ISSN 0091-1798 (print)
ISSN 2168-894X (online)

# THE ANNALS *of* PROBABILITY

*AN OFFICIAL JOURNAL OF THE*
INSTITUTE OF MATHEMATICAL STATISTICS

**Articles**



**Erratum**



Vol. 51, No. 6—November 2023

THE ANNALS OF PROBABILITY Vol. 51, No. 6, pp. 1993–2395 November 2023

# INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

*The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.*

---




***The Annals of Probability*** **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 51, Number 6, November 2023. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, OH 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

**POSTMASTER:** Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, MD 21769, USA.

*The Annals of Probability*
2023, Vol. 51, No. 6, 1993–2055
https://doi.org/10.1214/23-AOP1632


# PARKING ON CAYLEY TREES AND FROZEN ERDŐS–RÉNYI

BY ALICE CONTAT[a] AND NICOLAS CURIEN[b]

*Laboratoire de Mathématique d'Orsay, Université Paris-Saclay,* [a]*alice.contat@universite-paris-saclay.fr,* [b]*nicolas.curien@gmail.com*

Consider a uniform rooted Cayley tree $T_n$ with $n$ vertices and let $m$ cars arrive sequentially, independently, and uniformly on its vertices. Each car tries to park on its arrival node, and if the spot is already occupied, it drives towards the root of the tree and parks as soon as possible. Lackner and Panholzer (*J. Combin. Theory Ser. A* **142** (2016) 1–28) established a phase transition for this process when $m \approx \frac{n}{2}$. In this work, we couple this model with a variant of the classical Erdős–Rényi random graph process. This enables us to describe the phase transition for the size of the components of parked cars using a modification of the multiplicative coalescent which we name the *frozen multiplicative coalescent*. The geometry of critical parked clusters is also studied. Those trees are very different from Bienaymé–Galton–Watson trees and should converge towards the growth-fragmentation trees canonically associated to the 3/2-stable process that already appeared in the study of random planar maps.

## REFERENCES

[1] ADDARIO-BERRY, L. (2019). A probabilistic approach to block sizes in random maps. *ALEA Lat. Am. J. Probab. Math. Stat.* **16** 1–13. MR3903022 https://doi.org/10.30757/alea.v16-01

[2] ADDARIO-BERRY, L., BROUTIN, N., GOLDSCHMIDT, C. and MIERMONT, G. (2017). The scaling limit of the minimum spanning tree of the complete graph. *Ann. Probab.* **45** 3075–3144. MR3706739 https://doi.org/10.1214/16-AOP1132

[3] ALDOUS, D. (1997). Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* **25** 812–854. MR1434128 https://doi.org/10.1214/aop/1024404421

[4] ALDOUS, D., MIERMONT, G. and PITMAN, J. (2004). Brownian bridge asymptotics for random $p$-mappings. *Electron. J. Probab.* **9** 37–56. MR2041828 https://doi.org/10.1214/EJP.v9-186

[5] ALDOUS, D. J. (2000). The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.* **128** 465–477. MR1744108 https://doi.org/10.1017/S0305004199004326

[6] ANGEL, O. and SCHRAMM, O. (2003). Uniform infinite planar triangulations. *Comm. Math. Phys.* **241** 191–213. MR2013797 https://doi.org/10.1007/978-1-4419-9675-6_16

[7] ARMENDÁRIZ, I. (2005). Dual fragmentation and multiplicative coagulation. Unpublished preprint.

[8] ARMENDÁRIZ, I. and LOULAKIS, M. (2011). Conditional distribution of heavy tailed random variables on large deviations of their sum. *Stochastic Process. Appl.* **121** 1138–1147. MR2775110 https://doi.org/10.1016/j.spa.2011.01.011

[9] ASMUSSEN, S., FOSS, S. and KORSHUNOV, D. (2003). Asymptotics for sums of random variables with local subexponential behaviour. *J. Theoret. Probab.* **16** 489–518. MR1982040 https://doi.org/10.1023/A:1023535030388

[10] BAHL, R., BARNET, P., JOHNSON, T. and JUNGE, M. (2022). Diffusion-limited annihilating systems and the increasing convex order. *Electron. J. Probab.* **27** Paper No. 84, 19. MR4453592 https://doi.org/10.1214/22-ejp808

[11] BAHL, R., BARNET, P. and JUNGE, M. (2021). Parking on supercritical Galton–Watson trees. *ALEA Lat. Am. J. Probab. Math. Stat.* **18** 1801–1815. MR4332220 https://doi.org/10.30757/alea.v18-67

[12] BANDERIER, C., FLAJOLET, P., SCHAEFFER, G. and SORIA, M. (2001). Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms* **19** 194–246. MR1871555 https://doi.org/10.1002/rsa.10021

*MSC2020 subject classifications.* Primary 60C05; secondary 05C05, 05C80, 05C85.

*Key words and phrases.* Parking process, random trees, Erdős–Rényi random graph, multiplicative coalescent, planar maps.

[13] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564
[14] BERTOIN, J. (2017). Markovian growth-fragmentation processes. *Bernoulli* **23** 1082–1101. MR3606760 https://doi.org/10.3150/15-BEJ770
[15] BERTOIN, J., BUDD, T., CURIEN, N. and KORTCHEMSKI, I. (2018). Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probab. Theory Related Fields* **172** 663–724. MR3877545 https://doi.org/10.1007/s00440-017-0818-5
[16] BERTOIN, J. and CURIEN, N. Scaling limits for branching process with integer types and their conditional versions. (in preparation).
[17] BERTOIN, J., CURIEN, N. and KORTCHEMSKI, I. (2018). Random planar maps and growth-fragmentations. *Ann. Probab.* **46** 207–260. MR3758730 https://doi.org/10.1214/17-AOP1183
[18] BERTOIN, J., CURIEN, N. and KORTCHEMSKI, I. (2021). On conditioning a self-similar growth-fragmentation by its intrinsic area. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1136–1156. MR4260498 https://doi.org/10.1214/20-aihp1110
[19] BHAMIDI, S., BUDHIRAJA, A. and WANG, X. (2014). The augmented multiplicative coalescent, bounded size rules and critical dynamics of random graphs. *Probab. Theory Related Fields* **160** 733–796. MR3278920 https://doi.org/10.1007/s00440-013-0540-x
[20] BHAMIDI, S., VAN DER HOFSTAD, R. and SEN, S. (2018). The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs. *Probab. Theory Related Fields* **170** 387–474. MR3748328 https://doi.org/10.1007/s00440-017-0760-6
[21] BÖTTCHER, B., SCHILLING, R. and WANG, J. (2013). *Lévy Matters. III*: *Lévy-Type Processes*: *Construction*, *Approximation and Sample Path Properties*. *Lecture Notes in Math.* **2099**. Springer, Cham. With a short biography of Paul Lévy by Jean Jacod, Lévy Matters. MR3156646 https://doi.org/10.1007/978-3-319-02684-8
[22] BOUSQUET-MÉLOU, M. (2006). Rational and algebraic series in combinatorial enumeration. In *International Congress of Mathematicians*. *Vol. III* 789–826. Eur. Math. Soc., Zürich. MR2275707
[23] BRITIKOV, V. E. (1988). Asymptotics of the number of forests made up of nonrooted trees. *Mat. Zametki* **43** 672–684, 703. MR0954351 https://doi.org/10.1007/BF01158847
[24] BROUTIN, N., DUQUESNE, T. and WANG, M. (2021). Limits of multiplicative inhomogeneous random graphs and Lévy trees: Limit theorems. *Probab. Theory Related Fields* **181** 865–973. MR4344135 https://doi.org/10.1007/s00440-021-01075-z
[25] BROUTIN, N. and MARCKERT, J.-F. (2016). A new encoding of coalescent processes: Applications to the additive and multiplicative cases. *Probab. Theory Related Fields* **166** 515–552. MR3547745 https://doi.org/10.1007/s00440-015-0665-1
[26] BUTLER, S., GRAHAM, R. and YAN, C. H. (2017). Parking distributions on trees. *European J. Combin.* **65** 168–185. MR3679844 https://doi.org/10.1016/j.ejc.2017.06.003
[27] CHASSAING, P. and LOUCHARD, G. (2002). Phase transition for parking blocks, Brownian excursion and coalescence. *Random Structures Algorithms* **21** 76–119. MR1913079 https://doi.org/10.1002/rsa.10039
[28] CHEN, L. (2021). Enumeration of fully parked trees. arXiv:2103.15770.
[29] CHEN, Q. and GOLDSCHMIDT, C. (2021). Parking on a random rooted plane tree. *Bernoulli* **27** 93–106. MR4177362 https://doi.org/10.3150/20-BEJ1227
[30] CHEN, X., DAGARD, V., DERRIDA, B., HU, Y., LIFSHITS, M. and SHI, Z. (2021). The Derrida–Retaux conjecture on recursive models. *Ann. Probab.* **49** 637–670. MR4255128 https://doi.org/10.1214/20-aop1457
[31] CONCHON-KERJAN, G. and GOLDSCHMIDT, C. (2020). The stable graph: the metric space scaling limit of a critical random graph with iid power-law degrees. arXiv:2002.04954.
[32] CONTAT, A. Parking on random trees via oriented configuration models. (in preparation).
[33] CONTAT, A. (2022). Sharpness of the phase transition for parking on random trees. *Random Structures Algorithms* **61** 84–100. MR4448750 https://doi.org/10.1002/rsa.21061
[34] CONTAT, A. (2022). Surprising identities for the greedy independent set on Cayley trees. *J. Appl. Probab.* **59** 1042–1058. MR4507681 https://doi.org/10.1017/jpr.2022.3
[35] CORI, R. and SCHAEFFER, G. (2003). Description trees and Tutte formulas. *Theoret. Comput. Sci.* **292** 165–183. MR1964632 https://doi.org/10.1016/S0304-3975(01)00221-3
[36] CRANE, E., FREEMAN, N. and TÓTH, B. (2015). Cluster growth in the dynamical Erdős–Rényi process with forest fires. *Electron. J. Probab.* **20** no. 101, 33. MR3407218 https://doi.org/10.1214/EJP.v20-4035
[37] CURIEN, N. (2019). Peeling random planar maps, Saint-Flour course. Available at https://www.imo.universite-paris-saclay.fr/~curien/.

[38] CURIEN, N. and HÉNARD, O. (2022). The phase transition for parking on Galton–Watson trees. *Discrete Anal*. Paper No. 1, 17. MR4396227 https://doi.org/10.19086/da
[39] CURIEN, N. and KORTCHEMSKI, I. (2014). Random non-crossing plane configurations: A conditioned Galton–Watson tree approach. *Random Structures Algorithms* **45** 236–260. MR3245291 https://doi.org/10.1002/rsa.20481
[40] DUCHI, E., GUERRINI, V., RINALDI, S. and SCHAEFFER, G. (2017). Fighting fish: Enumerative properties. *Sém. Lothar. Combin*. **78B** Art. 43, 12. MR3678625
[41] FEDERICO, L., VAN DER HOFSTAD, R., DEN HOLLANDER, F. and HULSHOF, T. (2020). Expansion of percolation critical points for Hamming graphs. *Combin. Probab. Comput*. **29** 68–100. MR4052928 https://doi.org/10.1017/s0963548319000208
[42] FLAJOLET, P., KNUTH, D. E. and PITTEL, B. (1989). The first cycles in an evolving graph. *Discrete Math*. **75** 167–215. MR1001395 https://doi.org/10.1016/0012-365X(89)90087-3
[43] FLAJOLET, P. and SEDGEWICK, R. (2009). *Analytic Combinatorics*. Cambridge Univ. Press.
[44] FRIEZE, A. and KAROŃSKI, M. (2016). *Introduction to Random Graphs*. Cambridge Univ. Press, Cambridge. MR3675279 https://doi.org/10.1017/CBO9781316339831
[45] GOLDSCHMIDT, C. and PRZYKUCKI, M. (2019). Parking on a random tree. *Combin. Probab. Comput*. **28** 23–45. MR3917904 https://doi.org/10.1017/S0963548318000457
[46] GOULDEN, I. P. and JACKSON, D. M. (2008). The KP hierarchy, branched covers, and triangulations. *Adv. Math*. **219** 932–951. MR2442057 https://doi.org/10.1016/j.aim.2008.06.013
[47] IKEADA, I. N. and VATANABÈ, S. (2014). *Stochastic Differential Equations and Diffusion Processes*. Elsevier.
[48] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5
[49] JANSON, S. (2012). Simply generated trees, conditioned Galton–Watson trees, random allocations and condensation. *Probab. Surv.* **9** 103–252. MR2908619 https://doi.org/10.1214/11-PS188
[50] JANSON, S., KNUTH, D. E., ŁUCZAK, T. and PITTEL, B. (1993). The birth of the giant component. *Random Structures Algorithms* **4** 231–358. With an introduction by the editors. MR1220220 https://doi.org/10.1002/rsa.3240040303
[51] JANSON, S. and LUCZAK, M. J. (2008). Susceptibility in subcritical random graphs. *J. Math. Phys*. **49** 125207, 23. MR2484338 https://doi.org/10.1063/1.2982848
[52] JOSEPH, A. (2014). The component sizes of a critical random graph with given degree sequence. *Ann. Appl. Probab.* **24** 2560–2594. MR3262511 https://doi.org/10.1214/13-AAP985
[53] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8
[54] KING, W. and YAN, C. H. (2019). Prime parking functions on rooted trees. *J. Combin. Theory Ser. A* **168** 1–25. MR3960163 https://doi.org/10.1016/j.jcta.2019.05.015
[55] KING, W. and YAN, C. H. (2020). Parking functions on directed graphs and some directed trees. *Electron. J. Combin*. **27** Paper No. 2.48, 15. MR4245103 https://doi.org/10.37236/9051
[56] KISS, D. (2015). Frozen percolation in two dimensions. *Probab. Theory Related Fields* **163** 713–768. MR3418754 https://doi.org/10.1007/s00440-014-0603-7
[57] KONHEIM, A. G. and WEISS, B. (1966). An occupancy discipline and applications. *SIAM J. Appl. Math*. **14** 1266–1274.
[58] KORTCHEMSKI, I. (2015). Limit theorems for conditioned non-generic Galton–Watson trees. *Ann. Inst. Henri Poincaré Probab. Stat*. **51** 489–511. MR3335012 https://doi.org/10.1214/13-AIHP580
[59] LACKNER, M.-L. and PANHOLZER, A. (2016). Parking functions for mappings. *J. Combin. Theory Ser. A* **142** 1–28. MR3499489 https://doi.org/10.1016/j.jcta.2016.03.001
[60] LE GALL, J.-F. and RIERA, A. (2020). Growth-fragmentation processes in Brownian motion indexed by the Brownian tree. *Ann. Probab.* **48** 1742–1784. MR4124524 https://doi.org/10.1214/19-AOP1406
[61] LIGGETT, T. M. (1968). An invariance principle for conditioned sums of independent random variables. *J. Math. Mech*. **18** 559–570. MR0238373 https://doi.org/10.1512/iumj.1969.18.18043
[62] LIMIC, V. (2017). A playful note on spanning and surplus edges. arXiv preprint, arXiv:1703.02574.
[63] LIMIC, V. (2019). The eternal multiplicative coalescent encoding via excursions of Lévy-type processes. *Bernoulli* **25** 2479–2507. MR4003555 https://doi.org/10.3150/18-BEJ1060
[64] LOUF, B. (2019). A new family of bijections for planar maps. *J. Combin. Theory Ser. A* **168** 374–395. MR3979278 https://doi.org/10.1016/j.jcta.2019.06.006
[65] ŁUCZAK, T. and PITTEL, B. (1992). Components of random forests. *Combin. Probab. Comput*. **1** 35–52. MR1167294 https://doi.org/10.1017/S0963548300000067
[66] ŁUCZAK, T., PITTEL, B. and WIERMAN, J. C. (1994). The structure of a random graph at the point of the phase transition. *Trans. Amer. Math. Soc*. **341** 721–748. MR1138950 https://doi.org/10.2307/2154580

[67] MARTIN, J. B. and RÁTH, B. (2017). Rigid representations of the multiplicative coalescent with linear deletion. *Electron. J. Probab.* **22** Paper No. 83, 47. MR3718711 https://doi.org/10.1214/17-EJP100

[68] MARTIN, J. B. and YEO, D. (2018). Critical random forests. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 913–960. MR3845513 https://doi.org/10.30757/alea.v15-35

[69] MOON, J. W. (1970). *Counting Labelled Trees. Canadian Mathematical Monographs* **1**. Canadian Mathematical Congress, Montreal, Que. From lectures delivered to the Twelfth Biennial Seminar of the Canadian Mathematical Congress (Vancouver, 1969). MR0274333

[70] PANHOLZER, A. (2021). Parking function varieties for combinatorial tree models. *Adv. in Appl. Math.* **128** Paper No. 102191, 42. MR4233285 https://doi.org/10.1016/j.aam.2021.102191

[71] PITMAN, J. (1999). Coalescent random forests. *J. Combin. Theory Ser. A* **85** 165–193. MR1673928 https://doi.org/10.1006/jcta.1998.2919

[72] RÁTH, B. (2009). Mean field frozen percolation. *J. Stat. Phys.* **137** 459–499. MR2564286 https://doi.org/10.1007/s10955-009-9863-5

[73] RÁTH, B. and TÓTH, B. (2009). Erdős–Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab.* **14** 1290–1327. MR2511285 https://doi.org/10.1214/EJP.v14-653

[74] REMBART, F. and WINKEL, M. (2018). Recursive construction of continuum random trees. *Ann. Probab.* **46** 2715–2748. MR3846837 https://doi.org/10.1214/17-AOP1237

[75] RÉNYI, A. (1959). Some remarks on univalent functions. *Bulgar. Akad. Nauk. Izv. Mat. Inst.* **3** 111–121. MR0109890

[76] ROSSIGNOL, R. (2021). Scaling limit of dynamical percolation on critical Erdős–Rényi random graphs. *Ann. Probab.* **49** 322–399. MR4203340 https://doi.org/10.1214/20-AOP1472

[77] URIBE, G. (2007). Bravo, Markovian bridges, Brownian excursions, and stochastic fragmentation and coalescence. Ph.D. thesis, UNAM.

[78] ZOLOTAREV, V. M. (1986). *One-Dimensional Stable Distributions. Translations of Mathematical Monographs* **65**. Amer. Math. Soc., Providence, RI. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver. MR0854867 https://doi.org/10.1090/mmono/065

*The Annals of Probability*
2023, Vol. 51, No. 6, 2056–2085
https://doi.org/10.1214/23-AOP1638


# ON THE (NON)STATIONARY DENSITY OF FRACTIONAL-DRIVEN STOCHASTIC DIFFERENTIAL EQUATIONS

BY XUE-MEI LI[1,a], FABIEN PANLOUP[2,c] AND JULIAN SIEBER[1,b]

[1]*Department of Mathematics, Imperial College London,* [a]*xue-mei.li@epfl.ch,* [b]*j.sieber19@imperial.ac.uk*
[2]*Univ. Angers, CNRS, LAREMA, SFR MATHSTIC,* [c]*fabien.panloup@univ-angers.fr*

We investigate the stationary measure $\pi$ of SDEs driven by additive fractional noise with any Hurst parameter and establish that $\pi$ admits a smooth Lebesgue density obeying both Gaussian-type lower and upper bounds. The proofs are based on a novel representation of the stationary density in terms of a Wiener–Liouville bridge, which proves to be of independent interest: We show that it also allows to obtain Gaussian bounds on the nonstationary density, which extend previously known results in the additive setting. In addition, we study a parameter-dependent version of the SDE and prove smoothness of the stationary density, jointly in the parameter and the spatial coordinate. With this, we revisit the fractional averaging principle of Li and Sieber (*Ann. Appl. Probab.* **32** (2022) 3964–4003) and remove an ad hoc assumption on the limiting coefficients. Avoiding any use of Malliavin calculus in our arguments, we can prove our results under minimal regularity requirements.

## REFERENCES

[1] ARNOLD, L. (1998). *Random Dynamical Systems*. *Springer Monographs in Mathematics*. Springer, Berlin. MR1723992 https://doi.org/10.1007/978-3-662-12878-7

[2] ASSARAF, R., JOURDAIN, B., LELIÈVRE, T. and ROUX, R. (2018). Computation of sensitivities for the invariant measure of a parameter dependent diffusion. *Stoch*. *Partial Differ*. *Equ*. *Anal*. *Comput*. **6** 125–183. MR3818403 https://doi.org/10.1007/s40072-017-0105-6

[3] BAUDOIN, F. and COUTIN, L. (2007). Volterra bridges and applications. *Markov Process*. *Related Fields* **13** 587–596. MR2357390

[4] BAUDOIN, F., NUALART, E., OUYANG, C. and TINDEL, S. (2016). On probability laws of solutions to differential systems driven by a fractional Brownian motion. *Ann*. *Probab*. **44** 2554–2590. MR3531675 https://doi.org/10.1214/15-AOP1028

[5] BAUDOIN, F., OUYANG, C. and TINDEL, S. (2014). Upper bounds for the density of solutions to stochastic differential equations driven by fractional Brownian motions. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **50** 111–135. MR3161525 https://doi.org/10.1214/12-AIHP522

[6] BESALÚ, M., KOHATSU-HIGA, A. and TINDEL, S. (2016). Gaussian-type lower bounds for the density of solutions of SDEs driven by fractional Brownian motions. *Ann*. *Probab*. **44** 399–443. MR3456342 https://doi.org/10.1214/14-AOP977

[7] BOGACHEV, V. I. (1998). *Gaussian Measures*. *Mathematical Surveys and Monographs* **62**. Amer. Math. Soc., Providence, RI. MR1642391 https://doi.org/10.1090/surv/062

[8] BOGACHEV, V. I., VERETENNIKOV, A. Y. and SHAPOSHNIKOV, S. V. (2015). Differentiability of invariant measures of diffusions with respect to a parameter. *Dokl*. *Akad*. *Nauk* **460** 507–511. MR3410620 https://doi.org/10.1134/s106456241501024x

[9] CASS, T., HAIRER, M., LITTERER, C. and TINDEL, S. (2015). Smoothness of the density for solutions to Gaussian rough differential equations. *Ann*. *Probab*. **43** 188–239. MR3298472 https://doi.org/10.1214/13-AOP896

[10] CATELLIER, R. and GUBINELLI, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Process*. *Appl*. **126** 2323–2366. MR3505229 https://doi.org/10.1016/j.spa.2016.02.002

[11] CHERIDITO, P., KAWAGUCHI, H. and MAEJIMA, M. (2003). Fractional Ornstein–Uhlenbeck processes. *Electron*. *J*. *Probab*. **8** no. 3. MR1961165 https://doi.org/10.1214/EJP.v8-125

---

*MSC2020 subject classifications.* 60G22, 60H10, 37A25.
*Key words and phrases.* Fractional Brownian motion, parameter-dependent SDE, smoothness of the invariant density, Girsanov theorem, Wiener–Liouville bridge, Gaussian-type bounds, smooth density.

[12] DECREUSEFOND, L. and ÜSTÜNEL, A. S. (1999). Stochastic analysis of the fractional Brownian motion. *Potential Anal*. **10** 177–214. MR1677455 https://doi.org/10.1023/A:1008634027843

[13] DEYA, A., PANLOUP, F. and TINDEL, S. (2019). Rate of convergence to equilibrium of fractional driven stochastic differential equations with rough multiplicative noise. *Ann*. *Probab*. **47** 464–518. MR3909974 https://doi.org/10.1214/18-AOP1265

[14] FERNIQUE, X. (1970). Intégrabilité des vecteurs gaussiens. *C*. *R*. *Acad*. *Sci*. *Paris Sér*. *A-B* **270** A1698–A1699. MR0266263

[15] FONTBONA, J. and PANLOUP, F. (2017). Rate of convergence to equilibrium of fractional driven stochastic differential equations with some multiplicative noise. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **53** 503–538. MR3634264 https://doi.org/10.1214/15-AIHP724

[16] FRIEDMAN, A. (1975). *Stochastic Differential Equations and Applications*. *Vol*. 1. *Probability and Mathematical Statistics*, *Vol*. 28. Academic Press, New York. MR0494490

[17] GALEATI, L. (2023). Nonlinear Young differential equations: A review. *J*. *Dynam*. *Differential Equations* **35** 985–1046. MR4594437 https://doi.org/10.1007/s10884-021-09952-w

[18] GASBARRA, D., SOTTINEN, T. and VALKEILA, E. (2007). Gaussian bridges. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 361–382. Springer, Berlin. MR2397795 https://doi.org/10.1007/978-3-540-70847-6_15

[19] GENG, X., OUYANG, C. and TINDEL, S. (2022). Precise local estimates for differential equations driven by fractional Brownian motion: Hypoelliptic case. *Ann*. *Probab*. **50** 649–687. MR4399160 https://doi.org/10.1214/21-aop1542

[20] GENG, X., OUYANG, C. and TINDEL, S. (2022). Precise local estimates for differential equations driven by fractional Brownian motion: Hypoelliptic case. *Ann*. *Probab*. **50** 649–687. MR4399160 https://doi.org/10.1214/21-aop1542

[21] HAIRER, M. (2005). Ergodicity of stochastic differential equations driven by fractional Brownian motion. *Ann*. *Probab*. **33** 703–758. MR2123208 https://doi.org/10.1214/009117904000000892

[22] HAIRER, M. and OHASHI, A. (2007). Ergodic theory for SDEs with extrinsic memory. *Ann*. *Probab*. **35** 1950–1977. MR2349580 https://doi.org/10.1214/009117906000001141

[23] HAIRER, M. and PILLAI, N. S. (2011). Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **47** 601–628. MR2814425 https://doi.org/10.1214/10-AIHP377

[24] HAIRER, M. and PILLAI, N. S. (2013). Regularity of laws and ergodicity of hypoelliptic SDEs driven by rough paths. *Ann*. *Probab*. **41** 2544–2598. MR3112925 https://doi.org/10.1214/12-AOP777

[25] HARDY, G. H. and LITTLEWOOD, J. E. (1928). Some properties of fractional integrals. I. *Math*. *Z*. **27** 565–606. MR1544927 https://doi.org/10.1007/BF01171116

[26] HU, Y. and NUALART, D. (2007). Differential equations driven by Hölder continuous functions of order greater than 1/2. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 399–413. Springer, Berlin. MR2397797 https://doi.org/10.1007/978-3-540-70847-6_17

[27] LÉVY, P. (1951). Wiener's random function, and other Laplacian random functions. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 171–187. Univ. California Press, Berkeley-Los Angeles, CA. MR0044774

[28] LI, X.-M. (2018). Perturbation of conservation laws and averaging on manifolds. In *Computation and Combinatorics in Dynamics*, *Stochastics and Control*. *Abel Symp*. **13** 499–550. Springer, Cham. MR3967395

[29] LI, X.-M. and SIEBER, J. (2022). Slow-fast systems with fractional environment and dynamics. *Ann*. *Appl*. *Probab*. **32** 3964–4003. MR4498200 https://doi.org/10.1214/22-aap1779

[30] LIFSHITS, M. A., LINDE, W. and SHI, Z. (2006). Small deviations of Riemann–Liouville processes in $L_q$-spaces with respect to fractal measures. *Proc*. *Lond*. *Math*. *Soc*. (3) **92** 224–250. MR2192391 https://doi.org/10.1017/S002461150501556X

[31] MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev*. **10** 422–437. MR0242239 https://doi.org/10.1137/1010093

[32] NOURDIN, I. and SIMON, T. (2006). On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion. *Statist*. *Probab*. *Lett*. **76** 907–912. MR2268434 https://doi.org/10.1016/j.spl.2005.10.021

[33] NUALART, D. and OUKNINE, Y. (2002). Regularization of differential equations by fractional noise. *Stochastic Process*. *Appl*. **102** 103–116. MR1934157 https://doi.org/10.1016/S0304-4149(02)00155-2

[34] NUALART, D. and SAUSSEREAU, B. (2009). Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. *Stochastic Process*. *Appl*. **119** 391–409. MR2493996 https://doi.org/10.1016/j.spa.2008.02.016

[35] PANLOUP, F. and RICHARD, A. (2020). Sub-exponential convergence to equilibrium for Gaussian driven stochastic differential equations with semi-contractive drift. *Electron. J. Probab.* **25** Paper No. 62. MR4112766 https://doi.org/10.1214/20-ejp464

[36] PARDOUX, E. and VERETENNIKOV, A. Y. (2001). On the Poisson equation and diffusion approximation. I. *Ann. Probab.* **29** 1061–1085. MR1872736 https://doi.org/10.1214/aop/1015345596

[37] PARDOUX, È. and VERETENNIKOV, A. Y. (2003). On Poisson equation and diffusion approximation. II. *Ann. Probab.* **31** 1166–1192. MR1988467 https://doi.org/10.1214/aop/1055425774

[38] PARDOUX, E. and VERETENNIKOV, A. Y. (2005). On the Poisson equation and diffusion approximation. III. *Ann. Probab.* **33** 1111–1133. MR2135314 https://doi.org/10.1214/009117905000000062

[39] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[40] SAMKO, S. G., KILBAS, A. A. and MARICHEV, O. I. (1993). *Fractional Integrals and Derivatives*. Gordon & Breach, Yverdon. MR1347689

[41] VERETENNIKOV, A. Y. (2011). On Sobolev solutions of Poisson equations in $\mathbb{R}^d$ with a parameter. *J. Math. Sci.* **179** 48–79. MR3014098 https://doi.org/10.1007/s10958-011-0582-5

*The Annals of Probability*
2023, Vol. 51, No. 6, 2086–2130
https://doi.org/10.1214/23-AOP1639


# LOEWNER EVOLUTION DRIVEN BY COMPLEX BROWNIAN MOTION

BY EWAIN GWYNNE[1,a] JOSHUA PFEFFER[2,c] AND MINJAE PARK[1,b]

[1]*Department of Mathematics, University of Chicago,* [a]*ewain@uchicago.edu,* [b]*minjaep@uchicago.edu*
[2]*Department of Mathematics, Columbia University,* [c]*pfeffer@math.columbia.edu*

We study the Loewner evolution whose driving function is $W_t = B_t^1 + iB_t^2$, where $(B^1, B^2)$ is a pair of Brownian motions with a given covariance matrix. This model can be thought of as a generalization of Schramm–Loewner evolution (SLE) with complex parameter values. We show that our Loewner evolutions behave very differently from ordinary SLE. For example, if neither $B^1$ nor $B^2$ is identically equal to zero, then the set of points disconnected from $\infty$ by the Loewner hull has nonempty interior at each time. We also show that our model exhibits three phases analogous to the phases of SLE: a phase where the hulls have zero Lebesgue measure, a phase where points are swallowed but not hit by the hulls and a phase where the hulls are space-filling. The phase boundaries are expressed in terms of the signs of explicit integrals. These boundaries have a simple closed form when the correlation of the two Brownian motions is zero.

## REFERENCES

[1] ALBERTS, T. and SHEFFIELD, S. (2008). Hausdorff dimension of the SLE curve intersected with the real line. *Electron. J. Probab.* **13** 1166–1188. MR2430703 https://doi.org/10.1214/EJP.v13-515
[2] BEFFARA, V. (2008). The dimension of the SLE curves. *Ann. Probab.* **36** 1421–1452. MR2435854 https://doi.org/10.1214/07-AOP364
[3] BERESTYCKI, N. and NORRIS, J. R. Lectures on Schramm–Loewner evolution. Available at http://www.statslab.cam.ac.uk/~james/Lectures/.
[4] BERESTYCKI, N. and POWELL, E. Gaussian free field, Liouville quantum gravity, and Gaussian multiplicative chaos. Available at https://homepage.univie.ac.at/nathanael.berestycki/Articles/master.pdf.
[5] BRACCI, F., CONTRERAS, M. D. and DÍAZ-MADRIGAL, S. (2012). Evolution families and the Loewner equation I: The unit disc. *J. Reine Angew. Math.* **672** 1–37. MR2995431 https://doi.org/10.1515/crelle.2011.167
[6] CAMIA, F. and NEWMAN, C. M. (2006). Two-dimensional critical percolation: The full scaling limit. *Comm. Math. Phys.* **268** 1–38. MR2249794 https://doi.org/10.1007/s00220-006-0086-1
[7] DING, J. and GWYNNE, E. (2020). Tightness of supercritical Liouville first passage percolation. *J. Eur. Math. Soc. (JEMS)*. To appear.
[8] DING, J. and GWYNNE, E. (2021). Regularity and confluence of geodesics for the supercritical Liouville quantum gravity metric. ArXiv e-prints.
[9] DING, J. and GWYNNE, E. (2023). Uniqueness of the critical and supercritical Liouville quantum gravity metrics. *Proc. Lond. Math. Soc.* (3) **126** 216–333. MR4535021
[10] DUBÉDAT, J. (2009). Duality of Schramm–Loewner evolutions. *Ann. Sci. Éc. Norm. Supér.* (4) **42** 697–724. MR2571956 https://doi.org/10.24033/asens.2107
[11] DUBÉDAT, J. (2009). SLE and the free field: Partition functions and couplings. *J. Amer. Math. Soc.* **22** 995–1054. MR2525778 https://doi.org/10.1090/S0894-0347-09-00636-5
[12] DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2021). Liouville quantum gravity as a mating of trees. *Astérisque* **427** viii+257. MR4340069 https://doi.org/10.24033/ast
[13] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836 https://doi.org/10.1017/CBO9780511779398
[14] HAIRER, M. (2011). On Malliavin's proof of Hörmander's theorem. *Bull. Sci. Math.* **135** 650–666. MR2838095 https://doi.org/10.1016/j.bulsci.2011.07.007

*MSC2020 subject classifications.* 60J67, 30C20.
*Key words and phrases.* Loewner evolution, complex driving function, Schramm–Loewner evolution, conformal mapping, Brownian motion.

[15] HUANG, Y. (2018). Path integral approach to analytic continuation of Liouville theory: The pencil region. ArXiv e-prints.

[16] JUNNILA, J., SAKSMAN, E. and WEBB, C. (2019). Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.* **29** 3786–3820. MR4047992 https://doi.org/10.1214/19-AAP1492

[17] JUNNILA, J., SAKSMAN, E. and WEBB, C. (2020). Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model. *Ann. Appl. Probab.* **30** 2099–2164. MR4149524 https://doi.org/10.1214/19-AAP1553

[18] KENNEDY, T. (2009). Numerical computations for the Schramm–Loewner evolution. *J. Stat. Phys.* **137** 839–856. MR2570752 https://doi.org/10.1007/s10955-009-9866-2

[19] LACOIN, H. (2022). Convergence in law for complex Gaussian multiplicative chaos in phase III. *Ann. Probab.* **50** 950–983. MR4413209 https://doi.org/10.1214/21-aop1551

[20] LACOIN, H., RHODES, R. and VARGAS, V. (2015). Complex Gaussian multiplicative chaos. *Comm. Math. Phys.* **337** 569–632. MR3339158 https://doi.org/10.1007/s00220-015-2362-4

[21] LAWLER, G. F. (2005). *Conformally Invariant Processes in the Plane*. *Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588 https://doi.org/10.1090/surv/114

[22] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2004). Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32** 939–995. MR2044671 https://doi.org/10.1214/aop/1079021469

[23] LAWLER, G. F. and WERNESS, B. M. (2013). Multi-point Green's functions for SLE and an estimate of Beffara. *Ann. Probab.* **41** 1513–1555. MR3098683 https://doi.org/10.1214/11-AOP695

[24] LIND, J. and UTLEY, J. (2022). Phase transition for a family of complex-driven Loewner hulls. *Involve* **15** 447–474. MR4522857 https://doi.org/10.2140/involve.2022.15.447

[25] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[26] MILLER, J. and SHEFFIELD, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2

[27] MILLER, J. and WU, H. (2017). Intersections of SLE paths: The double and cut point dimension of SLE. *Probab. Theory Related Fields* **167** 45–105. MR3602842 https://doi.org/10.1007/s00440-015-0677-x

[28] PFEFFER, J. (2021). Weak Liouville quantum gravity metrics with matter central charge $\mathbf{c} \in (-\infty, 25)$. ArXiv e-prints.

[29] POMMERENKE, C. (1992). *Boundary Behaviour of Conformal Maps*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **299**. Springer, Berlin. MR1217706 https://doi.org/10.1007/978-3-662-02770-7

[30] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[31] ROHDE, S. and SCHRAMM, O. (2005). Basic properties of SLE. *Ann. of Math.* (2) **161** 883–924. MR2153402 https://doi.org/10.4007/annals.2005.161.883

[32] ROHDE, S. and SCHRAMM, S. Unpublished manuscript.

[33] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084 https://doi.org/10.1007/BF02803524

[34] SCHRAMM, O. and SHEFFIELD, S. (2013). A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields* **157** 47–80. MR3101840 https://doi.org/10.1007/s00440-012-0449-9

[35] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 https://doi.org/10.1007/s00440-006-0050-1

[36] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[37] SMIRNOV, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 239–244. MR1851632 https://doi.org/10.1016/S0764-4442(01)01991-7

[38] SMIRNOV, S. (2010). Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math.* (2) **172** 1435–1467. MR2680496 https://doi.org/10.4007/annals.2010.172.1441

[39] TRAN, H. (2017). Loewner equation driven by complex-valued functions. ArXiv e-prints.

[40] WANG, Y. (2022). Large deviations of Schramm–Loewner evolutions: A survey. *Probab. Surv.* **19** 351–403. MR4417203 https://doi.org/10.1214/22-ps9

[41] WERNER, W. (2004). Random planar curves and Schramm–Loewner evolutions. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math.* **1840** 107–195. Springer, Berlin. MR2079672 https://doi.org/10.1007/978-3-540-39982-7_2

[42] ZHAN, D. (2008). Duality of chordal SLE. *Invent. Math.* **174** 309–353. MR2439609 https://doi.org/10.1007/s00222-008-0132-z

[43] ZHAN, D. (2010). Duality of chordal SLE, II. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 740–759. MR2682265 https://doi.org/10.1214/09-AIHP340

*The Annals of Probability*
2023, Vol. 51, No. 6, 2131–2157
https://doi.org/10.1214/23-AOP1641


# MULTISOURCE INVASION PERCOLATION ON THE COMPLETE GRAPH

BY LOUIGI ADDARIO-BERRY[a] AND JORDAN BARRETT[b]

*Department of Mathematics and Statistics, McGill University,* [a]*louigi.addario@mcgill.ca,* [b]*jordan.barrett@mail.mcgill.ca*

We consider invasion percolation on the complete graph $K_n$, started from some number $k(n)$ of distinct source vertices. The outcome of the process is a forest consisting of $k(n)$ trees, each containing exactly one source. Let $M_n$ be the size of the largest tree in this forest. Logan, Molloy and Pralat (2018) proved that if $k(n)/n^{1/3} \to 0$ then $M_n/n \to 1$ in probability. In this paper, we prove a complementary result: if $k(n)/n^{1/3} \to \infty$, then $M_n/n \to 0$ in probability. This establishes the existence of a phase transition in the structure of the invasion percolation forest around $k(n) \asymp n^{1/3}$.

Our arguments rely on the connection between invasion percolation and critical percolation, and on a coupling between multisource invasion percolation with differently-sized source sets. A substantial part of the proof is devoted to showing that, with high probability, a certain fragmentation process on large random binary trees leaves no components of macroscopic size.

## REFERENCES

[1] ADDARIO-BERRY, L. (2013). The local weak limit of the minimum spanning tree of the complete graph. Available at arXiv:1301.1667 [math.PR].

[2] ADDARIO-BERRY, L., BRANDENBERGER, A., HAMDAN, J. and KERRIOU, C. (2022). Universal height and width bounds for random trees. *Electron. J. Probab.* **27** 118. MR4479914 https://doi.org/10.1214/22-ejp842

[3] ADDARIO-BERRY, L., BROUTIN, N. and GOLDSCHMIDT, C. (2010). Critical random graphs: Limiting constructions and distributional properties. *Electron. J. Probab.* **15** 741–775. MR2650781 https://doi.org/10.1214/EJP.v15-772

[4] ADDARIO-BERRY, L., BROUTIN, N. and GOLDSCHMIDT, C. (2012). The continuum limit of critical random graphs. *Probab. Theory Related Fields* **152** 367–406. MR2892951 https://doi.org/10.1007/s00440-010-0325-4

[5] ADDARIO-BERRY, L., BROUTIN, N., GOLDSCHMIDT, C. and MIERMONT, G. (2017). The scaling limit of the minimum spanning tree of the complete graph. *Ann. Probab.* **45** 3075–3144. MR3706739 https://doi.org/10.1214/16-AOP1132

[6] ADDARIO-BERRY, L., BROUTIN, N. and HOLMGREN, C. (2014). Cutting down trees with a Markov chainsaw. *Ann. Appl. Probab.* **24** 2297–2339. MR3262504 https://doi.org/10.1214/13-AAP978

[7] ADDARIO-BERRY, L., GRIFFITHS, S. and KANG, R. J. (2012). Invasion percolation on the Poisson-weighted infinite tree. *Ann. Appl. Probab.* **22** 931–970. MR2977982 https://doi.org/10.1214/11-AAP761

[8] ADDARIO-BERRY, L. and SEN, S. (2021). Geometry of the minimal spanning tree of a random 3-regular graph. *Probab. Theory Related Fields* **180** 553–620. MR4288328 https://doi.org/10.1007/s00440-021-01071-3

[9] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326

[10] ALDOUS, D. (1997). Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* **25** 812–854. MR1434128 https://doi.org/10.1214/aop/1024404421

[11] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*. *Encyclopaedia Math. Sci.* **110** 1–72. Springer, Berlin. MR2023650 https://doi.org/10.1007/978-3-662-09444-0_1

[12] ALDOUS, D. J. (1985). Exchangeability and related topics. In *École D'été de Probabilités de Saint-Flour, XIII—1983*. *Lecture Notes in Math.* **1117** 1–198. Springer, Berlin. MR0883646 https://doi.org/10.1007/BFb0099421

*MSC2020 subject classifications.* Primary 60K35; secondary 60C05, 05C80, 82B43, 82C43.

*Key words and phrases.* Invasion percolation, critical percolation, minimum spanning trees, Erdős–Rényi random graph.

[13] ANGEL, O., GOODMAN, J., DEN HOLLANDER, F. and SLADE, G. (2008). Invasion percolation on regular trees. *Ann. Probab.* **36** 420–466. MR2393988 https://doi.org/10.1214/07-AOP346

[14] ANGEL, O., GOODMAN, J. and MERLE, M. (2013). Scaling limit of the invasion percolation cluster on a regular tree. *Ann. Probab.* **41** 229–261. MR3059198 https://doi.org/10.1214/11-AOP731

[15] BHAMIDI, S. and SEN, S. (2020). Geometry of the vacant set left by random walk on random graphs, Wright's constants, and critical random graphs with prescribed degrees. *Random Structures Algorithms* **56** 676–721. MR4084187 https://doi.org/10.1002/rsa.20880

[16] BHAMIDI, S., VAN DER HOFSTAD, R. and SEN, S. (2018). The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs. *Probab. Theory Related Fields* **170** 387–474. MR3748328 https://doi.org/10.1007/s00440-017-0760-6

[17] CHANDLER, R., KOPLIK, J., LERMAN, K. and WILLEMSEN, J. F. (1982). Capillary displacement and percolation in porous media. *J. Fluid Mech.* **119** 249–267.

[18] DAMRON, M. and SAPOZHNIKOV, A. (2012). Limit theorems for 2D invasion percolation. *Ann. Probab.* **40** 893–920. MR2962082 https://doi.org/10.1214/10-AOP641

[19] GARBAN, C., PETE, G. and SCHRAMM, O. (2018). The scaling limits of the minimal spanning tree and invasion percolation in the plane. *Ann. Probab.* **46** 3501–3557. MR3857861 https://doi.org/10.1214/17-AOP1252

[20] GARBAN, C., PETE, G. and SCHRAMM, O. (2018). The scaling limits of near-critical and dynamical percolation. *J. Eur. Math. Soc.* (*JEMS*) **20** 1195–1268. MR3790067 https://doi.org/10.4171/JEMS/786

[21] KRUSKAL, J. B. JR. (1956). On the shortest spanning subtree of a graph and the traveling salesman problem. *Proc. Amer. Math. Soc.* **7** 48–50. MR0078686 https://doi.org/10.2307/2033241

[22] LOGAN, A., MOLLOY, M. and PRALAT, P. (2018). A variant of the Erdos–Renyi random graph process. Available at arXiv:1806.10975 [math.CO].

[23] ŁUCZAK, T. (1990). Component behavior near the critical point of the random graph process. *Random Structures Algorithms* **1** 287–310. MR1099794 https://doi.org/10.1002/rsa.3240010305

[24] MCDIARMID, C., JOHNSON, T. and STONE, H. S. (1997). On finding a minimum spanning tree in a network with random weights **10** 187–204. MR1611522 https://doi.org/10.1002/(SICI)1098-2418(199701/03)10:1/2<187::AID-RSA10>3.3.CO;2-Y

[25] MICHELEN, M., PEMANTLE, R. and ROSENBERG, J. (2019). Invasion percolation on Galton–Watson trees. *Electron. J. Probab.* **24** 31. MR3940761 https://doi.org/10.1214/19-EJP281

[26] NEWMAN, C. M. and STEIN, D. L. (1995). Random walk in a strongly inhomogeneous environment and invasion percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **31** 249–261. MR1340039

[27] NEWMAN, C. M. and STEIN, D. L. (1996). Ground-state structure in a highly disordered spin-glass model. *J. Stat. Phys.* **82** 1113–1132. MR1372437 https://doi.org/10.1007/BF02179805

[28] NICKEL, B. and WILKINSON, D. (1983). Invasion percolation on the Cayley tree: Exact solution of a modified percolation model. *Phys. Rev. Lett.* **51** 71–74. MR0708864 https://doi.org/10.1103/PhysRevLett.51.71

[29] PRIM, R. C. (1957). Shortest connection networks and some generalizations. *Bell Syst. Tech. J.* **36** 1389–1401.

[30] STARK, C. P. (1991). An invasion percolation model of drainage network evolution. *Nature* **352** 423–425.

[31] VAN DEN BERG, J., JÁRAI, A. A. and VÁGVÖLGYI, B. (2007). The size of a pond in 2D invasion percolation. *Electron. Commun. Probab.* **12** 411–420. MR2350578 https://doi.org/10.1214/ECP.v12-1327

[32] WILKINSON, D. and WILLEMSEN, J. F. (1983). Invasion percolation: A new form of percolation theory. *J. Phys. A* **16** 3365–3376. MR0725616

[33] ZHANG, Y. (1995). The fractal volume of the two-dimensional invasion percolation cluster. *Comm. Math. Phys.* **167** 237–254. MR1316507

*The Annals of Probability*
2023, Vol. 51, No. 6, 2158–2191
https://doi.org/10.1214/23-AOP1642


# ISOMORPHISMS OF POISSON SYSTEMS OVER LOCALLY COMPACT GROUPS

BY AMANDA WILKENS[a]

*For my brother, Daniel, and in memory of him.*

*Mathematics, University of Texas at Austin,* [a]*amanda.wilkens@math.utexas.edu*

A Poisson system is a Poisson point process and a group action, together forming a measure-preserving dynamical system. Ornstein and Weiss proved Poisson systems over many amenable groups were isomorphic in their 1987 paper. We consider Poisson systems over nondiscrete, noncompact, locally compact Polish groups, and we prove by construction all Poisson systems over such a group are finitarily isomorphic, producing examples of isomorphisms for nonamenable group actions. As a corollary, we prove Poisson systems and products of Poisson systems are finitarily isomorphic.

For a Poisson system over a group belonging to a slightly more restrictive class than above, we further prove it splits into two Poisson systems whose intensities sum to the intensity of the original, generalizing the same result for Poisson systems over Euclidean space proved by Holroyd, Lyons and Soo.

## REFERENCES

[1] ABÉRT, M. and MELLICK, S. (2022). Point processes, cost, and the growth of rank in locally compact groups. *Israel J. Math.* **251** 47–154. MR4555892 https://doi.org/10.1007/s11856-022-2445-9

[2] BOWEN, L. (2010). Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc.* **23** 217–245. MR2552252 https://doi.org/10.1090/S0894-0347-09-00637-7

[3] BOWEN, L. (2011). Weak isomorphisms between Bernoulli shifts. *Israel J. Math.* **183** 93–102. MR2811154 https://doi.org/10.1007/s11856-011-0043-3

[4] BOWEN, L. (2012). Every countably infinite group is almost Ornstein. In *Dynamical Systems and Group Actions*. *Contemp. Math.* **567** 67–78. Amer. Math. Soc., Providence, RI. MR2931910 https://doi.org/10.1090/conm/567/11234

[5] CORNULIER, Y. and DE LA HARPE, P. (2016). *Metric Geometry of Locally Compact Groups*. *EMS Tracts in Mathematics* **25**. European Mathematical Society (EMS), Zürich. Winner of the 2016 EMS Monograph Award. MR3561300 https://doi.org/10.4171/166

[6] EVANS, S. N. (2010). A zero-one law for linear transformations of Lévy noise. In *Algebraic Methods in Statistics and Probability II*. *Contemp. Math.* **516** 189–197. Amer. Math. Soc., Providence, RI. MR2730749 https://doi.org/10.1090/conm/516/10175

[7] FOLLAND, G. B. (1995). *A Course in Abstract Harmonic Analysis*. *Studies in Advanced Mathematics*. CRC Press, Boca Raton, FL. MR1397028

[8] HOLROYD, A. E., LYONS, R. and SOO, T. (2011). Poisson splitting by factors. *Ann. Probab.* **39** 1938–1982. MR2884878 https://doi.org/10.1214/11-AOP651

[9] KATOK, A. (2007). Fifty years of entropy in dynamics: 1958–2007. *J. Mod. Dyn.* **1** 545–596. MR2342699 https://doi.org/10.3934/jmd.2007.1.545

[10] KEANE, M. and SMORODINSKY, M. (1977). A class of finitary codes. *Israel J. Math.* **26** 352–371. MR0450514 https://doi.org/10.1007/BF03007652

[11] KEANE, M. and SMORODINSKY, M. (1979). Bernoulli schemes of the same entropy are finitarily isomorphic. *Ann. of Math.* (2) **109** 397–406. MR0528969 https://doi.org/10.2307/1971117

[12] KECHRIS, A. S. (1995). *Classical Descriptive Set Theory*. *Graduate Texts in Mathematics* **156**. Springer, New York. MR1321597 https://doi.org/10.1007/978-1-4612-4190-4

*MSC2020 subject classifications.* Primary 37A35; secondary 60G10, 60G55.
*Key words and phrases.* Poisson point process, Poisson system, isomorphism, finitary isomorphism, nonamenable, locally compact second countable, Poisson splitting.

[13] KERR, D. and LI, H. (2011). Bernoulli actions and infinite entropy. *Groups Geom. Dyn.* **5** 663–672. MR2813530 https://doi.org/10.4171/GGD/142

[14] KIEFFER, J. C. (1975). A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space. *Ann. Probab.* **3** 1031–1037. MR0393422 https://doi.org/10.1214/aop/1176996230

[15] KINGMAN, J. F. C. (1993). *Poisson Processes*. *Oxford Studies in Probability* **3**. The Clarendon Press, Oxford University Press, New York. Oxford Science Publications. MR1207584

[16] ORNSTEIN, D. (1970). Bernoulli shifts with the same entropy are isomorphic. *Adv. Math.* **4** 337–352. MR0257322 https://doi.org/10.1016/0001-8708(70)90029-0

[17] ORNSTEIN, D. S. and WEISS, B. (1987). Entropy and isomorphism theorems for actions of amenable groups. *J. Anal. Math.* **48** 1–141. MR0910005 https://doi.org/10.1007/BF02790325

[18] REISS, R.-D. (1993). *A Course on Point Processes*. *Springer Series in Statistics*. Springer, New York. MR1199815 https://doi.org/10.1007/978-1-4613-9308-5

[19] SERAFIN, J. (2006). Finitary codes, a short survey. In *Dynamics & Stochastics*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **48** 262–273. IMS, Beachwood, OH. MR2306207 https://doi.org/10.1214/lnms/1196285827

[20] SEWARD, B. (2019). Krieger's finite generator theorem for actions of countable groups II. *J. Mod. Dyn.* **15** 1–39. MR3959054 https://doi.org/10.3934/jmd.2019012

[21] SEWARD, B. (2022). Bernoulli shifts with bases of equal entropy are isomorphic. *J. Mod. Dyn.* **18** 345–362. MR4459630 https://doi.org/10.3934/jmd.2022011

[22] SOO, T. and WILKENS, A. (2019). Finitary isomorphisms of Poisson point processes. *Ann. Probab.* **47** 3055–3081. MR4021244 https://doi.org/10.1214/18-AOP1332

[23] STEPIN, A. M. (1975). Bernoulli shifts on groups. *Dokl. Akad. Nauk SSSR* **223** 300–302. MR0409769

[24] STRUBLE, R. A. (1974). Metrics in locally compact groups. *Compos. Math.* **28** 217–222. MR0348037

*The Annals of Probability*
2023, Vol. 51, No. 6, 2192–2242
https://doi.org/10.1214/23-AOP1643


# ON THE RIGHTMOST EIGENVALUE OF NON-HERMITIAN RANDOM MATRICES

BY GIORGIO CIPOLLONI[1,a], LÁSZLÓ ERDŐS[2,b], DOMINIK SCHRÖDER[3,d] AND YUANYUAN XU[2,c]

[1]*Princeton Center for Theoretical Science, Princeton University,* [a]*gc4233@princeton.edu*
[2]*Institute of Science and Technology Austria,* [b]*laszlo.erdoes@ist.ac.at,* [c]*yuanyuan.xu@ist.ac.at*
[3]*Institute for Theoretical Studies, ETH Zurich,* [d]*dschroeder@ethz.ch*

We establish a precise three-term asymptotic expansion, with an optimal estimate of the error term, for the rightmost eigenvalue of an $n \times n$ random matrix with independent identically distributed complex entries as $n$ tends to infinity. All terms in the expansion are universal.

## REFERENCES

[1] AKEMANN, G. and BENDER, M. (2010). Interpolation between Airy and Poisson statistics for unitary chiral non-Hermitian random matrix ensembles. *J. Math. Phys.* **51** 103524.
[2] AKEMANN, G. and PHILLIPS, M. J. (2014). The interpolating Airy kernels for the $\beta = 1$ and $\beta = 4$ elliptic Ginibre ensembles. *J. Stat. Phys.* **155** 421–465.
[3] ALJADEFF, J., STERN, M. and SHARPEE, T. (2015). Transition to chaos in random networks with cell-type-specific connectivity. *Phys. Rev. Lett.* **114** 088101.
[4] ALLESINA, S., GRILLI, J., BARABÁS, G., TANG, S., ALJADEFF, J. and MARITAN, A. (2015). Predicting the stability of large structured food webs. *Nat. Commun.* **6** 7842.
[5] ALLESINA, S. and TANG, S. (2015). The stability–complexity relationship at age 40: A random matrix perspective. *Popul. Ecol.* **57** 63–75.
[6] ALT, J., ERDŐS, L. and KRÜGER, T. (2018). Local inhomogeneous circular law. *Ann. Appl. Probab.* **28** 148–203.
[7] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751.
[8] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2019). Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.* **72** 500–535.
[9] ARGUIN, L.-P., BOURGADE, P. and RADZIWIŁŁ, M. (2020). The Fyodorov–Hiary–Keating conjecture. I. ArXiv preprint.
[10] BAI, Z. D. (1997). Circular law. *Ann. Probab.* **25** 494–529.
[11] BAI, Z. D. and YIN, Y. Q. (1986). Limiting behavior of the norm of products of random matrices and two problems of Geman–Hwang. *Probab. Theory Related Fields* **73** 555–569.
[12] BENDER, M. (2010). Edge scaling limits for a family of non-Hermitian random matrix ensembles. *Probab. Theory Related Fields* **147** 241–271.
[13] BEN AROUS, G., FYODOROV, Y. V. and KHORUZHENKO, B. A. (2021). Counting equilibria of large complex systems by instability index. *Proc. Natl. Acad. Sci. USA* **118** e2023719118.
[14] BEN AROUS, G. and PÉCHÉ, S. (2005). Universality of local eigenvalue statistics for some sample covariance matrices. *Comm. Pure Appl. Math.* **58** 1316–1357.
[15] BORDENAVE, C., CAPUTO, P., CHAFAÏ, D. and TIKHOMIROV, K. (2018). On the spectral radius of a random matrix: An upper bound without fourth moment. *Ann. Probab.* **46** 2268–2286.
[16] BORDENAVE, C., CHAFAÏ, D. and GARCÍA-ZELADA, D. (2022). Convergence of the spectral radius of a random matrix through its characteristic polynomial. *Probab. Theory Related Fields* **182** 1163–1181.
[17] BOURGADE, P. (2022). Extreme gaps between eigenvalues of Wigner matrices. *J. Eur. Math. Soc.* (*JEMS*) **24** 2823–2873.
[18] BOURGADE, P., YAU, H.-T. and YIN, J. (2014). The local circular law II: The edge case. *Probab. Theory Related Fields* **159** 619–660.

*MSC2020 subject classifications.* 60B20, 15B52.
*Key words and phrases.* Ginibre ensemble, Girko's formula, Gumbel distribution, SUSY method.

[19] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymptotic distribution of smoothed eigenvalue density. II. Wigner random matrices. *Random Oper. Stoch. Equ.* **7** 149–168.

[20] CHALKER, J. T. and MEHLIG, B. (1998). Eigenvector statistics in non-Hermitian random matrix ensembles. *Phys. Rev. Lett.* **81** 3367–3370.

[21] CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the C$\beta$E field. *Duke Math. J.* **167** 2243–2345.

[22] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2020). Optimal lower bound on the least singular value of the shifted Ginibre ensemble. *Probab. Math. Phys.* **1** 101–146.

[23] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2021). Edge universality for non-Hermitian random matrices. *Probab. Theory Related Fields* **179** 1–28.

[24] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2021). Fluctuation around the circular law for random matrices with real entries. *Electron. J. Probab.* **26** 24.

[25] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2021). Central limit theorem for linear eigenvalue statistics of non-Hermitian random matrices. *Comm. Pure Appl. Math.*

[26] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2022). Density of small singular values of the shifted real Ginibre ensemble. *Ann. Henri Poincaré* **23** 3981–4002.

[27] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2022). On the condition number of the shifted real Ginibre ensemble. *SIAM J. Matrix Anal. Appl.* **43** 1469–1487.

[28] CIPOLLONI, G., ERDŐS, L., SCHRÖDER, D. and XU, Y. (2022). Directional extremal statistics for Ginibre eigenvalues. *J. Math. Phys.* **63** 103303.

[29] ERDŐS, L., KNOWLES, A. and YAU, H.-T. (2013). Averaging fluctuations in resolvents of random band matrices. *Ann. Henri Poincaré* **14** 1837–1926.

[30] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). The local semicircle law for a general class of random matrices. *Electron. J. Probab.* **18** 59.

[31] ERDŐS, L., KRÜGER, T. and RENFREW, D. (2018). Power law decay for systems of randomly coupled differential equations. *SIAM J. Math. Anal.* **50** 3271–3290.

[32] ERDŐS, L., KRÜGER, T. and RENFREW, D. (2019). Randomly coupled differential equations with elliptic correlations. *Ann. Appl. Probab.*, to appear.

[33] ERDŐS, L., KRÜGER, T. and SCHRÖDER, D. (2019). Random matrices with slow correlation decay. *Forum Math. Sigma* **7** e8.

[34] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory. Courant Lecture Notes in Mathematics* **28**. Amer. Math. Soc., Providence, RI.

[35] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Bulk universality for generalized Wigner matrices. *Probab. Theory Related Fields* **154** 341–407.

[36] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** 1435–1515.

[37] FENG, R., TIAN, G., WEI, D. and YAO, D. (2022). Principal minors of Gaussian orthogonal ensemble. ArXiv preprint.

[38] FYODOROV, Y. V., HIARY, G. A. and KEATING, J. P. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.* **108** 170601.

[39] FYODOROV, Y. V. and KEATING, J. P. (2014). Freezing transitions and extreme values: Random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** 20120503.

[40] FYODOROV, Y. V. and SIMM, N. J. (2016). On the distribution of the maximum value of the characteristic polynomial of GUE random matrices. *Nonlinearity* **29** 2837–2855.

[41] GEMAN, S. (1986). The spectral radius of large random matrices. *Ann. Probab.* **14** 1318–1328.

[42] GINIBRE, J. (1965). Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.* **6** 440–449.

[43] GIRKO, V. L. (1984). The circular law. *Teor. Veroyatn. Primen.* **29** 669–679.

[44] HARPER, A. J. (2019). On the partition function of the Riemann zeta function, and the Fyodorov–Hiary–Keating conjecture. ArXiv preprint.

[45] HE, Y. and KNOWLES, A. (2017). Mesoscopic eigenvalue statistics of Wigner matrices. *Ann. Appl. Probab.* **27** 1510–1550.

[46] HE, Y. and KNOWLES, A. (2021). Fluctuations of extreme eigenvalues of sparse Erdős–Rényi graphs. *Probab. Theory Related Fields* **180** 985–1056.

[47] HUANG, J., LANDON, B. and YAU, H.-T. (2020). Transition from Tracy-Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 916–962.

[48] JOHANSSON, K. (2007). From Gumbel to Tracy-Widom. *Probab. Theory Related Fields* **138** 75–112.

[49] KHORUNZHY, A. M., KHORUZHENKO, B. A. and PASTUR, L. A. (1996). Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.* **37** 5033–5060.

[50] KOPEL, P. (2015). Linear statistics of non-Hermitian matrices matching the real or complex Ginibre ensemble to four moments. ArXiv preprint.
[51] LAMBERT, G. (2020). Maximum of the characteristic polynomial of the Ginibre ensemble. *Comm. Math. Phys.* **378** 943–985.
[52] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv. Math.* **346** 1137–1332.
[53] LEE, J. O. and SCHNELLI, K. (2015). Edge universality for deformed Wigner matrices. *Rev. Math. Phys.* **27** 1550018.
[54] LEE, J. O. and SCHNELLI, K. (2016). Tracy-Widom distribution for the largest eigenvalue of real sample covariance matrices with general population. *Ann. Appl. Probab.* **26** 3786–3839.
[55] LEE, J. O. and SCHNELLI, K. (2018). Local law and Tracy–Widom limit for sparse random matrices. *Probab. Theory Related Fields* **171** 543–616.
[56] LYTOVA, A. and PASTUR, L. (2009). Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Probab.* **37** 1778–1840.
[57] MAY, R. M. (1972). Will a large complex system be stable? *Nature* **238** 413–4.
[58] MEHLIG, B. and CHALKER, J. T. (2000). Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles. *J. Math. Phys.* **41** 3233–3256.
[59] NAJNUDEL, J. (2018). On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Related Fields* **172** 387–452.
[60] PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int. Math. Res. Not. IMRN* **16** 5028–5119.
[61] RAJAN, K. and ABBOTT, L. F. (2006). Eigenvalue spectra of random matrices for neural networks. *Phys. Rev. Lett.* **97** 188104.
[62] SAKSMAN, E. and WEBB, C. (2020). The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line. *Ann. Probab.* **48** 2680–2754.
[63] SCHNELLI, K. and XU, Y. (2021). Convergence rate to the Tracy–Widom laws for the largest eigenvalue of sample covariance matrices. ArXiv preprint.
[64] SCHNELLI, K. and XU, Y. (2022). Convergence rate to the Tracy–Widom laws for the largest eigenvalue of Wigner matrices. *Comm. Math. Phys.* **393** 839–907.
[65] SOMPOLINSKY, H., CRISANTI, A. and SOMMERS, H. J. (1988). Chaos in random neural networks. *Phys. Rev. Lett.* **61** 259–262. https://doi.org/10.1103/PhysRevLett.61.259
[66] TAO, T. and VU, V. (2008). Random matrices: The circular law. *Commun. Contemp. Math.* **10** 261–307.
[67] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math.* **206** 127–204.
[68] TAO, T. and VU, V. (2015). Random matrices: Universality of local spectral statistics of non-Hermitian matrices. *Ann. Probab.* **43** 782–874.

*The Annals of Probability*
2023, Vol. 51, No. 6, 2243–2264
https://doi.org/10.1214/23-AOP1647


# ESSENTIAL ENHANCEMENTS IN ABELIAN NETWORKS: CONTINUITY AND UNIFORM STRICT MONOTONICITY

BY LORENZO TAGGI[a]

*Dipartimento di Matematica 'Guido Castelnuovo', Sapienza Università di Roma,* [a]*lorenzo.taggi@uniroma1.it*

We prove that in wide generality the critical curve of the activated random walk model is a continuous function of the deactivation rate, and we provide a bound on its slope, which is uniform with respect to the choice of the graph. Moreover, we derive strict monotonicity properties for the probability of a wide class of "increasing" events, extending previous results of (*Invent. Math.* **188** (2012) 127–150). Our proof method is of independent interest and can be viewed as a reformulation of the 'essential enhancements' technique, which was introduced for percolation, in the framework of abelian networks.

## REFERENCES

[1] AIZENMAN, M. and GRIMMETT, G. (1991). Strict monotonicity for critical points in percolation and ferromagnetic models. *J. Stat. Phys.* **63** 817–835. MR1116036 https://doi.org/10.1007/BF01029985

[2] ASSELAH, A., ROLLA, L. T. and SCHAPIRA, B. (2022). Diffusive bounds for the critical density of activated random walks. *ALEA Lat. Am. J. Probab. Math. Stat.* **19** 457–465. MR4394304 https://doi.org/10.30757/alea.v19-17

[3] BALISTER, P., BOLLOBÁS, B. and RIORDAN, O. (2014). Essential enhancements revisited. Preprint. Available at arXiv:1402.0834.

[4] BASU, R., GANGULY, S. and HOFFMAN, C. (2018). Non-fixation for conservative stochastic dynamics on the line. *Comm. Math. Phys.* **358** 1151–1185. MR3778354 https://doi.org/10.1007/s00220-017-3059-7

[5] BOND, B. and LEVINE, L. (2016). Abelian networks I. Foundations and examples. *SIAM J. Discrete Math.* **30** 856–874. MR3493110 https://doi.org/10.1137/15M1030984

[6] CANDELLERO, E., GANGULY, S., HOFFMAN, C. and LEVINE, L. (2017). Oil and water: A two-type internal aggregation model. *Ann. Probab.* **45** 4019–4070. MR3729622 https://doi.org/10.1214/16-AOP1157

[7] CANDELLERO, E., STAUFFER, A. and TAGGI, L. (2021). Abelian oil and water dynamics does not have an absorbing-state phase transition. *Trans. Amer. Math. Soc.* **374** 2733–2752. MR4223031 https://doi.org/10.1090/tran/8276

[8] DICKMAN, R., ROLLA, L. T. and SIDORAVICIUS, V. (2010). Activated random walkers: Facts, conjectures and challenges. *J. Stat. Phys.* **138** 126–142. MR2594894 https://doi.org/10.1007/s10955-009-9918-7

[9] FORIEN, N. and GAUDILLAIRE, A. Active phase for activated random walks on the lattice in all dimensions. Available at arXiv:2203.02476.

[10] HOFFMAN, C., JOHNSON, T. and JUNGE, M. (2017). Recurrence and transience for the frog model on trees. *Ann. Probab.* **45** 2826–2854. MR3706732 https://doi.org/10.1214/16-AOP1125

[11] JÁRAI, A. A. (2012). Abelian sandpiles: An overview and results on certain transitive graphs. *Markov Process. Related Fields* **18** 111–156. MR2952022

[12] ROLLA, L. T. (2020). Activated random walks on $\mathbb{Z}^d$. *Probab. Surv.* **17** 478–544. MR4152668 https://doi.org/10.1214/19-PS339

[13] ROLLA, L. T. and SIDORAVICIUS, V. (2012). Absorbing-state phase transition for driven-dissipative stochastic dynamics on $\mathbb{Z}$. *Invent. Math.* **188** 127–150. MR2897694 https://doi.org/10.1007/s00222-011-0344-5

[14] ROLLA, L. T., SIDORAVICIUS, V. and ZINDY, O. (2019). Universality and sharpness in activated random walks. *Ann. Henri Poincaré* **20** 1823–1835. MR3956161 https://doi.org/10.1007/s00023-019-00797-0

[15] ROLLA, L. T. and TOURNIER, L. (2018). Non-fixation for biased activated random walks. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 938–951. MR3795072 https://doi.org/10.1214/17-AIHP827

*MSC2020 subject classifications.* Primary 82C22, 60K35; secondary 82C26.

*Key words and phrases.* Activated random walks, abelian networks, essential enhancements, self-organised criticality, absorbing-state phase transition.

[16] RUSSO, L. (1978). A note on percolation. *Z. Wahrsch. Verw. Gebiete* **43** 39–48. MR0488383 https://doi.org/10.1007/BF00535274

[17] RUSSO, L. (1981). On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete* **56** 229–237. MR0618273 https://doi.org/10.1007/BF00535742

[18] SHELLEF, E. (2010). Nonfixation for activated random walks. *ALEA Lat. Am. J. Probab. Math. Stat.* **7** 137–149. MR2651824

[19] SIDORAVICIUS, V. and TEIXEIRA, A. (2017). Absorbing-state transition for stochastic sandpiles and activated random walks. *Electron. J. Probab.* **22** Paper No. 33, 35. MR3646059 https://doi.org/10.1214/17-EJP50

[20] STAUFFER, A. and TAGGI, L. (2018). Critical density of activated random walks on transitive graphs. *Ann. Probab.* **46** 2190–2220. MR3813989 https://doi.org/10.1214/17-AOP1224

[21] TAGGI, L. (2016). Absorbing-state phase transition in biased activated random walk. *Electron. J. Probab.* **21** Paper No. 13, 15. MR3485355 https://doi.org/10.1214/16-EJP4275

[22] TAGGI, L. (2019). Active phase for activated random walks on $\mathbb{Z}^d$, $d \geq 3$, with density less than one and arbitrary sleeping rate. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1751–1764. MR4010950 https://doi.org/10.1214/18-aihp933

*The Annals of Probability*
2023, Vol. 51, No. 6, 2265–2300
https://doi.org/10.1214/23-AOP1648


# THE CRITICAL 2D STOCHASTIC HEAT FLOW IS NOT A GAUSSIAN MULTIPLICATIVE CHAOS

BY FRANCESCO CARAVENNA[1,a], RONGFENG SUN[2,b] AND NIKOS ZYGOURAS[3,c]

[1]*Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca,* [a]*francesco.caravenna@unimib.it*
[2]*Department of Mathematics, National University of Singapore,* [b]*matsr@nus.edu.sg*
[3]*Department of Mathematics, University of Warwick,* [c]*N.Zygouras@warwick.ac.uk*

The critical 2$d$ stochastic heat flow (SHF) is a stochastic process of random measures on $\mathbb{R}^2$, recently constructed in (*Invent. Math.* **233** (2023) 325–460). We show that this process falls outside the class of Gaussian multiplicative chaos (GMC), in the sense that it cannot be realised as the exponential of a (generalised) Gaussian field. We achieve this by deriving strict lower bounds on the moments of the SHF that are of independent interest.

## REFERENCES

[1] ANDERSON, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** 170–176. MR0069229 https://doi.org/10.2307/2032333

[2] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman–Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. MR1316109 https://doi.org/10.1007/BF02180136

[3] BERTINI, L. and CANCRINI, N. (1998). The two-dimensional stochastic heat equation: Renormalizing a multiplicative noise. *J. Phys. A* **31** 615–622. MR1629198 https://doi.org/10.1088/0305-4470/31/2/019

[4] BRÖKER, Y. and MUKHERJEE, C. (2020). Geometry of the Gaussian multiplicative chaos in the Wiener space. Available at arXiv:2008.04290.

[5] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2017). Universality in marginally relevant disordered systems. *Ann. Appl. Probab.* **27** 3050–3112. MR3719953 https://doi.org/10.1214/17-AAP1276

[6] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2019). The Dickman subordinator, renewal theorems, and disordered systems. *Electron. J. Probab.* **24** Paper No. 101, 40. MR4017119 https://doi.org/10.1214/19-ejp353

[7] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2019). On the moments of the $(2+1)$-dimensional directed polymer and stochastic heat equation in the critical window. *Comm. Math. Phys.* **372** 385–440. MR4032870 https://doi.org/10.1007/s00220-019-03527-z

[8] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2020). The two-dimensional KPZ equation in the entire subcritical regime. *Ann. Probab.* **48** 1086–1127. MR4112709 https://doi.org/10.1214/19-AOP1383

[9] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2023). The critical 2d stochastic heat flow. *Invent. Math.* **233** 325–460. MR4602000 https://doi.org/10.1007/s00222-023-01184-7

[10] CHEN, Y.-T. (2021). The critical 2D delta-Bose gas as mixed-order asymptotics of planar Brownian motion. Available at arXiv:2105.05154.

[11] CLARK, J. T. (2022). Continuum models of directed polymers on disordered diamond fractals in the critical case. *Ann. Appl. Probab.* **32** 4186–4250. MR4522350 https://doi.org/10.1214/22-aap1783

[12] CLARK, J. T. (2023). The conditional Gaussian multiplicative chaos structure underlying a critical continuum random polymer model on a diamond fractal. *Ann. Inst. Henri Poincaré Probab. Stat.* **59** 1203–1222. MR4635708 https://doi.org/10.1214/22-aihp1312

[13] CLARK, J. T. and MIAN, B. On the correlation measure for the critical 2$d$ continuum polymer. Unpublished manuscript.

[14] COSCO, C. and ZEITOUNI, O. (2021). Moments of partition functions of 2D Gaussian polymers in the weak disorder regime. Available at arXiv:2112.03767.

[15] FENG, Z. S. (2016). Rescaled directed random polymer in random environment in dimension $1+2$. Ph.D. thesis, Ann Arbor, MI. Available at https://www.proquest.com/docview/1820736587.

*MSC2020 subject classifications.* Primary 82B44; secondary 35R60, 60H15, 82D60.
*Key words and phrases.* Directed polymer in random environment, Gaussian multiplicative chaos, Gaussian correlation inequality, stochastic heat equation, stochastic heat flow, KPZ equation.

[16] GU, Y. (2020). Gaussian fluctuations from the 2D KPZ equation. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** 150–185. MR4058958 https://doi.org/10.1007/s40072-019-00144-8

[17] GU, Y., QUASTEL, J. and TSAI, L.-C. (2021). Moments of the 2D SHE at criticality. *Probab. Math. Phys.* **2** 179–219. MR4404819 https://doi.org/10.2140/pmp.2021.2.179

[18] GUBINELLI, M., IMKELLER, P. and PERKOWSKI, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6, 75. MR3406823 https://doi.org/10.1017/fmp.2015.2

[19] HAIRER, M. (2014). A theory of regularity structures. *Invent. Math.* **198** 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4

[20] KALLIANPUR, G. and ROBBINS, H. (1953). Ergodic property of the Brownian motion process. *Proc. Natl. Acad. Sci. USA* **39** 525–533. MR0056233 https://doi.org/10.1073/pnas.39.6.525

[21] KUPIAINEN, A. (2016). Renormalization group and stochastic PDEs. *Ann. Henri Poincaré* **17** 497–535. MR3459120 https://doi.org/10.1007/s00023-015-0408-y

[22] LATAŁA, R. and MATLAK, D. (2017). Royen's proof of the Gaussian correlation inequality. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **2169** 265–275. Springer, Cham. MR3645127

[23] LYGKONIS, D. and ZYGOURAS, N. (2023). Moments of the 2D directed polymer in the subcritical regime and a generalisation of the Erdös–Taylor theorem. *Comm. Math. Phys.* **401** 2483–2520. MR4616647 https://doi.org/10.1007/s00220-023-04694-w

[24] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218

[25] ROYEN, T. (2014). A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions. *Far East J. Theor. Stat.* **48** 139–145. MR3289621

*The Annals of Probability*
2023, Vol. 51, No. 6, 2301–2344
https://doi.org/10.1214/23-AOP1653


# PARTICLE DENSITY IN DIFFUSION-LIMITED ANNIHILATING SYSTEMS

BY TOBIAS JOHNSON[1,a], MATTHEW JUNGE[2,b], HANBAEK LYU[3,c] AND DAVID SIVAKOFF[4,d]

[1]*Mathematics, CUNY College of Staten Island,* [a]*Tobias.Johnson@csi.cuny.edu*
[2]*Mathematics, CUNY Baruch College,* [b]*Matthew.Junge@baruch.cuny.edu*
[3]*Mathematics, University of Wisconsin—Madison,* [c]*hlyu@math.wisc.edu*
[4]*Mathematics and Statistics, Ohio State University,* [d]*dsivakoff@stat.osu.edu*

Place an $A$-particle at each site of a graph independently with probability $p$, and otherwise place a $B$-particle. $A$- and $B$-particles perform independent continuous time random walks at rates $\lambda_A$ and $\lambda_B$, respectively, and annihilate upon colliding with a particle of opposite type. Bramson and Lebowitz studied the setting $\lambda_A = \lambda_B$ in the early 1990s. Despite recent progress, many basic questions remain unanswered when $\lambda_A \neq \lambda_B$. For the critical case $p = 1/2$ on low-dimensional integer lattices, we give a lower bound on the expected number of particles at the origin that matches physicists' predictions. For the process with $\lambda_B = 0$ on the integers and on the bidirected regular tree, we give sharp upper and lower bounds for the expected total occupation time of the root at and approaching criticality.

## REFERENCES

[1] AHLBERG, D., GRIFFITHS, S. and JANSON, S. (2021). To fixate or not to fixate in two-type annihilating branching random walks. *Ann. Probab.* **49** 2637–2667. MR4317715 https://doi.org/10.1214/21-aop1521

[2] ARRATIA, R. (1981). Limiting point processes for rescalings of coalescing and annihilating random walks on $\mathbf{Z}^d$. *Ann. Probab.* **9** 909–936. MR0632966

[3] ARRATIA, R. and BAXENDALE, P. (2015). Bounded size bias coupling: A Gamma function bound, and universal Dickman-function behavior. *Probab. Theory Related Fields* **162** 411–429. MR3383333 https://doi.org/10.1007/s00440-014-0572-x

[4] ARRATIA, R., GOLDSTEIN, L. and KOCHMAN, F. (2019). Size bias for one and all. *Probab. Surv.* **16** 1–61. MR3896143 https://doi.org/10.1214/13-ps221

[5] AUFFINGER, A., DAMRON, M. and HANSON, J. (2017). 50 *Years of First-Passage Percolation*. *University Lecture Series* **68**. Amer. Math. Soc., Providence, RI. MR3729447 https://doi.org/10.1090/ulect/068

[6] BAHL, R., BARNET, P., JOHNSON, T. and JUNGE, M. (2022). Diffusion-limited annihilating systems and the increasing convex order. *Electron. J. Probab.* **27** Paper No. 84, 19. MR4453592 https://doi.org/10.1214/22-ejp808

[7] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities*: *A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 https://doi.org/10.1093/acprof:oso/9780199535255.001.0001

[8] BRAMSON, M. and GRIFFEATH, D. (1980). Asymptotics for interacting particle systems on $\mathbf{Z}^d$. *Z. Wahrsch. Verw. Gebiete* **53** 183–196. MR0580912 https://doi.org/10.1007/BF01013315

[9] BRAMSON, M. and LEBOWITZ, J. L. (1988). Asymptotic behavior of densities in diffusion-dominated annihilation reactions. *Phys. Rev. Lett.* **61** 2397–2400. https://doi.org/0.1103/PhysRevLett.61.2397

[10] BRAMSON, M. and LEBOWITZ, J. L. (1991). Asymptotic behavior of densities for two-particle annihilating random walks. *J. Stat. Phys.* **62** 297–372. MR1105266 https://doi.org/10.1007/BF01020872

[11] BRENTI, F. (1989). Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.* **81** viii+106. MR0963833 https://doi.org/10.1090/memo/0413

[12] CABEZAS, M., ROLLA, L. T. and SIDORAVICIUS, V. (2014). Non-equilibrium phase transitions: Activated random walks at criticality. *J. Stat. Phys.* **155** 1112–1125. MR3207731 https://doi.org/10.1007/s10955-013-0909-3

*MSC2020 subject classifications.* 60K35.
*Key words and phrases.* Interacting particle system, critical behavior.

[13] CABEZAS, M., ROLLA, L. T. and SIDORAVICIUS, V. (2018). Recurrence and density decay for diffusion-limited annihilating systems. *Probab. Theory Related Fields* **170** 587–615. MR3773795 https://doi.org/10.1007/s00440-017-0763-3

[14] CHEN, X., DAGARD, V., DERRIDA, B., HU, Y., LIFSHITS, M. and SHI, Z. (2021). The Derrida–Retaux conjecture on recursive models. *Ann. Probab.* **49** 637–670. MR4255128 https://doi.org/10.1214/20-aop1457

[15] CHEN, X., DERRIDA, B., HU, Y., LIFSHITS, M. and SHI, Z. (2019). A max-type recursive model: Some properties and open questions. In *Sojourns in Probability Theory and Statistical Physics*. *III*. *Interacting Particle Systems and Random Walks*, *a Festschrift for Charles M*. *Newman* (Sidoravicius, V. ed.). *Springer Proc*. *Math*. *Stat*. **300** 166–186. Springer, Singapore. MR4044392 https://doi.org/10.1007/978-981-15-0302-3_6

[16] COOK, N., GOLDSTEIN, L. and JOHNSON, T. (2018). Size biased couplings and the spectral gap for random regular graphs. *Ann. Probab.* **46** 72–125. MR3758727 https://doi.org/10.1214/17-AOP1180

[17] CRISTALI, I., JIANG, Y., JUNGE, M., KASSEM, R., SIVAKOFF, D. and YORK, G. (2021). Two-type annihilating systems on the complete and star graph. *Stochastic Process*. *Appl*. **139** 321–342. MR4275051 https://doi.org/10.1016/j.spa.2021.05.004

[18] DAMRON, M., GRAVNER, J., JUNGE, M., LYU, H. and SIVAKOFF, D. (2019). Parking on transitive unimodular graphs. *Ann*. *Appl*. *Probab*. **29** 2089–2113. MR3983336 https://doi.org/10.1214/18-AAP1443

[19] DAMRON, M., LYU, H. and SIVAKOFF, D. (2021). Stretched exponential decay for subcritical parking times on $\mathbb{Z}^d$. *Random Structures Algorithms* **59** 143–154. MR4284168 https://doi.org/10.1002/rsa.21001

[20] DAUVERGNE, D. and SLY, A. (2023). Spread of infections in a heterogeneous moving population. *Probab. Theory Related Fields* **187** 73–131. MR4634337 https://doi.org/10.1007/s00440-023-01216-6

[21] DERRIDA, B. and RETAUX, M. (2014). The depinning transition in presence of disorder: A toy model. *J. Stat*. *Phys*. **156** 268–290. https://doi.org/10.1007/s10955-014-1006-y

[22] GHOSH, S. and GOLDSTEIN, L. (2011). Concentration of measures via size-biased couplings. *Probab. Theory Related Fields* **149** 271–278. MR2773032 https://doi.org/10.1007/s00440-009-0253-3

[23] GOLDSCHMIDT, C. and PRZYKUCKI, M. (2019). Parking on a random tree. *Combin*. *Probab. Comput*. **28** 23–45. MR3917904 https://doi.org/10.1017/S0963548318000457

[24] HARDY, R. and HARRIS, S. C. (2009). A spine approach to branching diffusions with applications to $\mathscr{L}^p$-convergence of martingales. In *Séminaire de Probabilités XLII*. *Lecture Notes in Math*. **1979** 281–330. Springer, Berlin. MR2599214 https://doi.org/10.1007/978-3-642-01763-6_11

[25] HU, Y. and SHI, Z. (2018). The free energy in the Derrida–Retaux recursive model. *J. Stat*. *Phys*. **172** 718–741. MR3827300 https://doi.org/10.1007/s10955-018-2066-1

[26] KALLENBERG, O. (1997). *Foundations of Modern Probability*. *Probability and Its Applications* (*New York*). Springer, New York. MR1464694

[27] KOZA, Z. (1996). The long-time behavior of initially separated $A + B \to 0$ reaction-diffusion systems with arbitrary diffusion constants. *J. Stat*. *Phys*. **85** 179–191. MR1413241 https://doi.org/10.1007/BF02175561

[28] LACKNER, M.-L. and PANHOLZER, A. (2016). Parking functions for mappings. *J. Combin*. *Theory Ser. A* **142** 1–28. MR3499489 https://doi.org/10.1016/j.jcta.2016.03.001

[29] LEE, B. P. and CARDY, J. (1995). Renormalization group study of the $A + B \to \varnothing$ diffusion-limited reaction. *J. Stat*. *Phys*. **80** 971–1007. MR1349773 https://doi.org/10.1007/BF02179861

[30] LIGGETT, T. M. (1997). Ultra logconcave sequences and negative dependence. *J. Combin*. *Theory Ser. A* **79** 315–325. MR1462561 https://doi.org/10.1006/jcta.1997.2790

[31] LYONS, R., PEMANTLE, R. and PERES, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann*. *Probab.* **23** 1125–1138. MR1349164

[32] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. *Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[33] OVCHINNIKOV, A. A. and ZELDOVICH, Y. B. (1978). Role of density fluctuations in bimolecular reaction kinetics. *Chem*. *Phys*. **28** 215–218.

[34] PEKÖZ, E. A. and RÖLLIN, A. (2011). New rates for exponential approximation and the theorems of Rényi and Yaglom. *Ann*. *Probab.* **39** 587–608. MR2789507 https://doi.org/10.1214/10-AOP559

[35] PRZYKUCKI, M., ROBERTS, A. and SCOTT, A. (2023). Parking on the integers. *Ann. Appl*. *Probab.* **33** 876–901. MR4564421 https://doi.org/10.1214/22-aap1836

[36] KANG, K. and REDNER, S. (1984). Scaling approach for the kinetics of recombination processes. *Phys. Rev. Lett*. **52** 955–958. https://doi.org/10.1103/PhysRevLett.52.955

[37] RIVERA, N., SAUERWALD, T., STAUFFER, A. and SYLVESTER, J. (2019). The dispersion time of random walks on finite graphs. In *The* 31*st ACM Symposium on Parallelism in Algorithms and Architectures* 103–113.

[38] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[39] SHAKED, M. and SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. *Springer Series in Statistics*. Springer, New York. MR2265633 https://doi.org/10.1007/978-0-387-34675-5

[40] SHI, Z. (2015). *Branching Random Walks*. *Lecture Notes in Math*. **2151**. Springer, Cham. MR3444654 https://doi.org/10.1007/978-3-319-25372-5

[41] STANLEY, R. P. (1989). Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph Theory and Its Applications*: *East and West* (*Jinan*, 1986). *Ann*. *New York Acad*. *Sci*. **576** 500–535. New York Acad. Sci., New York. MR1110850 https://doi.org/10.1111/j.1749-6632.1989.tb16434.x

[42] TOUSSAINT, D. and WILCZEK, F. (1983). Particle–antiparticle annihilation in diffusive motion. *J. Chem. Phys*. **78** 2642–2647.

*The Annals of Probability*
2023, Vol. 51, No. 6, 2345–2386
https://doi.org/10.1214/23-AOP1654
This research was funded, in whole or in part, by [EPSEC/UKRI, EP/W026899]. A CC BY 4.0 license is applied to this article arising from this submission, in accordance with the grant's open access conditions.

# SCALING LIMIT OF THE FLEMING–VIOT MULTICOLOR PROCESS

BY OLIVER TOUGH[a]

*Department of Mathematics, University of Bath,* [a]*okt24@bath.ac.uk*

We consider the $N$-particle Fleming–Viot process associated to a normally reflected diffusion with soft catalyst killing. The Fleming–Viot multicolor process is obtained by attaching genetic information to the particles in the Fleming–Viot process. We establish that, after rescaling time by $t \mapsto Nt$, this genetic information converges to the (very different) Fleming–Viot process from population genetics, as $N \to \infty$. An extension is provided to dynamics given by Brownian motion with hard catalyst killing at the boundary of its domain.

## REFERENCES

[1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446 https://doi.org/10.1214/aop/1176995579

[2] ASSELAH, A., FERRARI, P. A. and GROISMAN, P. (2011). Quasistationary distributions and Fleming–Viot processes in finite spaces. *J. Appl. Probab.* **48** 322–332. MR2840302 https://doi.org/10.1239/jap/1308662630

[3] ASSELAH, A., FERRARI, P. A., GROISMAN, P. and JONCKHEERE, M. (2016). Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 647–668. MR3498004 https://doi.org/10.1214/14-AIHP635

[4] BALL, K., KURTZ, T. G., POPOVIC, L. and REMPALA, G. (2006). Asymptotic analysis of multiscale approximations to reaction networks. *Ann. Appl. Probab.* **16** 1925–1961. MR2288709 https://doi.org/10.1214/105051606000000420

[5] BERESTYCKI, J., BERESTYCKI, N. and SCHWEINSBERG, J. (2013). The genealogy of branching Brownian motion with absorption. *Ann. Probab.* **41** 527–618. MR3077519 https://doi.org/10.1214/11-AOP728

[6] BERESTYCKI, J., BRUNET, É., NOLEN, J. and PENINGTON, S. (2022). Brownian bees in the infinite swarm limit. *Ann. Probab.* **50** 2133–2177. MR4499276 https://doi.org/10.1214/22-aop1578

[7] BIENIEK, M. and BURDZY, K. (2018). The distribution of the spine of a Fleming–Viot type process. *Stochastic Process. Appl.* **128** 3751–3777. MR3860009 https://doi.org/10.1016/j.spa.2017.12.003

[8] BIENIEK, M., BURDZY, K. and FINCH, S. (2012). Non-extinction of a Fleming–Viot particle model. *Probab. Theory Related Fields* **153** 293–332. MR2925576 https://doi.org/10.1007/s00440-011-0372-5

[9] BROWN, S., JENKINS, P. A., JOHANSEN, A. M. and KOSKELA, J. (2021). Simple conditions for convergence of sequential Monte Carlo genealogies with applications. *Electron. J. Probab.* **26** Paper No. 1. MR4216514 https://doi.org/10.1214/20-ejp561

[10] BRUNET, É. and DERRIDA, B. (1997). Shift in the velocity of a front due to a cutoff. *Phys. Rev. E* (3) **56** 2597–2604. MR1473413 https://doi.org/10.1103/PhysRevE.56.2597

[11] BRUNET, É. and DERRIDA, B. (2001). Effect of microscopic noise on front propagation. *J. Stat. Phys.* **103** 269–282. MR1828730 https://doi.org/10.1023/A:1004875804376

[12] BRUNET, É., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2006). Noisy traveling waves: Effect of selection on genealogies. *Europhys. Lett.* **76** 1–7. MR2299937 https://doi.org/10.1209/epl/i2006-10224-4

[13] BRUNET, É., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2007). Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization. *Phys. Rev. E* (3) **76** 041104. MR2365627 https://doi.org/10.1103/PhysRevE.76.041104

[14] BURDZY, K. List of open problems.

[15] BURDZY, K. and ENGLÄNDER, J. (2021). The spine of the Fleming–Viot process driven by Brownian motion. Available at arXiv:2112.01720.

[16] BURDZY, K., HOŁYST, R. and MARCH, P. (2000). A Fleming–Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.* **214** 679–703. MR1800866 https://doi.org/10.1007/s002200000294

*MSC2020 subject classifications.* Primary 60K35, 60J80; secondary 92D15.
*Key words and phrases.* Fleming–Viot process (particle system), Fleming–Viot process (population genetics).

[17] BURDZY, K., KOŁODZIEJEK, B. and TADIĆ, T. (2019). Inverse exponential decay: Stochastic fixed point equation and ARMA models. *Bernoulli* **25** 3939–3977. MR4010978 https://doi.org/10.3150/19-bej1116
[18] BURDZY, K., KOŁODZIEJEK, B. and TADIĆ, T. (2022). Stochastic fixed-point equation and local dependence measure. *Ann. Appl. Probab.* **32** 2811–2840. MR4474520 https://doi.org/10.1214/21-aap1749
[19] CHAMPAGNAT, N. and VILLEMONAIS, D. (2016). Exponential convergence to quasi-stationary distribution and $Q$-process. *Probab. Theory Related Fields* **164** 243–283. MR3449390 https://doi.org/10.1007/s00440-014-0611-7
[20] CHAMPAGNAT, N. and VILLEMONAIS, D. (2017). Uniform convergence to the $Q$-process. *Electron. Commun. Probab.* **22** Paper No. 33. MR3663104 https://doi.org/10.1214/17-ECP63
[21] DE MULATIER, C., DUMONTEIL, E., ROSSO, A. and ZOIA, A. (2015). The critical catastrophe revisited. *J. Stat. Mech. Theory Exp.* **8** P08021. MR3400257 https://doi.org/10.1088/1742-5468/2015/08/p08021
[22] DURRETT, R. and REMENIK, D. (2011). Brunet–Derrida particle systems, free boundary problems and Wiener–Hopf equations. *Ann. Probab.* **39** 2043–2078. MR2932664 https://doi.org/10.1214/10-AOP601
[23] ETHIER, S. N. and KURTZ, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim.* **31** 345–386. MR1205982 https://doi.org/10.1137/0331019
[24] ETHIER, S. N. and KURTZ, T. G. (1994). Convergence to Fleming–Viot processes in the weak atomic topology. *Stochastic Process. Appl.* **54** 1–27. MR1302692 https://doi.org/10.1016/0304-4149(94)00006-9
[25] FLEMING, W. H. and VIOT, M. (1979). Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.* **28** 817–843. MR0542340 https://doi.org/10.1512/iumj.1979.28.28058
[26] FRANKHAM, R. (1995). Effective population size/adult population size ratios in wildlife: A review. *Genet. Res.* **66** 95–107.
[27] GIBBS, A. L. and SU, F. E. (2002). On choosing and bounding probability metrics. *Int. Stat. Rev.* **70** 419–435.
[28] GRIGORESCU, I. (2007). Large deviations for a catalytic Fleming–Viot branching system. *Comm. Pure Appl. Math.* **60** 1056–1080. MR2319055 https://doi.org/10.1002/cpa.20174
[29] GRIGORESCU, I. and KANG, M. (2012). Immortal particle for a catalytic branching process. *Probab. Theory Related Fields* **153** 333–361. MR2925577 https://doi.org/10.1007/s00440-011-0347-6
[30] KATZENBERGER, G. S. (1991). Solutions of a stochastic differential equation forced onto a manifold by a large drift. *Ann. Probab.* **19** 1587–1628. MR1127717
[31] KURTZ, T. G. (1992). Averaging for martingale problems and stochastic approximation. In *Applied Stochastic Analysis* (*New Brunswick*, *NJ*, 1991) (I. Karatzas and D. Ocone, eds.). *Lect. Notes Control Inf. Sci.* **177** 186–209. Springer, Berlin. MR1169928 https://doi.org/10.1007/BFb0007058
[32] LABBÉ, C. (2013). Flots stochastiques et représentation lookdown. Ph.D. thesis, Université Pierre-et-Marie-Curie.
[33] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330 https://doi.org/10.1002/cpa.3160370408
[34] LÖBUS, J.-U. (2009). A stationary Fleming–Viot type Brownian particle system. *Math. Z.* **263** 541–581. MR2545857 https://doi.org/10.1007/s00209-008-0430-6
[35] MAILLARD, P. (2016). Speed and fluctuations of $N$-particle branching Brownian motion with spatial selection. *Probab. Theory Related Fields* **166** 1061–1173. MR3568046 https://doi.org/10.1007/s00440-016-0701-9
[36] MALLEIN, B. (2017). Branching random walk with selection at critical rate. *Bernoulli* **23** 1784–1821. MR3624878 https://doi.org/10.3150/15-BEJ796
[37] MÉLÉARD, S. and TRAN, V. C. (2012). Slow and fast scales for superprocess limits of age-structured populations. *Stochastic Process. Appl.* **122** 250–276. MR2860449 https://doi.org/10.1016/j.spa.2011.08.007
[38] MÉLÉARD, S. and VILLEMONAIS, D. (2012). Quasi-stationary distributions and population processes. *Probab. Surv.* **9** 340–410. MR2994898 https://doi.org/10.1214/11-PS191
[39] PENINGTON, S., ROBERTS, M. I. and TALYIGÁS, Z. (2022). Genealogy and spatial distribution of the $N$-particle branching random walk with polynomial tails. *Electron. J. Probab.* **27** Paper No. 93. MR4456776 https://doi.org/10.1214/22-ejp806
[40] SCHWAB, C. (2005). Krein–Rutman theorem and the principal eigenvalue. In *Numerical Methods for Elliptic and Parabolic PDEs* (*Lecture Notes*).
[41] STROOCK, D. W. and VARADHAN, S. R. S. (1971). Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* **24** 147–225. MR0277037 https://doi.org/10.1002/cpa.3160240206
[42] TOUGH, O. (2021). Scaling limit of the Fleming–Viot multicolor process. Available at arXiv:2110.05049.
[43] TOUGH, O. (2021). Asymptotic behaviour of the Fleming–Viot process. Ph.D. thesis, Duke Univ.

[44] TOUGH, O. (2022). $L^\infty$-convergence to a quasi-stationary distribution. Available at arXiv:2210.13581.
[45] TOUGH, O. (2023). Selection principle for the Fleming–Viot process with drift $-1$. Available at arXiv:2306.03585.
[46] TOUGH, O. and NOLEN, J. (2022). The Fleming–Viot process with McKean–Vlasov dynamics. *Electron. J. Probab.* **27** Paper No. 101. MR4460269 https://doi.org/10.1214/22-ejp820
[47] VILLEMONAIS, D. (2014). General approximation method for the distribution of Markov processes conditioned not to be killed. *ESAIM Probab. Stat.* **18** 441–467. MR3333998 https://doi.org/10.1051/ps/2013045
[48] XU, X. (2009). Gradient estimates for the eigenfunctions on compact manifolds with boundary and Hörmander multiplier theorem. *Forum Math.* **21** 455–476. MR2526794 https://doi.org/10.1515/FORUM.2009.021

*The Annals of Probability*
2023, Vol. 51, No. 6, 2387–2395
https://doi.org/10.1214/23-AOP1650
Main article: https://doi.org/10.1214/20-AOP1482


# ERRATUM TO "AN OPTIMAL REGULARITY RESULT FOR KOLMOGOROV EQUATIONS AND WEAK UNIQUENESS FOR SOME CRITICAL SPDES"

BY ENRICO PRIOLA[a]

*Dipartimento di Matematica, Università di Pavia,* [a]*enrico.priola@unipv.it*

## REFERENCES

[1] ATHREYA, S. R., BASS, R. F., GORDINA, M. and PERKINS, E. A. (2006). Infinite dimensional stochastic differential equations of Ornstein–Uhlenbeck type. *Stochastic Process*. *Appl*. **116** 381–406. MR2199555 https://doi.org/10.1016/j.spa.2005.10.001

[2] BASS, R. F. and PERKINS, E. A. (2012). On uniqueness in law for parabolic SPDEs and infinite-dimensional SDEs. *Electron*. *J*. *Probab*. **17** no. 36, 54. MR2928719 https://doi.org/10.1214/EJP.v17-2049

[3] DA PRATO, G. (2003). A new regularity result for Ornstein–Uhlenbeck generators and applications. *J*. *Evol*. *Equ*. **3** 485–498. MR2019031 https://doi.org/10.1007/s00028-003-0114-x

[4] DOLERA, E. and PRIOLA, E. A counterexample to $L^\infty$-gradient type estimates for Ornstein–Uhlenbeck operators. *Ann. Mat. Pura Appl.* To appear. Available at arXiv:2210.06347.

[5] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam; Kodansha, Ltd., Tokyo. MR1011252

[6] PRIOLA, E. (2009). Global Schauder estimates for a class of degenerate Kolmogorov equations. *Studia Math*. **194** 117–153. MR2534181 https://doi.org/10.4064/sm194-2-2

[7] PRIOLA, E. (2021). An optimal regularity result for Kolmogorov equations and weak uniqueness for some critical SPDEs. *Ann*. *Probab*. **49** 1310–1346. MR4255146 https://doi.org/10.1214/20-aop1482

[8] WANG, M. and WANG, Y. (1996). Properties of positive solutions for non-local reaction–diffusion problems. *Math*. *Methods Appl*. *Sci*. **19** 1141–1156. MR1409544 https://doi.org/10.1002/(SICI)1099-1476(19960925)19:14<1141::AID-MMA811>3.0.CO;2-9

[9] ZAMBOTTI, L. (2000). An analytic approach to existence and uniqueness for martingale problems in infinite dimensions. *Probab. Theory Related Fields* **118** 147–168. MR1790079 https://doi.org/10.1007/s440-000-8012-6

*MSC2020 subject classifications.* Primary 60H15, 35R60; secondary 35R15.

*Key words and phrases.* Critical SPDEs, weak uniqueness in infinite dimensions, optimal regularity for Kolmogorov operators.

VOLUME 51

2023

## Articles

## Erratum

# The Annals of Probability

## Future Issues