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# THE STATIONARY HORIZON AND SEMI-INFINITE GEODESICS IN THE DIRECTED LANDSCAPE

BY OFER BUSANI<sup>1,a</sup>, TIMO SEPPÄLÄINEN<sup>2,b</sup> AND EVAN SORENSEN<sup>2,c</sup>

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The stationary horizon (SH) is a stochastic process of coupled Brownian motions indexed by their real-valued drifts. It was first introduced by the first author as the diffusive scaling limit of the Busemann process of exponential last-passage percolation. It was independently discovered as the Busemann process of Brownian last-passage percolation by the second and third authors. We show that SH is the unique invariant distribution and an attractor of the KPZ fixed point under conditions on the asymptotic spatial slopes. It follows that SH describes the Busemann process of the directed landscape. This gives control of semi-infinite geodesics simultaneously across all initial points and directions. The countable dense set  $\Xi$  of directions of discontinuity of the Busemann process is the set of directions in which not all geodesics coalesce and in which there exist at least two distinct geodesics from each initial point. This creates two distinct families of coalescing geodesics in each  $\Xi$  direction. In  $\Xi$  directions the Busemann difference profile is distributed like Brownian local time. We describe the point process of directions  $\xi \in \Xi$  and spatial locations where the  $\xi \pm$  Busemann functions separate.

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# ON THE COMING DOWN FROM INFINITY OF COALESCING BROWNIAN MOTIONS

BY CLAYTON BARNES<sup>1,a</sup>, LEONID MYTNIK<sup>1,b</sup> AND ZHENYAO SUN<sup>2,c</sup>

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Consider a system of Brownian particles on the real line where each pair of particles coalesces at a certain rate according to their intersection local time. Assume that there are infinitely many initial particles in the system. We give a necessary and sufficient condition for the number of particles to come down from infinity. We also identify the rate of this coming down from infinity for different initial configurations.

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# LIMIT THEOREMS FOR THE VOLUMES OF SMALL CODIMENSIONAL RANDOM SECTIONS OF $\ell_p^n$ -BALLS

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We establish central limit theorems for the volumes of intersections of  $B_p^n$  (the unit ball of  $\ell_p^n$ ) with uniform random subspaces of codimension  $d$  for fixed  $d$  and  $n \rightarrow \infty$ . As a corollary we obtain higher-order approximations for expected volumes, refining previous results by Koldobsky and Lifschitz and approximations obtained from the Eldan–Klartag version of CLT for convex bodies. We also obtain a central limit theorem for the Minkowski functional of the intersection body of  $B_p^n$ , evaluated on a random vector distributed uniformly on the unit sphere.

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# DELTA-BOSE GAS FROM THE VIEWPOINT OF THE TWO-DIMENSIONAL STOCHASTIC HEAT EQUATION

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In order to give a dual, annealed description of the two-dimensional stochastic heat equation (SHE) from regularizing the noise, we consider the Schrödinger semigroup of the many-body delta-Bose gas from mollifying the delta potentials. The main theorem proves the convergences of the corresponding approximate semigroups when they act on bounded functions. For the proof we introduce a mean-field Poisson system to expand the Feynman–Kac formula of the approximate semigroups. This expansion yields infinite series, showing certain Markovian decompositions of the summands into nonconcurrent, nonconsecutive two-body interactions. Components in these decompositions are then grouped nonlinearly in time to establish the dominated convergence of the infinite series. With regards to the two-dimensional SHE, the main theorem also characterizes the  $N$ th moments for all  $N \geq 3$  under any bounded initial condition. A particular example is the constant initial condition under which the solution of the SHE has the interpretation of the partition function of the continuum directed random polymer.

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# THE MARKOV PROPERTY OF LOCAL TIMES OF BROWNIAN MOTION INDEXED BY THE BROWNIAN TREE

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We consider the model of Brownian motion indexed by the Brownian tree, which has appeared in a variety of different contexts in probability, statistical physics and combinatorics. For this model the total occupation measure is known to have a continuously differentiable density  $(\ell^x)_{x \in \mathbb{R}}$ , and we write  $(\dot{\ell}^x)_{x \in \mathbb{R}}$  for its derivative. Although the process  $(\ell^x)_{x \geq 0}$  is not Markov, we prove that the pair  $(\ell^x, \dot{\ell}^x)_{x \geq 0}$  is a time-homogeneous Markov process. We also establish a similar result for the local times of one-dimensional super-Brownian motion. Our methods rely on the excursion theory for Brownian motion indexed by the Brownian tree.

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# LONG-TIME DERIVATION AT EQUILIBRIUM OF THE FLUCTUATING BOLTZMANN EQUATION

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We study the deterministic evolution of a gas of hard spheres, initially distributed according to the equilibrium measure, and prove that, in the low-density limit, the fluctuation field converges to a Gaussian process governed by the fluctuating Boltzmann equation. This result holds for arbitrarily long times.

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## SCALING LIMIT OF AN ADAPTIVE CONTACT PROCESS

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We introduce and study an interacting particle system evolving on the  $d$ -dimensional torus  $(\mathbb{Z}/N\mathbb{Z})^d$ . Each vertex of the torus can be either empty or occupied by an individual of type  $\lambda \in (0, \infty)$ . An individual of type  $\lambda$  dies with rate one and gives birth at each neighboring empty position with rate  $\lambda$ ; moreover, when the birth takes place, the newborn individual is likely to have the same type as the parent but has a small probability of being a mutant. A mutant child of an individual of type  $\lambda$  has type chosen according to a probability kernel. We consider the asymptotic behavior of this process when  $N \rightarrow \infty$  and, simultaneously, the mutation probability tends to zero fast enough that mutations are sufficiently separated in time so that the amount of time spent on configurations with more than one type becomes negligible. We show that, after a suitable time scaling and deletion of the periods of time spent on configurations with more than one type, the process converges to a Markov jump process on  $(0, \infty)$ , whose rates we characterize.

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# STOCHASTIC HOMOGENIZATION WITH SPACE-TIME ERGODIC DIVERGENCE-FREE DRIFT

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We prove that diffusion equations with a space-time stationary and ergodic, divergence-free drift homogenize in law to a deterministic stochastic partial differential equation with Stratonovich transport noise. In the absence of spatial ergodicity, the drift is only partially absorbed into the skew-symmetric part of the flux through the use of an appropriately defined stream matrix. This leaves a time-dependent, spatially-homogenous transport which, for mildly decorrelating fields, converges to a Brownian noise with deterministic covariance in the homogenization limit. The results apply to uniformly elliptic, stationary and ergodic environments in which the drift admits a suitably defined stationary and  $L^2$ -integrable stream matrix.

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## ERRATUM: “EXISTENCE OF AN UNBOUNDED NODAL HYPERSURFACE FOR SMOOTH GAUSSIAN FIELDS IN DIMENSION $d \geq 3$ ”

BY HUGO DUMINIL-COPIN<sup>1,a</sup>, ALEJANDRO RIVERA<sup>2,b</sup>,  
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This note is an erratum to the paper “Existence of an unbounded nodal hypersurface for smooth Gaussian fields in dimension  $d \geq 3$ ” (*Ann. Probab.* **51** 228–276 (2023) <https://doi.org/10.1214/22-AOP1594>). The published version of this paper contains one error: Proposition 1.12 therein is stated with a sprinkling  $R^{-2+\theta_0}$  for some (small)  $\theta_0 > 0$ , which we cannot afford in Section 5, where this result is applied at mesoscopic scales. We circumvent this issue by proving a stronger version of this proposition, interesting in its own right, which contains no sprinkling. We thus obtain that, for a general class of positively correlated smooth Gaussian fields  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  with rapid decay of correlations (including the Bargmann–Fock field), large planar clusters in  $\{f \geq 0\} \cap (\mathbb{R}^2 \times \{0\})$  typically belong to clusters in  $\{f \geq 0\}$  which are not confined in thin slabs.

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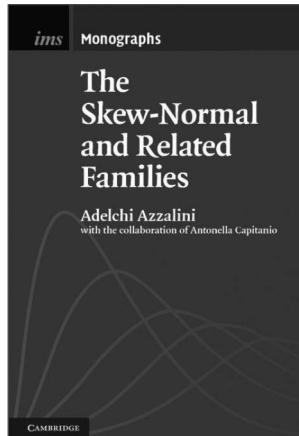
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