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DECAY OF CONVOLVED DENSITIES VIA LAPLACE TRANSFORM

BY SERGEY G. BOBKOV[a]

School of Mathematics, University of Minnesota, [a]*bobkov@math.umn.edu*

Upper pointwise bounds are considered for convolution of bounded densities in terms of the associated Laplace and Legendre transforms. Applications of these bounds are illustrated in the central limit theorem with respect to the Rényi divergence.

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UNIVERSALITY OF APPROXIMATE MESSAGE PASSING WITH SEMIRANDOM MATRICES

BY RISHABH DUDEJA[1,a], YUE M. LU[2,c] AND SUBHABRATA SEN[1,b]

[1]*Department of Statistics, Harvard University,* [a]*rd2714@columbia.edu,* [b]*subhabratasen@fas.harvard.edu*
[2]*John A. Paulson School of Engineering and Applied Sciences, Harvard University,* [c]*yuelu@seas.harvard.edu*

Approximate Message Passing (AMP) is a class of iterative algorithms that have found applications in many problems in high-dimensional statistics and machine learning. In its general form, AMP can be formulated as an iterative procedure driven by a matrix **M**. Theoretical analyses of AMP typically assume strong distributional properties on **M**, such as **M** has i.i.d. sub-Gaussian entries or is drawn from a rotational invariant ensemble. However, numerical experiments suggest that the behavior of AMP is universal as long as the eigenvectors of **M** are generic. In this paper we take the first step in rigorously understanding this universality phenomenon. In particular, we investigate a class of memory-free AMP algorithms (proposed by Çakmak and Opper for mean-field Ising spin glasses) and show that their asymptotic dynamics is universal on a broad class of semirandom matrices. In addition to having the standard rotational invariant ensemble as a special case, the class of semirandom matrices that we define in this work also includes matrices constructed with very limited randomness. One such example is a randomly signed version of the sine model, introduced by Marinari, Parisi, Potters, and Ritort for spin glasses with fully deterministic couplings.

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CONVERGENCE AND NONCONVERGENCE OF SCALED SELF-INTERACTING RANDOM WALKS TO BROWNIAN MOTION PERTURBED AT EXTREMA

BY ELENA KOSYGINA[1,a], THOMAS MOUNTFORD[2,b] AND JONATHON PETERSON[3,c]

[1]*Department of Mathematics, Baruch College,* [a]*elena.kosygina@baruch.cuny.edu*
[2]*Department of Mathematics, École Polytechnique Fédéral de Lausanne,* [b]*thomas.mountford@epfl.ch*
[3]*Department of Mathematics, Purdue University,* [c]*peterson@purdue.edu*

We use generalized Ray–Knight theorems, introduced by B. Tóth in 1996, together with techniques developed for excited random walks as main tools for establishing positive and negative results concerning convergence of some classes of diffusively scaled self-interacting random walks (SIRW) to Brownian motions perturbed at extrema (BMPE). Tóth's work studied two classes of SIRWs: asymptotically free and polynomially self-repelling walks. For both classes Tóth has shown, in particular, that the distribution function of a scaled SIRW observed at independent geometric times converges to that of a BMPE indicated by the generalized Ray–Knight theorem for this SIRW. The question of weak convergence of one-dimensional distributions of scaled SIRW remained open. In this paper, on the one hand, we prove a full functional limit theorem for a large class of *asymptotically free* SIRWs, which includes the asymptotically free walks considered by Tóth. On the other hand, we show that rescaled *polynomially self-repelling* SIRWs do not converge to the BMPE predicted by the corresponding generalized Ray–Knight theorems and hence do not converge to any BMPE.

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ON STRONG SOLUTIONS OF ITÔ'S EQUATIONS WITH $D\sigma$ AND b IN MORREY CLASSES CONTAINING L_d

BY N.V. KRYLOV[a]

School of Mathematics, University of Minnesota, [a]*nkrylov@umn.edu*

We consider Itô uniformly nondegenerate equations with time independent coefficients, the diffusion coefficient in $W^1_{2+\varepsilon,\text{loc}}$, and the drift in a Morrey class containing L_d. We prove the unique strong solvability in the class of admissible solutions for any starting point. The result is new even if the diffusion is constant.

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PERTURBATIONS OF PARABOLIC EQUATIONS AND DIFFUSION PROCESSES WITH DEGENERATION: BOUNDARY PROBLEMS, METASTABILITY, AND HOMOGENIZATION

BY MARK FREIDLIN[a] AND LEONID KORALOV[b]

Department of Mathematics, University of Maryland, [a]*mif@umd.edu,* [b]*koralov@umd.edu*

We study diffusion processes that are stopped or reflected on the boundary of a domain. The generator of the process is assumed to contain two parts: the main part that degenerates on the boundary in a direction orthogonal to the boundary and a small nondegenerate perturbation. The behavior of such processes determines the stabilization of solutions to the corresponding parabolic equations with a small parameter. Metastability effects arise in this case: the asymptotics of solutions, as the size of the perturbation tends to zero, depends on the time scale. Initial-boundary value problems with both the Dirichlet and the Neumann boundary conditions are considered. We also consider periodic homogenization for operators with degeneration.

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THE PHASE TRANSITION FOR PLANAR GAUSSIAN PERCOLATION MODELS WITHOUT FKG

BY STEPHEN MUIRHEAD[1,a], ALEJANDRO RIVERA[2,b], HUGO VANNEUVILLE[3,c] AND LAURIN KÖHLER-SCHINDLER[3,d]

[1]*School of Mathematical Sciences, Queen Mary University of London,* [a]*smui@unimelb.edu.au*
[2]*Institute of Mathematics, EPFL,* [b]*alejandro.rivera@epfl.ch*
[3]*Department of Mathematics, ETH Zürich,* [c]*hugo.vanneuville@univ-grenoble-alpes.fr,*
[d]*laurin.koehler-schindler@math.ethz.ch*

We develop techniques to study the phase transition for planar Gaussian percolation models that are *not* (necessarily) positively correlated. These models lack the property of positive associations (also known as the 'FKG inequality'), and hence many classical arguments in percolation theory do not apply. More precisely, we consider a smooth stationary centred planar Gaussian field f and, given a level $\ell \in \mathbb{R}$, we study the connectivity properties of the excursion set $\{f \geq -\ell\}$. We prove the existence of a phase transition at the critical level $\ell_{\text{crit}} = 0$ under only symmetry and (very mild) correlation decay assumptions, which are satisfied by the *random plane wave* for instance. As a consequence, all nonzero level lines are bounded almost surely, although our result does not settle the boundedness of zero level lines ('no percolation at criticality').

To show our main result: (i) we prove a general sharp threshold criterion, inspired by works of Chatterjee, that states that 'sharp thresholds are equivalent to the delocalisation of the threshold location'; (ii) we prove threshold delocalisation for crossing events at large scales—at this step we obtain a sharp threshold result but without being able to locate the threshold—and (iii) to identify the threshold, we adapt Tassion's RSW theory replacing the FKG inequality by a sprinkling procedure. Although some arguments are specific to the Gaussian setting, many steps are very general and we hope that our techniques may be adapted to analyse other models without FKG.

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STATIONARY MEASURES FOR THE LOG-GAMMA POLYMER AND KPZ EQUATION IN HALF-SPACE

BY GUILLAUME BARRAQUAND[1,a] AND IVAN CORWIN[2,b]

[1]*Laboratoire de Physique de l'École normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université Paris-Cité,* [a]*guillaume.barraquand@ens.fr*

[2]*Department of Mathematics, Columbia University,* [b]*ivan.corwin@gmail.com*

We construct explicit one-parameter families of stationary measures for the Kardar–Parisi–Zhang equation in half-space with Neumann boundary conditions at the origin, as well as for the log-gamma polymer model in a half-space. The stationary measures are stochastic processes that depend on the boundary condition as well as a parameter related to the drift at infinity. They are expressed in terms of exponential functionals of Brownian motions and gamma random walks. We conjecture that these constitute all extremal stationary measures for these models. The log-gamma polymer result is proved through a symmetry argument related to half-space Whittaker processes which we expect may be applicable to other integrable models. The KPZ result comes as an intermediate disorder limit of the log-gamma polymer result and confirms the conjectural description of these stationary measures from Barraquand and Le Doussal (2021). To prove the intermediate disorder limit, we provide a general half-space polymer convergence framework that extends works of (*J. Stat. Phys.* **181** (2020) 2372–2403; *Electron. J. Probab.* **27** (2022) Paper No. 45; *Ann. Probab.* **42** (2014) 1212–1256).

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SCALING LIMIT OF THE HEAVY TAILED BALLISTIC DEPOSITION MODEL WITH *p*-STICKING

BY FRANCIS COMETS[1,a], JOSEBA DALMAU[2,b] AND SANTIAGO SAGLIETTI[3,c]

This paper is dedicated to the memory of author Francis Comets, who passed away in June 2022

[1]*Mathématiques, Université de Paris,* [a]*comets@lpsm.paris*
[2]*NYU–ECNU Institute of Mathematics, NYU Shanghai,* [b]*dalmau.joseba@gmail.com*
[3]*Facultad de Matemáticas, Pontificia Universidad Católica de Chile,* [c]*sasaglietti@mat.uc.cl*

Ballistic deposition is a classical model for interface growth in which unit blocks fall down vertically at random on the different sites of $\mathbb{Z}$ and stick to the interface at the first point of contact, causing it to grow. We consider an alternative version of this model in which the blocks have random heights which are i.i.d. and heavy tailed, and where each block sticks to the interface at the first point of contact with probability p (otherwise, it falls straight down until it lands on a block belonging to the interface). We study scaling limits of the resulting interface for the different values of p and show that there is a phase transition as p goes from 1 to 0.

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MOST TRANSIENT RANDOM WALKS HAVE INFINITELY MANY CUT TIMES

BY NOAH HALBERSTAM[1,a] AND TOM HUTCHCROFT[2,b]

[1]*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge,* [a]*nh448@cam.ac.uk*
[2]*The Division of Physics, Mathematics and Astronomy, California Institute of Technology,* [b]*t.hutchcroft@caltech.edu*

We prove that if $(X_n)_{n\geq 0}$ is a random walk on a transient graph such that the Green's function decays at least polynomially along the random walk, then $(X_n)_{n\geq 0}$ has infinitely many cut times almost surely. This condition applies in particular to any graph of spectral dimension strictly larger than 2. In fact, our proof applies to general (possibly nonreversible) Markov chains satisfying a similar decay condition for the Green's function that is sharp for birth–death chains. We deduce that a conjecture of Diaconis and Freedman (*Ann. Probab.* **8** (1980) 115–130) holds for the same class of Markov chains, and resolve a conjecture of Benjamini, Gurel-Gurevich, and Schramm (*Ann. Probab.* **39** (2011) 1122–1136) on the existence of infinitely many cut times for random walks of positive speed.

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GLOBAL INFORMATION FROM LOCAL OBSERVATIONS OF THE NOISY VOTER MODEL ON A GRAPH

BY ITAI BENJAMINI[a], HAGAI HELMAN TOV[b] AND MAKSIM ZHUKOVSKII[c]

Weizmann Institute of Science, [a]*itai.benjamini@weizmann.ac.il,* [b]*hagai.helman@gmail.com,* [c]*zhukmax@gmail.com*

We observe the outcome of the discrete time noisy voter model at a single vertex of a graph. We show that certain pairs of graphs can be distinguished by the frequency of repetitions in the sequence of observations. We prove that this statistic is asymptotically normal and that it distinguishes between (asymptotically) almost all pairs of finite graphs. We conjecture that the noisy voter model distinguishes between any two graphs other than stars.

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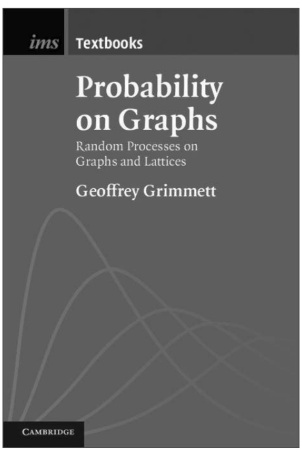

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