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# THE DELOCALIZED PHASE OF THE ANDERSON HAMILTONIAN IN 1-D

BY LAURE DUMAZ[1,a] AND CYRIL LABBÉ[2,b]

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We introduce a random differential operator that we call $\mathtt{CS}_\tau$ operator, whose spectrum is given by the $\mathrm{Sch}_\tau$ point process introduced by Kritchevski, Valkó and Virág (*Comm. Math Phys.* (2012) **314** 775–806) and whose eigenvectors match with the description provided by Rifkind and Virág (*Geom. Funct. Anal.* (2018) **28** 1394–1419). This operator acts on $\mathbf{R}^2$-valued functions from the interval [0, 1] and takes the form

$$2\begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + \sqrt{\tau}\begin{pmatrix} d\mathcal{B} + \frac{1}{\sqrt{2}}d\mathcal{W}_1 & \frac{1}{\sqrt{2}}d\mathcal{W}_2 \\ \frac{1}{\sqrt{2}}d\mathcal{W}_2 & d\mathcal{B} - \frac{1}{\sqrt{2}}d\mathcal{W}_1 \end{pmatrix},$$

where $d\mathcal{B}$, $d\mathcal{W}_1$ and $d\mathcal{W}_2$ are independent white noises. Then we investigate the high part of the spectrum of the Anderson Hamiltonian $\mathcal{H}_L := -\partial_t^2 + dB$ on the segment $[0, L]$ with white noise potential $dB$, when $L \to \infty$. We show that the operator $\mathcal{H}_L$, recentred around energy levels $E \sim L/\tau$ and unitarily transformed, converges in law as $L \to \infty$ to $\mathtt{CS}_\tau$ in an appropriate sense. This allows us to answer a conjecture of Rifkind and Virág on the behavior of the eigenvectors of $\mathcal{H}_L$. Our approach also explains how such an operator arises in the limit of $\mathcal{H}_L$. Finally we show that, at higher energy levels, the Anderson Hamiltonian matches (asymptotically in $L$) with the unperturbed Laplacian $-\partial_t^2$. In a companion paper, it is shown that, at energy levels much smaller than $L$, the spectrum is localized with Poisson statistics: the present paper, therefore, identifies the delocalized phase of the Anderson Hamiltonian.

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# UNIVERSALITY OF SPIN CORRELATIONS IN THE ISING MODEL ON ISORADIAL GRAPHS

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We prove universality of spin correlations in the scaling limit of the planar Ising model on isoradial graphs with uniformly bounded angles and Z-invariant weights. Specifically, we show that in the massive scaling limit, that is, as the mesh size tends to zero at the same rate as the Baxter elliptic parameter tends to 1, the two-point spin correlations in the full plane converge to a universal rotationally invariant limit.

These results, together with techniques developed to obtain them, are sufficient to extend to isoradial graphs, the convergence results for multipoint spin correlations in bounded planar domains which were previously known only on the square grid. We also give a simple proof of the fact that the infinite-volume magnetization in a subcritical Z-invariant Ising model is independent of the site and of the lattice.

As compared to techniques already existing in the literature, we streamline the analysis of discrete (massive) holomorphic spinors near their ramification points which also provides a solid ground for further generalizations.

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# UNICELLULAR MAPS VS. HYPERBOLIC SURFACES IN LARGE GENUS: SIMPLE CLOSED CURVES

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We study uniformly random maps with a single face, genus $g$, and size $n$, as $n, g \to \infty$ with $g = o(n)$, in continuation of several previous works on the geometric properties of "high genus maps." We calculate the number of short simple cycles, and we show convergence of their lengths (after a well-chosen rescaling of the graph distance) to a Poisson process, which happens to be exactly the same as the limit law obtained by Mirzakhani and Petri (*Comment. Math. Helv.* **94** (2019) 869–889) when they studied simple closed geodesics on random hyperbolic surfaces under the Weil–Petersson measure as $g \to \infty$.

This leads us to conjecture that these two models are somehow "the same" in the limit, which would allow to translate problems on hyperbolic surfaces in terms of random trees, thanks to a powerful bijection of Chapuy, Féray and Fusy (*J. Combin. Theory Ser. A* **2013** (120) 2064–2092).

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# LAW OF ITERATED LOGARITHMS AND FRACTAL PROPERTIES OF THE KPZ EQUATION

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We consider the Cole–Hopf solution of the $(1+1)$-dimensional KPZ equation started from the narrow wedge initial condition. In this article we ask how the peaks and valleys of the KPZ height function (centered by time/24) at any spatial point grow as time increases. Our first main result is about the law of iterated logarithms for the KPZ equation. As time variable $t$ goes to $\infty$, we show that the limsup of the KPZ height function with the scaling by $t^{1/3}(\log\log t)^{2/3}$ is almost surely equal to $(3/4\sqrt{2})^{2/3}$, whereas the liminf of the height function with the scaling by $t^{1/3}(\log\log t)^{1/3}$ is almost surely equal to $-6^{1/3}$. Our second main result concerns with the *macroscopic* fractal properties of the KPZ equation. Under exponential transformation of the time variable, we show that the peaks of KPZ height function mutate from being monofractal to multifractal, a property reminiscent of a similar phenomenon in Brownian motion (Theorem 1.4 in *Ann. Probab.* **45** (2017) 3697–3751).

The proofs of our main results hinge on the following three key tools: (1) a *multipoint composition law* of the KPZ equation which can be regarded as a generalization of the two point composition law from (Proposition 2.9 in *Ann. Probab.* **49** (2021) 832–876), (2) the Gibbsian line ensemble techniques from (*Invent. Math.* **195** (2014) 441–508; *Probab. Theory Related Fields* **166** (2016) 67–185; *Ann. Probab.* **49** (2021) 832–876), and (3) the tail probabilities of the KPZ height function in short time and its spatiotemporal modulus of continuity. We advocate this last tool as one of our new and important contributions which might garner independent interest.

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# IMPROVED LOG-CONCAVITY FOR ROTATIONALLY INVARIANT MEASURES OF SYMMETRIC CONVEX SETS

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We prove that the (B) conjecture and the Gardner–Zvavitch conjecture are true for all log-concave measures that are rotationally invariant, extending previous results known for Gaussian measures. Actually, our result apply beyond the case of log-concave measures, for instance, to Cauchy measures as well. For the proof, new sharp weighted Poincaré inequalities are obtained for even probability measures that are log-concave with respect to a rotationally invariant measure.

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# TAP APPROACH FOR MULTISPECIES SPHERICAL SPIN GLASSES II: THE FREE ENERGY OF THE PURE MODELS

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In a companion paper we developed the generalized TAP approach for general multispecies spherical mixed $p$-spin models. In this paper we use it to compute the limit of the free energy at any temperature for all pure multispecies spherical $p$-spin models, assuming that certain free energies converge. Importantly, the pure multispecies models do not satisfy the convexity assumption on the mixture which was crucial in the recent proofs of the Parisi formula for the multispecies Sherrington–Kirkpatrick model by Barra et al. (*Ann. Henri Poincaré* **16** (2015) 691–708) and Panchenko (*Ann. Probab.* **43** (2015) 3494–3513) and for the multispecies spherical mixed $p$-spin models by Bates and Sohn (*Electron. J. Probab.* **27** (2022) Paper No. 52; *Comm. Math. Phys.* **394** (2022) 1101–1152).

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# NUMBER-RIGIDITY AND $\beta$-CIRCULAR RIESZ GAS

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For an inverse temperature $\beta > 0$, we define the $\beta$-circular Riesz gas on $\mathbb{R}^d$ as any microscopic thermodynamic limit of Gibbs particle systems on the torus interacting via the Riesz potential $g(x) = \|x\|^{-s}$. We focus on the nonintegrable case $d - 1 < s < d$. Our main result ensures, for any dimension $d \geq 1$ and inverse temperature $\beta > 0$, the existence of a $\beta$-circular Riesz gas which is not number-rigid. Recall that a point process is said number rigid if the number of points in a bounded Borel set $\Delta$ is a function of the point configuration outside $\Delta$. It is the first time that the nonnumber-rigidity is proved for a Gibbs point process interacting via a nonintegrable potential. We follow a statistical physics approach based on the canonical DLR equations. It is inspired by the recent paper (*Comm. Pure Appl. Math.* **74** (2021) 172–222) where the authors prove the number-rigidity of the $\text{Sine}_\beta$ process.

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# LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF THE FRACTIONAL BROWNIAN MOTION

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We investigate first and second order fluctuations of additive functionals of a fractional Brownian motion (fBm) of the form

$$\left\{\int_0^t f\big(n^H(B_s-\lambda)\big)\,ds; t\geq 0\right\}, \tag{0.1}$$

where $B=\{B_t;t\geq 0\}$ is a fBm with Hurst parameter $H\in(0,1)$, $f$ is a suitable test function and $\lambda\in\mathbb{R}$. We develop our study by distinguishing two regimes which exhibit different behaviors. When $H\in(0,1/3)$, we show that a suitable renormalization of (0.1), compensated by a multiple of the local time of $B$, converges toward a constant multiple of the derivative of the local time of $B$. In contrast, in the case $H\in[1/3,1)$ we show that (0.1) converges toward an independent Brownian motion subordinated to the local time of $B$. Our results refine and complement those from (*Ann. Appl. Probab.* **31** (2021) 2143–2191), (Jeganathani (2006)), (*Ann. Probab.* **42** (2014) 168–203), (*Electron. Commun. Probab.* **74** (2013) 18) and solve at the same time the critical case $H=1/3$ which had remained open until now.

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# MONOTONICITY FOR CONTINUOUS-TIME RANDOM WALKS

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Consider continuous-time random walks on Cayley graphs where the rate assigned to each edge depends only on the corresponding generator. We show that the limiting speed is monotone increasing in the rates for infinite Cayley graphs that arise from Coxeter systems but not for all Cayley graphs. On finite Cayley graphs, we show that the distance—in various senses—to stationarity is monotone decreasing in the rates for Coxeter systems and for abelian groups but not for all Cayley graphs. We also find several examples of surprising behaviour in the dependence of the distance to stationarity on the rates. This includes a counterexample to a conjecture on entropy of Benjamini, Lyons, and Schramm. We also show that the expected distance at any fixed time for random walks on $\mathbb{Z}^+$ is monotone increasing in the rates for arbitrary rate functions, which is not true on all of $\mathbb{Z}$. Various intermediate results are also of interest.

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# KPZ-TYPE FLUCTUATION EXPONENTS FOR INTERACTING DIFFUSIONS IN EQUILIBRIUM

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We consider systems of $N$ diffusions in equilibrium interacting through a potential $V$. We study a "height function," which, for the special choice $V(x) = \mathrm{e}^{-x}$, coincides with the partition function of a stationary semidiscrete polymer, also known as the (stationary) O'Connell–Yor polymer. For a general class of smooth convex potentials (generalizing the O'Connell–Yor case), we obtain the order of fluctuations of the height function by proving matching upper and lower bounds for the variance of order $N^{2/3}$, the expected scaling for models lying in the KPZ universality class. The models we study are not expected to be integrable, and our methods are analytic and nonperturbative, making no use of explicit formulas or any results for the O'Connell–Yor polymer.

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