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SECULAR COEFFICIENTS AND THE HOLOMORPHIC MULTIPLICATIVE CHAOS

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We study the *secular coefficients* of $N \times N$ random unitary matrices U_N drawn from the Circular β-Ensemble which are defined as the coefficients of $\{z^n\}$ in the characteristic polynomial $\det(1 - zU_N^*)$. When $\beta > 4$, we obtain a new class of limiting distributions that arise when both n and N tend to infinity simultaneously. We solve an open problem of Diaconis and Gamburd (*Electron. J. Combin.* **11** (2004/06) 2) by showing that, for $\beta = 2$, the middle coefficient of degree $n = \lfloor \frac{N}{2} \rfloor$ tends to zero as $N \to \infty$. We show how the theory of Gaussian multiplicative chaos (GMC) plays a prominent role in these problems and in the explicit description of the obtained limiting distributions. We extend the remarkable *magic square formula* of (*Electron. J. Combin.* **11** (2004/06) 2) for the moments of secular coefficients to all $\beta > 0$ and analyse the asymptotic behaviour of the moments. We obtain estimates on the order of magnitude of the secular coefficients for all $\beta > 0$, and we prove these estimates are sharp when $\beta \geq 2$ and N is sufficiently large with respect to n. These insights motivated us to introduce a new stochastic object associated with the secular coefficients, which we call *Holomorphic Multiplicative Chaos (HMC)*. Viewing the HMC as a random distribution, we prove a sharp result about its regularity in an appropriate Sobolev space. Our proofs expose and exploit several novel connections with other areas, including random permutations, Tauberian theorems and combinatorics.

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LOCALITY OF RANDOM DIGRAPHS ON EXPANDERS

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We study random digraphs on sequences of expanders with a bounded average degree which converge locally in probability. We prove that the relative size and the threshold for the existence of a giant strongly connected component as well as the asymptotic fraction of nodes with giant fan-in or nodes with giant fan-out are local, in the sense that they are the same for two sequences with the same local limit. The digraph has a bow-tie structure, with all but a vanishing fraction of nodes lying either in the unique strongly connected giant and its fan-in and fan-out or in sets with small fan-in and small fan-out. All local quantities are expressed in terms of percolation on the limiting rooted graph, without any structural assumptions on the limit, allowing, in particular, for nontree-like graphs.

In the course of establishing these results, we generalize previous results on the locality of the size of the giant to expanders of bounded average degree with possibly nontree-like limits. We also show that, regardless of the local convergence of a sequence, the uniqueness of the giant and convergence of its relative size for unoriented percolation imply the bow-tie structure for directed percolation.

An application of our methods shows that the critical threshold for bond percolation and random digraphs on preferential attachment graphs is $p_c = 0$ with an infinite order phase transition at p_c.

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CONCURRENT DONSKER–VARADHAN AND HYDRODYNAMICAL LARGE DEVIATIONS

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We consider the weakly asymmetric exclusion process on the d-dimensional torus. We prove a large deviations principle for the time averaged empirical density and current in the joint limit in which both the time interval and the number of particles diverge. This result is obtained both by analyzing the variational convergence, as the number of particles diverges, of the Donsker–Varadhan functional for the empirical process and by considering the large time behavior of the hydrodynamical rate function. The large deviations asymptotic of the time averaged current is then deduced by contraction principle. The structure of the minimizers of this variational problem corresponds to the possible occurrence of dynamical phase transitions.

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MIXING TIMES FOR THE TASEP IN THE MAXIMAL CURRENT PHASE

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We study mixing times for the totally asymmetric simple exclusion process (TASEP) on a segment of size N with open boundaries. We focus on the maximal current phase and prove that the mixing time is of order $N^{3/2}$, up to logarithmic corrections. In the triple point, where the TASEP with open boundaries approaches the Uniform distribution on the state space, we show that the mixing time is precisely of order $N^{3/2}$. This is conjectured to be the correct order of the mixing time for a wide range of particle systems with maximal current. Our arguments rely on a connection to last passage percolation and recent results on moderate deviations of last passage times.

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LARGE DEVIATION EXPANSIONS FOR THE COEFFICIENTS OF RANDOM WALKS ON THE GENERAL LINEAR GROUP

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Consider $(g_n)_{n\geq 1}$ a sequence of independent and identically distributed random matrices and the left random walk $G_n := g_n \dots g_1$, $n \geq 1$ on the general linear group $\mathrm{GL}(d, \mathbb{R})$. Under suitable conditions we establish a Bahadur–Rao–Petrov type large deviation expansion for the coefficient $\langle f, G_n v\rangle$ of the product G_n, where $v \in \mathbb{R}^d$ and $f \in (\mathbb{R}^d)^*$. In particular, we obtain an explicit rate function in the large deviation principle, thus improving significantly the known large deviation bounds. A local limit theorem with large deviations for the coefficients and large deviation expansions under the change of probability measure are also proved.

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BALANCED EXCITED RANDOM WALK IN TWO DIMENSIONS

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We give nontrivial upper and lower bounds on the range of the so-called Balanced Excited Random Walk in two dimensions and verify a conjecture of Benjamini, Kozma and Schapira. To the best of our knowledge, these are the first nontrivial results for this two-dimensional model.

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A LANDSCAPE OF PEAKS: THE INTERMITTENCY ISLANDS OF THE STOCHASTIC HEAT EQUATION WITH LÉVY NOISE

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We show that the spatial profile of the solution to the stochastic heat equation features multiple layers of intermittency islands if the driving noise is non-Gaussian. On the one hand, as expected, if the noise is sufficiently heavy-tailed, the largest peaks of the solution will be taller under multiplicative than under additive noise. On the other hand, surprisingly, as soon as the noise has a finite moment of order $\frac{2}{d}$, where d is the spatial dimension, the largest peaks will be of the same order for both additive and multiplicative noise, which is in sharp contrast to the behavior of the solution under Gaussian noise. However, in this case a closer inspection reveals a second layer of peaks, beneath the largest peaks, that is exclusive to multiplicative noise and that can be observed by sampling the solution on the lattice. Finally, we compute the macroscopic Hausdorff and Minkowski dimensions of the intermittency islands of the solution. Under both additive and multiplicative noise, if it is not too heavy-tailed, the largest peaks will be self-similar in terms of their large-scale multifractal behavior. But under multiplicative noise, this type of self-similarity is not present in the peaks observed on the lattice.

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MONOTONE SUBSEQUENCES IN LOCALLY UNIFORM RANDOM PERMUTATIONS

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A locally uniform random permutation is generated by sampling n points independently from some absolutely continuous distribution ρ on the plane and interpreting them as a permutation by the rule that i maps to j if the ith point from the left is the jth point from below. As n tends to infinity, decreasing subsequences in the permutation will appear as curves in the plane, and by interpreting these as level curves, a union of decreasing subsequences give rise to a surface. We show that, under the correct scaling, for any $r \ge 0$, the largest union of $\lfloor r\sqrt{n} \rfloor$ decreasing subsequences approaches a limit surface as n tends to infinity, and the limit surface is a solution to a specific variational problem. As a corollary, we prove the existence of a limit shape for the Young diagram associated to the random permutation under the Robinson–Schensted correspondence. In the special case where ρ is the uniform distribution on the diamond $|x| + |y| < 1$, we conjecture that the limit shape is triangular, and assuming the conjecture is true, we find an explicit formula for the limit surfaces of a uniformly random permutation and recover the famous limit shape of Vershik, Kerov and Logan, Shepp.

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ON TAIL TRIVIALITY OF NEGATIVELY DEPENDENT STOCHASTIC PROCESSES

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We prove that every negatively associated sequence of Bernoulli random variables with "summable covariances" has a trivial tail σ-field. A corollary of this result is the tail triviality of strongly Rayleigh processes. This is a generalization of a result due to Lyons, which establishes tail triviality for discrete determinantal processes. We also study the tail behavior of negatively associated Gaussian and Gaussian threshold processes. We show that these processes are tail trivial though, in general, they do not satisfy the summable covariances property. Furthermore, we construct negatively associated Gaussian threshold vectors that are not strongly Rayleigh. This identifies a natural family of negatively associated measures that is not a subset of the class of strongly Rayleigh measures.

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EXPONENTIAL MIXING FOR RANDOM DYNAMICAL SYSTEMS AND AN EXAMPLE OF PIERREHUMBERT

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We consider the question of exponential mixing for random dynamical systems on arbitrary compact manifolds without boundary. We put forward a robust, dynamics-based framework that allows us to construct space-time smooth, uniformly bounded in time, universal exponential mixers. The framework is then applied to the problem of proving exponential mixing in a classical example proposed by Pierrehumbert in 1994, consisting of alternating periodic shear flows with randomized phases. This settles a longstanding open problem on proving the existence of a space-time smooth (universal) exponentially mixing incompressible velocity field on a two-dimensional periodic domain while also providing a toolbox for constructing such smooth universal mixers in all dimensions.

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Future Issues