Tan, N. (2012). Approximating CSPs with global cardinality constraints using SDP hierarchies. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms* 373–384. ACM, New York. MR3205223

[32] Saxton, D. and Thomason, A. (2015). Hypergraph containers. *Invent*. *Math*. **201** 925–992. MR3385638 https://doi.org/10.1007/s00222-014-0562-8

[33] Schacht, M. (2016). Extremal results for random discrete structures. *Ann*. *of Math*. (2) **184** 333–365. MR3548528 https://doi.org/10.4007/annals.2016.184.2.1

[34] Zhao, Y. (2017). On the lower tail variational problem for random graphs. *Combin*. *Probab*. *Comput*. **26** 301–320. MR3603970 https://doi.org/10.1017/S0963548316000262

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STABILITY OF SCHRÖDINGER POTENTIALS AND CONVERGENCE OF SINKHORN'S ALGORITHM

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We study the stability of entropically regularized optimal transport with respect to the marginals. Given marginals converging weakly, we establish a strong convergence for the Schrödinger potentials, describing the density of the optimal couplings. When the marginals converge in total variation, the optimal couplings also converge in total variation. This is applied to show that Sinkhorn's algorithm converges in total variation when costs are quadratic and marginals are subgaussian or, more generally, for all continuous costs satisfying an integrability condition.

REFERENCES

[1] ALVAREZ-MELIS, D. and JAAKKOLA, T. (2018). Gromov–Wasserstein alignment of word embedding spaces. In *Proceedings of the* 2018 *Conference on Empirical Methods in Natural Language Processing* 1881–1890.

[2] ARJOVSKY, M., CHINTALA, S. and BOTTOU, L. (2017). Wasserstein generative adversarial networks. In *Proceedings of Machine Learning Research* **70** 214–223.

[3] BERMAN, R. J. (2020). The Sinkhorn algorithm, parabolic optimal transport and geometric Monge–Ampère equations. *Numer. Math.* **145** 771–836. MR4125978 https://doi.org/10.1007/s00211-020-01127-x

[4] BERNTON, E., GHOSAL, P. and NUTZ, M. Entropic optimal transport: Geometry and large deviations. *Duke Math. J.* To appear.

[5] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[6] BLANCHET, J., JAMBULAPATI, A., KENT, C. and SIDFORD, A. (2018). Towards optimal running times for optimal transport. Preprint. Available at arXiv:1810.07717v1.

[7] CARLIER, G. (2022). On the linear convergence of the multimarginal Sinkhorn algorithm. *SIAM J. Optim.* **32** 786–794. MR4418040 https://doi.org/10.1137/21M1410634

[8] CARLIER, G. and LABORDE, M. (2020). A differential approach to the multi-marginal Schrödinger system. *SIAM J. Math. Anal.* **52** 709–717. MR4062805 https://doi.org/10.1137/19M1253800

[9] CHEN, Y., GEORGIOU, T. T. and PAVON, M. (2016). On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint. *J. Optim. Theory Appl.* **169** 671–691. MR3489825 https://doi.org/10.1007/s10957-015-0803-z

[10] CHERNOZHUKOV, V., GALICHON, A., HALLIN, M. and HENRY, M. (2017). Monge–Kantorovich depth, quantiles, ranks and signs. *Ann. Statist.* **45** 223–256. MR3611491 https://doi.org/10.1214/16-AOS1450

[11] CHIARINI, A., CONFORTI, G., GRECO, G. and TAMANINI, L. (2022). Gradient estimates for the Schrödinger potentials: Convergence to the Brenier map and quantitative stability. Preprint. Available at arXiv:2207.14262v1.

[12] COMINETTI, R. and SAN MARTÍN, J. (1994). Asymptotic analysis of the exponential penalty trajectory in linear programming. *Math. Program.* **67** 169–187. MR1305565 https://doi.org/10.1007/BF01582220

[13] CONFORTI, G. and TAMANINI, L. (2021). A formula for the time derivative of the entropic cost and applications. *J. Funct. Anal.* **280** 108964. MR4232667 https://doi.org/10.1016/j.jfa.2021.108964

[14] CUTURI, M. (2013). Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in Neural Information Processing Systems* **26** 2292–2300.

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Key words and phrases. Optimal transport, entropic regularization, Schrödinger potentials, Sinkhorn's algorithm, IPFP.

[15] DELIGIANNIDIS, G., DE BORTOLI, V. and DOUCET, A. (2021). Quantitative uniform stability of the iterative proportional fitting procedure. Preprint. Available at arXiv:2108.08129v1.

[16] DEMING, W. E. and STEPHAN, F. F. (1940). On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. *Ann. Math. Stat.* **11** 427–444. MR0003527 https://doi.org/10.1214/aoms/1177731829

[17] ECKSTEIN, S. and NUTZ, M. (2021). Quantitative stability of regularized optimal transport and convergence of Sinkhorn's algorithm. *SIAM J. Math. Anal.* To appear.

[18] FÖLLMER, H. (1988). Random fields and diffusion processes. In *École D'Été de Probabilités de Saint-Flour XV–XVII*, 1985–87. *Lecture Notes in Math.* **1362** 101–203. Springer, Berlin. MR0983373 https://doi.org/10.1007/BFb0086180

[19] FRANKLIN, J. and LORENZ, J. (1989). On the scaling of multidimensional matrices. *Linear Algebra Appl.* **114/115** 717–735. MR0986904 https://doi.org/10.1016/0024-3795(89)90490-4

[20] GHOSAL, P., NUTZ, M. and BERNTON, E. (2022). Stability of entropic optimal transport and Schrödinger bridges. *J. Funct. Anal.* **283** 109622. MR4460341 https://doi.org/10.1016/j.jfa.2022.109622

[21] GIGLI, N. and TAMANINI, L. (2021). Second order differentiation formula on $\mathrm{RCD}^*(K, N)$ spaces. *J. Eur. Math. Soc. (JEMS)* **23** 1727–1795. MR4244516 https://doi.org/10.4171/JEMS/1042

[22] IRELAND, C. T. and KULLBACK, S. (1968). Contingency tables with given marginals. *Biometrika* **55** 179–188. MR0229329 https://doi.org/10.1093/biomet/55.1.179

[23] KULLBACK, S. (1968). Probability densities with given marginals. *Ann. Math. Stat.* **39** 1236–1243. MR0229330 https://doi.org/10.1214/aoms/1177698249

[24] LÉONARD, C. (2012). From the Schrödinger problem to the Monge–Kantorovich problem. *J. Funct. Anal.* **262** 1879–1920. MR2873864 https://doi.org/10.1016/j.jfa.2011.11.026

[25] LÉONARD, C. (2014). A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst.* **34** 1533–1574. MR3121631 https://doi.org/10.3934/dcds.2014.34.1533

[26] MIKAMI, T. (2002). Optimal control for absolutely continuous stochastic processes and the mass transportation problem. *Electron. Commun. Probab.* **7** 199–213. MR1937905 https://doi.org/10.1214/ECP.v7-1061

[27] MIKAMI, T. (2004). Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. *Probab. Theory Related Fields* **129** 245–260. MR2063377 https://doi.org/10.1007/s00440-004-0340-4

[28] NUTZ, M. (2021). *Introduction to Entropic Optimal Transport. Lecture Notes*. Columbia Univ., New York.

[29] NUTZ, M. and WIESEL, J. (2022). Entropic optimal transport: Convergence of potentials. *Probab. Theory Related Fields* **184** 401–424. MR4498514 https://doi.org/10.1007/s00440-021-01096-8

[30] PAL, S. (2019). On the difference between entropic cost and the optimal transport cost. Preprint. Available at arXiv:1905.12206v1.

[31] PEYRÉ, G. and CUTURI, M. (2019). Computational optimal transport: With applications to data science. *Found. Trends Mach. Learn.* **11** 355–607.

[32] RUBNER, Y., TOMASI, C. and GUIBAS, L. J. (2000). The Earth mover's distance as a metric for image retrieval. *Int. J. Comput. Vis.* **40** 99–121.

[33] RÜSCHENDORF, L. (1995). Convergence of the iterative proportional fitting procedure. *Ann. Statist.* **23** 1160–1174. MR1353500 https://doi.org/10.1214/aos/1176324703

[34] SCHRÖDINGER, E. (1931). Über die Umkehrung der Naturgesetze. *Sitzungsberichte Preuss. Akad. Wiss. Akad. Wiss., Berlin. Phys. Math.* **144** 144–153.

[35] SINKHORN, R. (1964). A relationship between arbitrary positive matrices and doubly stochastic matrices. *Ann. Math. Stat.* **35** 876–879. MR0161868 https://doi.org/10.1214/aoms/1177703591

[36] SINKHORN, R. and KNOPP, P. (1967). Concerning nonnegative matrices and doubly stochastic matrices. *Pacific J. Math.* **21** 343–348. MR0210731

[37] WEED, J. (2018). An explicit analysis of the entropic penalty in linear programming. In *Proceedings of Machine Learning Research* **75** 1841–1855.

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STEIN'S METHOD FOR CONDITIONAL CENTRAL LIMIT THEOREM

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In the seventies, Charles Stein revolutionized the way of proving the central limit theorem by introducing a method that utilizes a characterization equation for Gaussian distribution. In the last 50 years, much research has been done to adapt and strengthen this method to a variety of different settings and other limiting distributions. However, it has not been yet extended to study conditional convergences. In this article we develop a novel approach, using Stein's method for exchangeable pairs, to find a rate of convergence in the conditional central limit theorem of the form $(X_n \mid Y_n = k)$, where (X_n, Y_n) are asymptotically jointly Gaussian, and extend this result to a multivariate version. We apply our general result to several concrete examples, including pattern count in a random binary sequence and subgraph count in Erdős–Rényi random graph.

REFERENCES

[1] ARRAS, B. and HOUDRÉ, C. (2019). On Stein's method for multivariate self-decomposable laws. *Electron. J. Probab.* **24** Paper No. 128, 63. MR4029431 https://doi.org/10.1214/19-ejp378

[2] BARBOUR, A. D. (1982). Poisson convergence and random graphs. *Math. Proc. Cambridge Philos. Soc.* **92** 349–359. MR0671189 https://doi.org/10.1017/S0305004100059995

[3] BARBOUR, A. D. (1990). Stein's method for diffusion approximations. *Probab. Theory Related Fields* **84** 297–322. MR1035659 https://doi.org/10.1007/BF01197887

[4] BARBOUR, A. D., KAROŃSKI, M. and RUCIŃSKI, A. (1989). A central limit theorem for decomposable random variables with applications to random graphs. *J. Combin. Theory Ser. B* **47** 125–145. MR1047781 https://doi.org/10.1016/0095-8956(89)90014-2

[5] BARBOUR, A. D., RÖLLIN, A. and ROSS, N. (2019). Error bounds in local limit theorems using Stein's method. *Bernoulli* **25** 1076–1104. MR3920366 https://doi.org/10.3150/17-bej1013

[6] BOLTHAUSEN, E. (1980). The Berry–Esseen theorem for functionals of discrete Markov chains. *Z. Wahrsch. Verw. Gebiete* **54** 59–73. MR0595481 https://doi.org/10.1007/BF00535354

[7] BULINSKIĬ, A. V. (2016). A conditional central limit theorem. *Teor. Veroyatn. Primen.* **61** 686–708. MR3632530 https://doi.org/10.1137/S0040585X97T98837X

[8] CHATTERJEE, S. (2007). Stein's method for concentration inequalities. *Probab. Theory Related Fields* **138** 305–321. MR2288072 https://doi.org/10.1007/s00440-006-0029-y

[9] CHATTERJEE, S. (2012). A new approach to strong embeddings. *Probab. Theory Related Fields* **152** 231–264. MR2875758 https://doi.org/10.1007/s00440-010-0321-8

[10] CHATTERJEE, S. and DEY, P. S. (2010). Applications of Stein's method for concentration inequalities. *Ann. Probab.* **38** 2443–2485. MR2683635 https://doi.org/10.1214/10-AOP542

[11] CHATTERJEE, S., FULMAN, J. and RÖLLIN, A. (2011). Exponential approximation by Stein's method and spectral graph theory. *ALEA Lat. Am. J. Probab. Math. Stat.* **8** 197–223. MR2802856

[12] CHATTERJEE, S. and MECKES, E. (2008). Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 257–283. MR2453473

[13] CHATTERJEE, S. and SHAO, Q.-M. (2011). Nonnormal approximation by Stein's method of exchangeable pairs with application to the Curie–Weiss model. *Ann. Appl. Probab.* **21** 464–483. MR2807964 https://doi.org/10.1214/10-AAP712

[14] CHEN, L. H. Y. (1975). Poisson approximation for dependent trials. *Ann. Probab.* **3** 534–545. MR0428387 https://doi.org/10.1214/aop/1176996359

[15] CHEN, L. H. Y., FANG, X. and SHAO, Q.-M. (2013). From Stein identities to moderate deviations. *Ann. Probab.* **41** 262–293. MR3059199 https://doi.org/10.1214/12-AOP746

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[16] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2732624 https://doi.org/10.1007/978-3-642-15007-4

[17] CHEN, L. H. Y. and RÖLLIN, A. (2010). Stein couplings for normal approximation. Preprint. Available at http://arxiv.org/abs/1003.6039v2.

[18] CHEN, L. H. Y. and SHAO, Q.-M. (2004). Normal approximation under local dependence. *Ann*. *Probab*. **32** 1985–2028. MR2073183 https://doi.org/10.1214/009117904000000450

[19] DEDECKER, J. and MERLEVÈDE, F. (2002). Necessary and sufficient conditions for the conditional central limit theorem. *Ann*. *Probab*. **30** 1044–1081. MR1920101 https://doi.org/10.1214/aop/1029867121

[20] DIACONIS, P. and HOLMES, S., eds. (2004) *Stein's Method*: *Expository Lectures and Applications*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **46**. IMS, Beachwood, OH. MR2118599

[21] DURRETT, R. (2010). *Probability*: *Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836 https://doi.org/10.1017/CBO9780511779398

[22] FANG, X. (2016). A multivariate CLT for bounded decomposable random vectors with the best known rate. *J*. *Theoret*. *Probab*. **29** 1510–1523. MR3571252 https://doi.org/10.1007/s10959-015-0619-7

[23] FANG, X. and KOIKE, Y. (2022). New error bounds in multivariate normal approximations via exchangeable pairs with applications to Wishart matrices and fourth moment theorems. *Ann*. *Appl*. *Probab*. **32** 602–631. MR4386537 https://doi.org/10.1214/21-aap1690

[24] GHOSH, S. and GOLDSTEIN, L. (2011). Concentration of measures via size-biased couplings. *Probab*. *Theory Related Fields* **149** 271–278. MR2773032 https://doi.org/10.1007/s00440-009-0253-3

[25] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, MA. MR0062975

[26] GOLDSTEIN, L. and IŞLAK, Ü. (2014). Concentration inequalities via zero bias couplings. *Statist*. *Probab*. *Lett*. **86** 17–23. MR3162712 https://doi.org/10.1016/j.spl.2013.12.001

[27] GOLDSTEIN, L. and REINERT, G. (1997). Stein's method and the zero bias transformation with application to simple random sampling. *Ann*. *Appl*. *Probab*. **7** 935–952. MR1484792 https://doi.org/10.1214/aoap/1043862419

[28] GOLDSTEIN, L. and REINERT, G. (2005). Zero biasing in one and higher dimensions, and applications. In *Stein's Method and Applications*. *Lect*. *Notes Ser*. *Inst*. *Math*. *Sci*. *Natl*. *Univ*. *Singap*. **5** 1–18. Singapore Univ. Press, Singapore. MR2201883 https://doi.org/10.1142/9789812567673_0001

[29] GOLDSTEIN, L. and RINOTT, Y. (1996). Multivariate normal approximations by Stein's method and size bias couplings. *J*. *Appl*. *Probab*. **33** 1–17. MR1371949 https://doi.org/10.1017/s0021900200103675

[30] GÖTZE, F. (1991). On the rate of convergence in the multivariate CLT. *Ann*. *Probab*. **19** 724–739. MR1106283

[31] GUO, X. and PETERSON, J. (2019). Berry–Esseen estimates for regenerative processes under weak moment assumptions. *Stochastic Process*. *Appl*. **129** 1379–1412. MR3926560 https://doi.org/10.1016/j.spa.2018.05.001

[32] HOLST, L. (1979). Two conditional limit theorems with applications. *Ann*. *Statist*. **7** 551–557. MR0527490

[33] KAROŃSKI, M. and RUCIŃSKI, A. (1987). Poisson convergence and semi-induced properties of random graphs. *Math*. *Proc*. *Cambridge Philos*. *Soc*. **101** 291–300. MR0870602 https://doi.org/10.1017/S0305004100066664

[34] MCDONALD, D. R. (1979). On local limit theorem for integer valued random variables. *Teor*. *Veroyatn*. *Primen*. **24** 607–614. MR0541375

[35] MECKES, E. (2006). *An Infinitesimal Version of Stein's Method of Exchangeable Pairs*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—Stanford University. MR2708805

[36] MECKES, E. (2009). On Stein's method for multivariate normal approximation. In *High Dimensional Probability V*: *The Luminy Volume*. *Inst*. *Math*. *Stat*. (*IMS*) *Collect*. **5** 153–178. IMS, Beachwood, OH. MR2797946 https://doi.org/10.1214/09-IMSCOLL511

[37] NOURDIN, I., PECCATI, G. and RÉVEILLAC, A. (2010). Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **46** 45–58. MR2641769 https://doi.org/10.1214/08-AIHP308

[38] RAIČ, M. (2018). A multivariate central limit theorem for Lipschitz and smooth test functions. https://arxiv.org/abs/1812.08268. https://doi.org/10.48550/ARXIV.1812.08268

[39] REINERT, G. and RÖLLIN, A. (2009). Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *Ann*. *Probab*. **37** 2150–2173. MR2573554 https://doi.org/10.1214/09-AOP467

[40] RINOTT, Y. (1994). On normal approximation rates for certain sums of dependent random variables. *J*. *Comput*. *Appl*. *Math*. **55** 135–143. MR1327369 https://doi.org/10.1016/0377-0427(94)90016-7

[41] RINOTT, Y. and ROTAR, V. (1996). A multivariate CLT for local dependence with $n^{-1/2}\log n$ rate and applications to multivariate graph related statistics. *J. Multivariate Anal.* **56** 333–350. MR1379533 https://doi.org/10.1006/jmva.1996.0017

[42] RÖLLIN, A. (2005). Approximation of sums of conditionally independent variables by the translated Poisson distribution. *Bernoulli* **11** 1115–1128. MR2189083 https://doi.org/10.3150/bj/1137421642

[43] RÖLLIN, A. (2008). A note on the exchangeability condition in Stein's method. *Statist. Probab. Lett.* **78** 1800–1806. MR2453918 https://doi.org/10.1016/j.spl.2008.01.043

[44] RÖLLIN, A. (2013). Stein's method in high dimensions with applications. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 529–549. MR3088380 https://doi.org/10.1214/11-aihp473

[45] RÖLLIN, A. (2018). On quantitative bounds in the mean martingale central limit theorem. *Statist. Probab. Lett.* **138** 171–176. MR3788734 https://doi.org/10.1016/j.spl.2018.03.004

[46] RÖLLIN, A. and ROSS, N. (2015). Local limit theorems via Landau–Kolmogorov inequalities. *Bernoulli* **21** 851–880. MR3338649 https://doi.org/10.3150/13-BEJ590

[47] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[48] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California, Berkeley, CA*, 1970/1971), *Vol. II*: *Probability Theory* 583–602. Univ. California Press, Berkeley, CA. MR0402873

[49] STEIN, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

[50] YUAN, D.-M., WEI, L.-R. and LEI, L. (2014). Conditional central limit theorems for a sequence of conditional independent random variables. *J. Korean Math. Soc.* **51** 1–15. MR3159314 https://doi.org/10.4134/JKMS.2014.51.1.001

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MORE LIMITING DISTRIBUTIONS FOR EIGENVALUES OF WIGNER MATRICES

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The Tracy–Widom distributions are among the most famous laws in probability theory, partly due to their connection with Wigner matrices. In particular, for $A = \frac{1}{\sqrt{n}}(a_{ij})_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$ symmetric with $(a_{ij})_{1 \le i \le j \le n}$ i.i.d. standard normal, the fluctuations of its largest eigenvalue $\lambda_1(A)$ are asymptotically described by a real-valued Tracy–Widom distribution TW_1 : $n^{2/3}(\lambda_1(A) - 2) \Rightarrow TW_1$. As it often happens, Gaussianity can be relaxed, and this results holds when $\mathbb{E}[a_{11}] = 0$, $\mathbb{E}[a_{11}^2] = 1$ and the tail of a_{11} decays sufficiently fast: $\lim_{x \to \infty} x^4 \mathbb{P}(|a_{11}| > x) = 0$, whereas when the law of a_{11} is regularly varying with index $\alpha \in (0, 4)$, $c_a(n)n^{1/2-2/\alpha}\lambda_1(A)$ converges to a Fréchet distribution for $c_a : (0, \infty) \to (0, \infty)$, slowly varying and depending solely on the law of a_{11}. This paper considers a family of edge cases, $\lim_{x \to \infty} x^4 \mathbb{P}(|a_{11}| > x) = c \in (0, \infty)$, and unveils a new type of limiting behavior for $\lambda_1(A)$: a continuous function of a Fréchet distribution in which 2, the almost sure limit of $\lambda_1(A)$ in the light-tailed case, plays a pivotal role:

$$f(x) = \begin{cases} 2, & 0 < x < 1, \\ x + \dfrac{1}{x}, & x \ge 1. \end{cases}$$

REFERENCES

[1] AUFFINGER, A., BEN AROUS, G. and PÉCHÉ, S. (2009). Poisson convergence for the largest eigenvalues of heavy tailed random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 589–610. MR2548495 https://doi.org/10.1214/08-AIHP188

[2] BAI, Z. and SILVERSTEIN, J. W. (2010). *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR2567175 https://doi.org/10.1007/978-1-4419-0661-8

[3] BAI, Z. D., SILVERSTEIN, J. W. and YIN, Y. Q. (1988). A note on the largest eigenvalue of a large-dimensional sample covariance matrix. *J. Multivariate Anal.* **26** 166–168. MR0963829 https://doi.org/10.1016/0047-259X(88)90078-4

[4] BAI, Z. D. and YIN, Y. Q. (1988). Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. *Ann. Probab.* **16** 1729–1741. MR0958213

[5] BAI, Z. D. and YIN, Y. Q. (1993). Limit of the smallest eigenvalue of a large-dimensional sample covariance matrix. *Ann. Probab.* **21** 1275–1294. MR1235416

[6] BENAYCH-GEORGES, F. and PÉCHÉ, S. (2014). Localization and delocalization for heavy tailed band matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 1385–1403. MR3269999 https://doi.org/10.1214/13-AIHP562

[7] DIACONU, S. (2022). Finite rank perturbations of heavy-tailed random matrices. Preprint. Available at arXiv:2208.02756.

[8] DIACONU, S. (2022). Two CLTs for sparse random matrices. Preprint. Available at arXiv:2210.09625.

[9] KHORUNZHIY, O. (2012). High moments of large Wigner random matrices and asymptotic properties of the spectral norm. *Random Oper. Stoch. Equ.* **20** 25–68. MR2899796 https://doi.org/10.1515/rose-2012-0002

[10] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. *Springer Series in Statistics*. Springer, New York. MR0691492

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[11] LEE, J. O. and YIN, J. (2014). A necessary and sufficient condition for edge universality of Wigner matrices. *Duke Math. J.* **163** 117–173. MR3161313 https://doi.org/10.1215/00127094-2414767

[12] RUZMAIKINA, A. (2006). Universality of the edge distribution of eigenvalues of Wigner random matrices with polynomially decaying distributions of entries. *Comm. Math. Phys.* **261** 277–296. MR2191882 https://doi.org/10.1007/s00220-005-1386-6

[13] SINAI, Y. G. and SOSHNIKOV, A. B. (1998). A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices. *Funct. Anal. Appl.* **32** 56–79, 96. MR1647832 https://doi.org/10.1007/BF02482597

[14] SINAI, Y. and SOSHNIKOV, A. (1998). Central limit theorem for traces of large random symmetric matrices with independent matrix elements. *Bol. Soc. Bras. Mat.* **29** 1–24. MR1620151 https://doi.org/10.1007/BF01245866

[15] SOSHNIKOV, A. (1999). Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.* **207** 697–733. MR1727234 https://doi.org/10.1007/s002200050743

[16] SOSHNIKOV, A. (2004). Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Commun. Probab.* **9** 82–91. MR2081462 https://doi.org/10.1214/ECP.v9-1112

[17] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math.* **206** 127–204. MR2784665 https://doi.org/10.1007/s11511-011-0061-3

[18] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[19] WIGNER, E. P. (1958). On the distribution of the roots of certain symmetric matrices. *Ann. of Math.* (2) **67** 325–327. MR0095527 https://doi.org/10.2307/1970008

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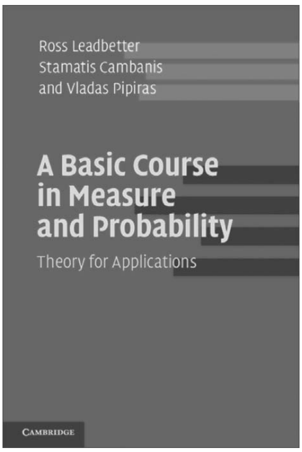
Articles



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