ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

An Official Journal of the
Institute of Mathematical Statistics

Articles

Vol. 52, No. 2—March 2024

THE ANNALS OF PROBABILITY Vol. 52, No. 2, pp. 387–763 March 2024

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Probability **[ISSN 0091-1798 (print); ISSN 2168-894X (online)],** Volume 52, Number 2, March 2024. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, OH 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, MD 21769, USA.

The Annals of Probability
2024, Vol. 52, No. 2, 387–433
https://doi.org/10.1214/23-AOP1664

UNIVERSALITY CLASSES FOR THE COALESCENT STRUCTURE OF HEAVY-TAILED GALTON–WATSON TREES

BY SIMON HARRIS[1,a], SAMUEL G. G. JOHNSTON[2,b] AND JUAN CARLOS PARDO[3,c]

[1]*Department of Mathematics, University of Auckland,* [a]*simon.harris@auckland.ac.nz*
[2]*Department of Mathematics, King's College London,* [b]*samuel.g.johnston@kcl.ac.uk*
[3]*Centro de Investigación en Matemáticas,* [c]*jcpardo@cimat.mx*

Consider a population evolving as a critical continuous-time Galton–Watson (GW) tree. Conditional on the population surviving until a large time T, sample k individuals uniformly at random (without replacement) from amongst those alive at time T. What is the genealogy of this sample of individuals? In cases where the offspring distribution has finite variance, the probabilistic properties of the joint ancestry of these k particles are well understood, as seen in (*Ann. Appl. Probab.* **30** (2020) 1368–1414; *Electron. J. Probab.* **24** (2019) 1–35). In the present article, we study the joint ancestry of a sample of k particles under the following regime: the offspring distribution has mean 1 (critical) and the tails of the offspring distribution are *heavy* in that $\alpha \in (1, 2]$ is the supremum over indices β such that the βth moment is finite. We show that for each α, after rescaling time by $1/T$, there is a universal stochastic process describing the joint coalescent structure of the k distinct particles. The special case $\alpha = 2$ generalises the known case of sampling from critical GW trees with finite variance where only pairwise mergers are observed and the genealogical tree is, roughly speaking, some kind of mixture of time-changed Kingman coalescents. The cases $\alpha \in (1, 2)$ introduce new universal limiting partition-valued stochastic processes with interesting probabilistic structures, which, in particular, have representations connected to the Lauricella function and the Dirichlet distribution and whose coalescent structures exhibit multiple-mergers of family lines. Moreover, in the case $\alpha \in (1, 2)$, we show that the coalescent events of the ancestry of the k particles are associated with birth events that produce giant numbers of offspring of the same order of magnitude as the entire population size, and we compute the joint law of the ancestry together with the sizes of these giant births.

REFERENCES

[1] ABRAHAM, R. and DEBS, P. (2020). Penalization of Galton–Watson processes. *Stochastic Process. Appl.* **130** 3095–3119. MR4080739 https://doi.org/10.1016/j.spa.2019.09.005

[2] ALDOUS, D. and POPOVIC, L. (2005). A critical branching process model for biodiversity. *Adv. in Appl. Probab.* **37** 1094–1115. MR2193998 https://doi.org/10.1239/aap/1134587755

[3] ATHREYA, K. B. (2012). Coalescence in the recent past in rapidly growing populations. *Stochastic Process. Appl.* **122** 3757–3766. MR2965924 https://doi.org/10.1016/j.spa.2012.06.015

[4] ATHREYA, K. B. (2012). Coalescence in critical and subcritical Galton–Watson branching processes. *J. Appl. Probab.* **49** 627–638. MR3012088 https://doi.org/10.1239/jap/1346955322

[5] ATHREYA, K. B. (2016). Coalescence in branching processes. In *Branching Processes and Their Applications*. *Lect. Notes Stat.* **219** 3–22. Springer, Cham. MR3587978

[6] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes*. *Die Grundlehren der Mathematischen Wissenschaften*, *Band* 196. Springer, New York. MR0373040

[7] BERESTYCKI, J., BERESTYCKI, N. and SCHWEINSBERG, J. (2007). Beta-coalescents and continuous stable random trees. *Ann. Probab.* **35** 1835–1887. MR2349577 https://doi.org/10.1214/009117906000001114

MSC2020 subject classifications. Primary 60J80, 60G09.
Key words and phrases. Galton–Watson tree, coalescent process, genealogy, spines, regularly varying functions.

[8] BERTOIN, J. and LE GALL, J.-F. (2000). The Bolthausen–Sznitman coalescent and the genealogy of continuous-state branching processes. *Probab. Theory Related Fields* **117** 249–266. MR1771663 https://doi.org/10.1007/s004400050006

[9] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871 https://doi.org/10.1017/CBO9780511721434

[10] BOENKOST, F., FOUTEL-RODIER, F. and SCHERTZER, E. (2022). The genealogy of nearly critical branching processes in varying environment. Available at arXiv:2207.11612.

[11] BÜHLER, W. J. (1971). Generations and degree of relationship in supercritical Markov branching processes. *Z. Wahrsch. Verw. Gebiete* **18** 141–152. MR0295444 https://doi.org/10.1007/BF00569184

[12] CHAMAYOU, J.-F. and WESOŁOWSKI, J. (2009). Lauricella and Humbert functions through probabilistic tools. *Integral Transforms Spec. Funct.* **20** 529–538. MR2543763 https://doi.org/10.1080/10652460802645750

[13] CHEEK, D. and JOHNSTON, S. G. G. (2023). Ancestral reproductive bias in branching processes. *J. Math. Biol.* **86** Paper No. 70, 24. MR4572129 https://doi.org/10.1007/s00285-023-01907-7

[14] DURRETT, R. (1978). The genealogy of critical branching processes. *Stochastic Process. Appl.* **8** 101–116. MR0511879 https://doi.org/10.1016/0304-4149(78)90071-6

[15] GERNHARD, T. (2008). The conditioned reconstructed process. *J. Theoret. Biol.* **253** 769–778. MR2964590 https://doi.org/10.1016/j.jtbi.2008.04.005

[16] GROSJEAN, N. and HUILLET, T. (2018). On the genealogy and coalescence times of Bienaymé–Galton–Watson branching processes. *Stoch. Models* **34** 1–24. MR3776534 https://doi.org/10.1080/15326349.2017.1375958

[17] HARRIS, S. C., JOHNSTON, S. G. G. and PARDO, J.-C. (2023). Universality classes for the coalescent structure of critical Galton–Watson trees with infinite mean. In preparation.

[18] HARRIS, S. C., JOHNSTON, S. G. G. and ROBERTS, M. I. (2020). The coalescent structure of continuous-time Galton–Watson trees. *Ann. Appl. Probab.* **30** 1368–1414. MR4133376 https://doi.org/10.1214/19-AAP1532

[19] HARRIS, S. C., PALAU, S. and PARDO, J.-C. (2022). The coalescent structure of Galton–Watson trees in varying environments. Available at arXiv:2207.10923.

[20] HARRIS, S. C. and ROBERTS, M. I. (2017). The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 226–242. MR3606740 https://doi.org/10.1214/15-AIHP714

[21] JOHNSTON, S. G. G. (2019). The genealogy of Galton–Watson trees. *Electron. J. Probab.* **24** 1–35. MR4003147 https://doi.org/10.1214/19-ejp355

[22] JOHNSTON, S. G. G. and LAMBERT, A. (2023). The coalescent structure of uniform and Poisson samples from multitype branching processes. *Ann. Appl. Probab.* To appear.

[23] JOHNSTON, S. G. G. and PROCHNO, J. (2022). Faà di Bruno's formula and inversion of power series. *Adv. Math.* **395** 108080. MR4363579 https://doi.org/10.1016/j.aim.2021.108080

[24] KINGMAN, J. F. C. (1982). The coalescent. *Stochastic Process. Appl.* **13** 235–248. MR0671034 https://doi.org/10.1016/0304-4149(82)90011-4

[25] LAGERÅS, A. N. and SAGITOV, S. (2008). Reduced branching processes with very heavy tails. *J. Appl. Probab.* **45** 190–200. MR2409320 https://doi.org/10.1239/jap/1208358961

[26] LAMBERT, A. (2003). Coalescence times for the branching process. *Adv. in Appl. Probab.* **35** 1071–1089. MR2014270 https://doi.org/10.1239/aap/1067436335

[27] LAMBERT, A. and POPOVIC, L. (2013). The coalescent point process of branching trees. *Ann. Appl. Probab.* **23** 99–144. MR3059232 https://doi.org/10.1214/11-AAP820

[28] LAMBERT, A. and STADLER, T. (2013). Birth-death models and coalescent point proceses: The shape and probability of reconstructed phylogenies. *Theor. Popul. Biol.* **90** 113–128.

[29] LAURICELLA, G. (1893). Sulle funzioni ipergeometriche a più variabili. *Rend. Circ. Mat. Palermo* **7** 111–158.

[30] LE, V. (2014). Coalescence times for the Bienaymé–Galton–Watson process. *J. Appl. Probab.* **51** 209–218. MR3189452 https://doi.org/10.1239/jap/1395771424

[31] O'CONNELL, N. (1995). The genealogy of branching processes and the age of our most recent common ancestor. *Adv. in Appl. Probab.* **27** 418–442. MR1334822 https://doi.org/10.2307/1427834

[32] PAKES, A. G. (2010). Critical Markov branching process limit theorems allowing infinite variance. *Adv. in Appl. Probab.* **42** 460–488. MR2675112 https://doi.org/10.1239/aap/1275055238

[33] SLACK, R. S. (1968). A branching process with mean one and possibly infinite variance. *Z. Wahrsch. Verw. Gebiete* **9** 139–145. MR0228077 https://doi.org/10.1007/BF01851004

[34] YAKYMIV, A. L. (1980). Reduced branching processes. *Theory Probab. Appl.* **20** 602–612.

[35] ZUBKOV, A. M. (1975). Limit distributions of the distance to the nearest common ancestor. *Theory Probab. Appl.* **20** 602–612. MR0397915

The Annals of Probability
2024, Vol. 52, No. 2, 434–501
https://doi.org/10.1214/23-AOP1665

MUTUAL INFORMATION FOR THE SPARSE STOCHASTIC BLOCK MODEL

BY TOMAS DOMINGUEZ[1,a] AND JEAN-CHRISTOPHE MOURRAT[2,b]

[1]*Department of Mathematics, University of Toronto,* [a]*tomas.dominguezchiozza@mail.utoronto.ca*
[2]*Department of Mathematics, ENS Lyon,* [b]*jean-christophe.mourrat@ens-lyon.fr*

We consider the problem of recovering the community structure in the stochastic block model with two communities. We aim to describe the mutual information between the observed network and the actual community structure in the sparse regime, where the total number of nodes diverges while the average degree of a given node remains bounded. Our main contributions are a conjecture for the limit of this quantity, which we express in terms of a Hamilton–Jacobi equation posed over a space of probability measures, and a proof that this conjectured limit provides a lower bound for the asymptotic mutual information. The well-posedness of the Hamilton–Jacobi equation is obtained in our companion paper. In the case when links across communities are more likely than links within communities, the asymptotic mutual information is known to be given by a variational formula. We also show that our conjectured limit coincides with this formula in this case.

REFERENCES

[1] ABBE, E. (2018). *Community Detection and Stochastic Block Models*. *Foundations and Trends in Communications and Information Theory*. Now Publishers, Hanover.

[2] ABBE, E., BANDEIRA, A. S. and HALL, G. (2016). Exact recovery in the stochastic block model. *IEEE Trans. Inf. Theory* **62** 471–487. MR3447993 https://doi.org/10.1109/TIT.2015.2490670

[3] ABBE, E. and SANDON, C. (2015). Detection in the stochastic block model with multiple clusters: proof of the achievability conjectures, acyclic BP, and the information-computation gap. arXiv e-prints.

[4] ABBE, E., CORNACCHIA, E., GU, Y. and POLYANSKIY, Y. (2021). Stochastic block model entropy and broadcasting on trees with survey. In *Proceedings of Thirty Fourth Conference on Learning Theory* **134** 1–25. PMLR.

[5] ABBE, E. and MONTANARI, A. (2015). Conditional random fields, planted constraint satisfaction, and entropy concentration. *Theory Comput.* **11** 413–443. MR3446024 https://doi.org/10.4086/toc.2015.v011a017

[6] ABBE, E. and SANDON, C. (2015). Recovering communities in the general stochastic block model without knowing the parameters. In *Advances in Neural Information Processing Systems* **28**. Curran Associates, Red Hook.

[7] AIROLDI, E. M., BLEI, D. M., FIENBERG, S. E. and XING, E. P. (2008). Mixed membership stochastic blockmodels. *J. Mach. Learn. Res.* **9** 1981–2014.

[8] BALL, B., KARRER, B. and NEWMAN, M. E. J. (2011). Efficient and principled method for detecting communities in networks. *Phys. Rev. E* **84**.

[9] BARBIER, J., CHAN, C. L. and MACRIS, N. (2019). Mutual information for the stochastic block model by the adaptive interpolation method. In 2019 *IEEE International Symposium on Information Theory* 405–409. IEEE Press, New York.

[10] BARBIER, J., DIA, M., MACRIS, N., KRZAKALA, F., LESIEUR, T. and ZDEBOROVÁ, L. (2016). Mutual information for symmetric rank-one matrix estimation: A proof of the replica formula. In *Advances in Neural Information Processing Systems* (*NIPS*) **29** 424–432.

[11] BARBIER, J. and MACRIS, N. (2019). The adaptive interpolation method: A simple scheme to prove replica formulas in Bayesian inference. *Probab. Theory Related Fields* **174** 1133–1185. MR3980313 https://doi.org/10.1007/s00440-018-0879-0

MSC2020 subject classifications. Primary 60K35, 82B44; secondary 35D40, 35R15.
Key words and phrases. Stochastic block model, mutual information, free energy, Hamilton–Jacobi equation.

[12] BARBIER, J., MACRIS, N. and MIOLANE, L. (2017). The layered structure of tensor estimation and its mutual information. In 2017 55*th Annual Allerton Conference on Communication*, *Control*, *and Computing* (*Allerton*) 1056–1063. IEEE.

[13] BARBIER, J. and PANCHENKO, D. Free Energy in the Stochastic Block Model In Sparse Regime: Disassortative case. Personal Communication.

[14] BARBIER, J. and PANCHENKO, D. (2022). Strong replica symmetry in high-dimensional optimal Bayesian inference. *Comm. Math. Phys.* **393** 1199–1239. MR4453233 https://doi.org/10.1007/s00220-022-04387-w

[15] BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396 https://doi.org/10.1002/rsa.20168

[16] BOPPANA, R. B. (1987). Eigenvalues and graph bisection: An average-case analysis. In 28*th Annual Symposium on Foundations of Computer Science* 280–285.

[17] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities*: *A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 https://doi.org/10.1093/acprof:oso/9780199535255.001.0001

[18] BUI, T. N., CHAUDHURI, S., LEIGHTON, F. T. and SIPSER, M. (1987). Graph bisection algorithms with good average case behavior. *Combinatorica* **7** 171–191. MR0905164 https://doi.org/10.1007/BF02579448

[19] CABREROS, I., ABBE, E. and TSIRIGOS, A. (2016). Detecting community structures in Hi-C genomic data. In *Conference on Information Science and Systems* 584–589.

[20] CALTAGIRONE, F., LELARGE, M. and MIOLANE, L. (2018). Recovering asymmetric communities in the stochastic block model. *IEEE Trans. Netw. Sci. Eng.* **5** 237–246. MR3859911 https://doi.org/10.1109/TNSE.2017.2758201

[21] CHEN, H., MOURRAT, J.-C. and XIA, J. (2022). Statistical inference of finite-rank tensors. *Ann. Henri Lebesgue* **5** 1161–1189. MR4526249 https://doi.org/10.5802/ahl.146

[22] CHEN, H.-B. (2022). Hamilton–Jacobi equations for nonsymmetric matrix inference. *Ann. Appl. Probab.* **32** 2540–2567. MR4474513 https://doi.org/10.1214/21-aap1739

[23] CHEN, H.-B. and XIA, J. (2022). Hamilton–Jacobi equations for inference of matrix tensor products. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 755–793. MR4421607 https://doi.org/10.1214/21-aihp1183

[24] CHEN, H.-B. and XIA, J. (2022). Hamilton–Jacobi equations with monotone nonlinearities on convex cones. ArXiv E-prints.

[25] CHEN, H.-B. and XIA, J. (2023). Free energy of multi-layer generalized linear models. *Comm. Math. Phys.* **400** 1861–1913. MR4595611 https://doi.org/10.1007/s00220-022-04630-4

[26] CHEN, J. and YUAN, B. (2006). Detecting functional modules in the yeast protein-protein interaction network. *Bioinformatics* **22** 2283–2290.

[27] COJA-OGHLAN, A., KRZAKALA, F., PERKINS, W. and ZDEBOROVÁ, L. (2018). Information-theoretic thresholds from the cavity method. *Adv. Math.* **333** 694–795. MR3818090 https://doi.org/10.1016/j.aim.2018.05.029

[28] DECELLE, A., KRZAKALA, F., MOORE, C. and ZDEBOROVÁ, L. (2011). Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. *Phys. Rev. E* **84** 066106.

[29] DESHPANDE, Y., ABBE, E. and MONTANARI, A. (2017). Asymptotic mutual information for the balanced binary stochastic block model. *Inf. Inference* **6** 125–170. MR3671474 https://doi.org/10.1093/imaiai/iaw017

[30] DOMINGUEZ, T. and MOURRAT, J.-C. (2022). Infinite-dimensional Hamilton–Jacobi equations for statistical inference on sparse graphs. Preprint. Available at arXiv:2209.04516.

[31] DOMINGUEZ, T. and MOURRAT, J.-C. (2023). Statistical mechanics of mean-field disordered systems: a Hamilton-Jacobi approach. Preprint. Available at arXiv:2311.08976.

[32] DYER, M. E. and FRIEZE, A. M. (1989). The solution of some random NP-hard problems in polynomial expected time. *J. Algorithms* **10** 451–489. MR1022107 https://doi.org/10.1016/0196-6774(89)90001-1

[33] GOPALAN, P. K. and BLEI, D. M. (2013). Efficient discovery of overlapping communities in massive networks. *Proc. Natl. Acad. Sci. USA* **110** 14534–14539. MR3105375 https://doi.org/10.1073/pnas.1221839110

[34] GU, Y. and POLYANSKIY, Y. (2023). Uniqueness of BP fixed point for the Potts model and applications to community detection. Preprint. Available at arXiv:2303.14688.

[35] HOLLAND, P. W., LASKEY, K. B. and LEINHARDT, S. (1983). Stochastic blockmodels: First steps. *Soc. Netw.* **5** 109–137. MR0718088 https://doi.org/10.1016/0378-8733(83)90021-7

[36] KADMON, J. and GANGULI, S. (2018). Statistical mechanics of low-rank tensor decomposition. In *Advances in Neural Information Processing Systems* 8201–8212.

[37] KANADE, V., MOSSEL, E. and SCHRAMM, T. (2016). Global and local information in clustering labeled block models. *IEEE Trans. Inf. Theory* **62** 5906–5917. MR3552430 https://doi.org/10.1109/TIT.2016.2516564

[38] KINGMAN, J. F. C. (1993). *Poisson Processes. Oxford Studies in Probability* **3**. The Clarendon Press, New York. MR1207584

[39] KIREEVA, A. and MOURRAT, J.-C. (2023). Breakdown of a concavity property of mutual information for non-Gaussian channels. Preprint. Available at arXiv:2304.05129.

[40] LELARGE, M. and MIOLANE, L. (2019). Fundamental limits of symmetric low-rank matrix estimation. *Probab. Theory Related Fields* **173** 859–929. MR3936148 https://doi.org/10.1007/s00440-018-0845-x

[41] MIOLANE, L. (2017). Fundamental limits of low-rank matrix estimation: the non-symmetric case. Preprint. Available at arXiv:1702.00473.

[42] LESIEUR, T., MIOLANE, L., LELARGE, M., KRZAKALA, F. and ZDEBOROVÁ, L. (2017). Statistical and computational phase transitions in spiked tensor estimation. In 2017 *IEEE International Symposium on Information Theory* (*ISIT*) 511–515. IEEE.

[43] LINDEN, G., SMITH, B. and YORK, J. (2003). Amazon.com recommendations: Item-to-item collaborative filtering. *IEEE Internet Comput.* **7** 76–80.

[44] LUNEAU, C., BARBIER, J. and MACRIS, N. (2021). Mutual information for low-rank even-order symmetric tensor estimation. *Inf. Inference* **10** 1167–1207. MR4353122 https://doi.org/10.1093/imaiai/iaaa022

[45] LUNEAU, C., MACRIS, N. and BARBIER, J. (2020). High-dimensional rank-one nonsymmetric matrix decomposition: The spherical case. In 2020 *IEEE International Symposium on Information Theory* (*ISIT*) 2646–2651. IEEE.

[46] MASSOULIÉ, L. (2014). Community detection thresholds and the weak Ramanujan property. In *STOC'*14—*Proceedings of the* 2014 *ACM Symposium on Theory of Computing* 694–703. ACM, New York. MR3238997

[47] MAYYA, V. and REEVES, G. (2019). Mutual information in community detection with covariate information and correlated networks. In 2019 57*th Annual Allerton Conference on Communication, Control, and Computing* (*Allerton*) 602–607. IEEE.

[48] MOSSEL, E., NEEMAN, J. and SLY, A. (2015). Reconstruction and estimation in the planted partition model. *Probab. Theory Related Fields* **162** 431–461. MR3383334 https://doi.org/10.1007/s00440-014-0576-6

[49] MOSSEL, E., NEEMAN, J. and SLY, A. (2016). Consistency thresholds for the planted bisection model. *Electron. J. Probab.* **21** Paper No. 21, 24. MR3485363 https://doi.org/10.1214/16-EJP4185

[50] MOSSEL, E., NEEMAN, J. and SLY, A. (2016). Belief propagation, robust reconstruction and optimal recovery of block models. *Ann. Appl. Probab.* **26** 2211–2256. MR3543895 https://doi.org/10.1214/15-AAP1145

[51] MOSSEL, E., NEEMAN, J. and SLY, A. (2018). A proof of the block model threshold conjecture. *Combinatorica* **38** 665–708. MR3876880 https://doi.org/10.1007/s00493-016-3238-8

[52] MOSSEL, E. and XU, J. (2016). Local algorithms for block models with side information [extended abstract]. In *ITCS'*16—*Proceedings of the* 2016 *ACM Conference on Innovations in Theoretical Computer Science* 71–80. ACM, New York. MR3629812

[53] MOURRAT, J.-C. (2020). Hamilton–Jacobi equations for finite-rank matrix inference. *Ann. Appl. Probab.* **30** 2234–2260. MR4149527 https://doi.org/10.1214/19-AAP1556

[54] MOURRAT, J.-C. (2021). Hamilton–Jacobi equations for mean-field disordered systems. *Ann. Henri Lebesgue* **4** 453–484. MR4275243 https://doi.org/10.5802/ahl.77

[55] MOURRAT, J.-C. (2021). Nonconvex interactions in mean-field spin glasses. *Probab. Math. Phys.* **2** 281–339. MR4408014 https://doi.org/10.2140/pmp.2021.2.281

[56] MOURRAT, J.-C. (2023). Free energy upper bound for mean-field vector spin glasses. *Ann. Inst. Henri Poincaré Probab. Stat.* **59** 1143–1182. MR4635706 https://doi.org/10.1214/22-aihp1292

[57] NEWMAN, M. E. J., WATTS, D. J. and STROGATZ, S. H. (2002). Random graph models of social networks. *Proc. Natl. Acad. Sci. USA* **99** 2566–2572. https://doi.org/10.1073/pnas.012582999

[58] PANCHENKO, D. (2013). Spin glass models from the point of view of spin distributions. *Ann. Probab.* **41** 1315–1361. MR3098679 https://doi.org/10.1214/11-AOP696

[59] PANCHENKO, D. (2013). *The Sherrington–Kirkpatrick Model. Springer Monographs in Mathematics*. Springer, New York. MR3052333 https://doi.org/10.1007/978-1-4614-6289-7

[60] PANCHENKO, D. (2014). On the replica symmetric solution of the K-sat model. *Electron. J. Probab.* **19** no. 67, 17. MR3248196 https://doi.org/10.1214/EJP.v19-2963

[61] PANCHENKO, D. (2014). Structure of 1-RSB asymptotic Gibbs measures in the diluted p-spin models. *J. Stat. Phys.* **155** 1–22. MR3180967 https://doi.org/10.1007/s10955-014-0955-5

[62] PANCHENKO, D. (2016). Structure of finite-RSB asymptotic Gibbs measures in the diluted spin glass models. *J. Stat. Phys.* **162** 1–42. MR3439329 https://doi.org/10.1007/s10955-015-1385-8

[63] PANCHENKO, D. (2019). Lecture Notes on Probability Theory.
[64] REEVES, G. (2020). Information-theoretic limits for the matrix tensor product. *IEEE J. Sel. Areas Inf. Theory* **1** 777–798.
[65] REEVES, G., MAYYA, V. and VOLFOVSKY, A. (2019). The geometry of community detection via the mmse matrix. In 2019 *IEEE International Symposium on Information Theory* (*ISIT*) 400–404. IEEE.
[66] SHI, J. and MALIK, J. (2000). Normalized cuts and image segmentation. *IEEE Trans. Pattern Anal. Mach. Intell.* **22** 888–905.
[67] SØRLIE, T., PEROU, C. M. et al. (2001). Gene expression patterns of breast carcinomas distinguish tumor subclasses with clinical implications. *Proc. Natl. Acad. Sci. USA* **98** 10869–10874.
[68] CHEN, H.-B. and XIA, J. (2022). Hamilton-Jacobi equations from mean-field spin glasses. arXiv e-prints.
[69] WANG, Y. J. and WONG, G. Y. (1987). Stochastic blockmodels for directed graphs. *J. Amer. Statist. Assoc.* **82** 8–19. MR0883333
[70] WASSERMAN, L. (2017). Stephen Fienberg: Superman of statistics. *Proc. Natl. Acad. Sci. USA* **114** 3002–3003. MR3632013 https://doi.org/10.1073/pnas.1700160114
[71] WHITE, H. C., BOORMAN, S. A. and BREIGER, R. L. (1976). Social structure from multiple networks. I. Blockmodels of roles and positions. *Am. J. Sociol.* **81** 730–780.
[72] YU, Q. and POLYANSKIY, Y. (2023). Ising model on locally tree-like graphs: Uniqueness of solutions to cavity equations. Preprint. Available at arXiv:2211.15242.

The Annals of Probability
2024, Vol. 52, No. 2, 502–544
https://doi.org/10.1214/23-AOP1644

A LIMIT LAW FOR THE MOST FAVORITE POINT OF SIMPLE RANDOM WALK ON A REGULAR TREE

BY MAREK BISKUP[1,a] AND OREN LOUIDOR[2,b]

[1]*Department of Mathematics, UCLA,* [a]*biskup@math.ucla.edu*
[2]*Faculty of Industrial Engineering and Management, Technion,* [b]*oren.louidor@gmail.com*

We consider a continuous-time random walk on a regular tree of finite depth and study its favorite points among the leaf vertices. For the walk started from a leaf vertex and stopped upon hitting the root, we prove that, in the limit as the depth of the tree tends to infinity, the suitably scaled and centered maximal time spent at any leaf converges to a randomly-shifted Gumbel law. The random shift is characterized using a derivative-martingale like object associated with square-root local-time process on the tree.

REFERENCES

[1] ABE, Y. (2015). Maximum and minimum of local times for two-dimensional random walk. *Electron. Commun. Probab.* **20** no. 22, 14. MR3320410 https://doi.org/10.1214/ECP.v20-3877

[2] ABE, Y. (2018). Extremes of local times for simple random walks on symmetric trees. *Electron. J. Probab.* **23** Paper No. 40, 41. MR3806408 https://doi.org/10.1214/18-EJP164

[3] ABE, Y. and BISKUP, M. (2022). Exceptional points of two-dimensional random walks at multiples of the cover time. *Probab. Theory Related Fields* **183** 1–55. MR4421170 https://doi.org/10.1007/s00440-022-01113-4

[4] ABE, Y., BISKUP, M. and LEE, S. (2023). Exceptional points of discrete-time random walks in planar domains. *Electron. J. Probab.* **28** Paper No. 1. MR4664460 https://doi.org/10.1214/23-ejp988

[5] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680 https://doi.org/10.1214/12-AOP750

[6] AÏDÉKON, E., BERESTYCKI, J., BRUNET, É. and SHI, Z. (2013). Branching Brownian motion seen from its tip. *Probab. Theory Related Fields* **157** 405–451. MR3101852 https://doi.org/10.1007/s00440-012-0461-0

[7] AÏDÉKON, É., BERESTYCKI, N., JEGO, A. and LUPU, T. (2023). Multiplicative chaos of the Brownian loop soup. *Proc. Lond. Math. Soc.* (3) **126** 1254–1393. MR4574830 https://doi.org/10.1112/plms.12511

[8] AÏDÉKON, E., HU, Y. and SHI, Z. (2020). Points of infinite multiplicity of planar Brownian motion: Measures and local times. *Ann. Probab.* **48** 1785–1825. MR4124525 https://doi.org/10.1214/19-AOP1407

[9] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2011). Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.* **64** 1647–1676. MR2838339 https://doi.org/10.1002/cpa.20387

[10] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2012). Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.* **22** 1693–1711. MR2985174 https://doi.org/10.1214/11-AAP809

[11] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2013). The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** 535–574. MR3129797 https://doi.org/10.1007/s00440-012-0464-x

[12] ARGUIN, L.-P. and OUIMET, F. (2016). Extremes of the two-dimensional Gaussian free field with scale-dependent variance. *ALEA Lat. Amer. J. Probab. Math. Stat.* **13** 779–808. MR3541850 https://doi.org/10.30757/alea.v13-31

[13] BASS, R. F., BURDZY, K. and KHOSHNEVISAN, D. (1994). Intersection local time for points of infinite multiplicity. *Ann. Probab.* **22** 566–625. MR1288124

[14] BELIUS, D., ROSEN, J. and ZEITOUNI, O. (2019). Barrier estimates for a critical Galton–Watson process and the cover time of the binary tree. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 127–154. MR3901643 https://doi.org/10.1214/17-aihp878

[15] BIGGINS, J. D. and KYPRIANOU, A. E. (2004). Measure change in multitype branching. *Adv. in Appl. Probab.* **36** 544–581. MR2058149 https://doi.org/10.1239/aap/1086957585

MSC2020 subject classifications. Primary 60G70, 60G50; secondary 05C81.
Key words and phrases. Random walk, favorite point, extremal value, multiplicative chaos.

[16] BISKUP, M. (2020). Extrema of the two-dimensional discrete Gaussian free field. In *Random Graphs*, *Phase Transitions*, *and the Gaussian Free Field*. *Springer Proc*. *Math*. *Stat*. **304** 163–407. Springer, Cham. MR4043225 https://doi.org/10.1007/978-3-030-32011-9_3

[17] BISKUP, M., GUFLER, S. and LOUIDOR, O. (2023). Near-maxima of the two-dimensional discrete Gaussian free field. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. To appear. Available at arXiv:2010.13939.

[18] BISKUP, M. and LOUIDOR, O. (2016). Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm*. *Math*. *Phys*. **345** 271–304. MR3509015 https://doi.org/10.1007/s00220-015-2565-8

[19] BISKUP, M. and LOUIDOR, O. (2018). Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv*. *Math*. **330** 589–687. MR3787554 https://doi.org/10.1016/j.aim.2018.02.018

[20] BISKUP, M. and LOUIDOR, O. (2019). On intermediate level sets of two-dimensional discrete Gaussian free field. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **55** 1948–1987. MR4029145 https://doi.org/10.1214/18-AIHP939

[21] BISKUP, M. and LOUIDOR, O. (2020). Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. *Comm*. *Math*. *Phys*. **375** 175–235. MR4082182 https://doi.org/10.1007/s00220-020-03698-0

[22] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem*. *Amer*. *Math*. *Soc*. **44** 285, iv+190. MR0705746 https://doi.org/10.1090/memo/0285

[23] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of nonlattice branching random walk. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **52** 1897–1924. MR3573300 https://doi.org/10.1214/15-AIHP703

[24] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm*. *Pure Appl*. *Math*. **69** 62–123. MR3433630 https://doi.org/10.1002/cpa.21621

[25] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm*. *Pure Appl*. *Math*. **31** 531–581. MR0494541 https://doi.org/10.1002/cpa.3160310502

[26] CHEN, X. and HE, H. (2020). Lower deviation and moderate deviation probabilities for maximum of a branching random walk. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **56** 2507–2539. MR4164846 https://doi.org/10.1214/20-AIHP1048

[27] CORTINES, A., HARTUNG, L. and LOUIDOR, O. (2019). The structure of extreme level sets in branching Brownian motion. *Ann*. *Probab*. **47** 2257–2302. MR3980921 https://doi.org/10.1214/18-AOP1308

[28] CORTINES, A., HARTUNG, L. and LOUIDOR, O. (2019). Decorated random walk restricted to stay below a curve (supplementary material to [27]). Available at https://doi.org/10.1214/18-AOP1308SUPP.

[29] CORTINES, A., LOUIDOR, O. and SAGLIETTI, S. (2021). A scaling limit for the cover time of the binary tree. *Adv*. *Math*. **391** Paper No. 107974, 78. MR4303734 https://doi.org/10.1016/j.aim.2021.107974

[30] DEMBO, A., PERES, Y., ROSEN, J. and ZEITOUNI, O. (2001). Thick points for planar Brownian motion and the Erdős–Taylor conjecture on random walk. *Acta Math*. **186** 239–270. MR1846031 https://doi.org/10.1007/BF02401841

[31] DING, J. (2014). Asymptotics of cover times via Gaussian free fields: Bounded-degree graphs and general trees. *Ann*. *Probab*. **42** 464–496. MR3178464 https://doi.org/10.1214/12-AOP822

[32] DING, J., ROY, R. and ZEITOUNI, O. (2017). Convergence of the centered maximum of log-correlated Gaussian fields. *Ann*. *Probab*. **45** 3886–3928. MR3729618 https://doi.org/10.1214/16-AOP1152

[33] DYNKIN, E. B. (1984). Gaussian and non-Gaussian random fields associated with Markov processes. *J*. *Funct*. *Anal*. **55** 344–376. MR0734803 https://doi.org/10.1016/0022-1236(84)90004-1

[34] EISENBAUM, N., KASPI, H., MARCUS, M. B., ROSEN, J. and SHI, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann*. *Probab*. **28** 1781–1796. MR1813843 https://doi.org/10.1214/aop/1019160507

[35] ERDŐS, P. and TAYLOR, S. J. (1960). Some problems concerning the structure of random walk paths. *Acta Math*. *Acad*. *Sci*. *Hung*. **11** 137–162. MR0121870 https://doi.org/10.1007/BF02020631

[36] FELS, M. and HARTUNG, L. (2021). Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Convergence of the maximum in the regime of weak correlations. *ALEA Lat*. *Amer*. *J*. *Probab*. *Math*. *Stat*. **18** 1891–1930. MR4332224 https://doi.org/10.30757/alea.v18-71

[37] JEGO, A. (2020). Thick points of random walk and the Gaussian free field. *Electron*. *J*. *Probab*. **25** Paper No. 32, 39. MR4073693 https://doi.org/10.1214/20-ejp433

[38] JEGO, A. (2020). Planar Brownian motion and Gaussian multiplicative chaos. *Ann*. *Probab*. **48** 1597–1643. MR4124521 https://doi.org/10.1214/19-AOP1399

[39] JEGO, A. (2021). Critical Brownian multiplicative chaos. *Probab*. *Theory Related Fields* **180** 495–552. MR4265027 https://doi.org/10.1007/s00440-021-01051-7

[40] JEGO, A. (2023). Characterisation of planar Brownian multiplicative chaos. *Comm*. *Math*. *Phys*. **399** 971–1019. MR4576762 https://doi.org/10.1007/s00220-022-04570-z

[41] KAC, M. (1949). On distributions of certain Wiener functionals. *Trans. Amer. Math. Soc.* **65** 1–13. MR0027960 https://doi.org/10.2307/1990512

[42] KAC, M. (1951). On some connections between probability theory and differential and integral equations. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 189–215. Univ. California Press, Berkeley. MR0045333

[43] MADAULE, T. (2017). Convergence in law for the branching random walk seen from its tip. *J. Theoret. Probab.* **30** 27–63. MR3615081 https://doi.org/10.1007/s10959-015-0636-6

[44] ROSEN, J. (2023). Tightness for thick points in two dimensions. *Electron. J. Probab.* **28** Paper No. 18, 45. MR4543442 https://doi.org/10.1214/23-ejp910

[45] SCHWEIGER, F. (2020). The maximum of the four-dimensional membrane model. *Ann. Probab.* **48** 714–741. MR4089492 https://doi.org/10.1214/19-AOP1372

[46] ZHAI, A. (2018). Exponential concentration of cover times. *Electron. J. Probab.* **23** Paper No. 32, 22. MR3785402 https://doi.org/10.1214/18-EJP149

The Annals of Probability
2024, Vol. 52, No. 2, 545–575
https://doi.org/10.1214/23-AOP1657

TOWARDS OPTIMAL SPECTRAL GAPS IN LARGE GENUS

BY MICHAEL LIPNOWSKI[1,a] AND ALEX WRIGHT[2,b]

[1]*Department of Mathematics, Ohio State University,,* [a]*lipnowski.1.osu.edu*
[2]*Department of Mathematics, University of Michigan,* [b]*alexmw@umich.edu*

We show that the Weil–Petersson probability that a random surface has first eigenvalue of the Laplacian less than $3/16 - \epsilon$ goes to zero as the genus goes to infinity.

REFERENCES

[1] ABERT, M., BERGERON, N., BIRINGER, I., GELANDER, T., NIKOLOV, N., RAIMBAULT, J. and SAMET, I. (2017). On the growth of L^2-invariants for sequences of lattices in Lie groups. *Ann. of Math.* (2) **185** 711–790. MR3664810 https://doi.org/10.4007/annals.2017.185.3.1

[2] ALON, N. (1997). On the edge-expansion of graphs. *Combin. Probab. Comput.* **6** 145–152. MR1447810 https://doi.org/10.1017/S096354839700299X

[3] ANANTHARAMAN, N. and MONK, L. (2022). A high-genus asymptotic expansion of Weil–Petersson volume polynomials. *J. Math. Phys.* **63** Paper No. 043502, 26. MR4402875 https://doi.org/10.1063/5.0039385

[4] BORDENAVE, C. (2020). A new proof of Friedman's second eigenvalue theorem and its extension to random lifts. *Ann. Sci. Éc. Norm. Supér.* (4) **53** 1393–1439. MR4203039 https://doi.org/10.24033/asens.2450

[5] BUDZINSKI, T., CURIEN, N. and PETRI, B. (2022). On Cheeger constants of hyperbolic surfaces. Available at arXiv:2207.00469.

[6] BUSER, P. (1992). *Geometry and Spectra of Compact Riemann Surfaces*. *Progress in Mathematics* **106**. Birkhäuser Boston, Cambridge. MR1183224

[7] CHENG, S. Y. (1975). Eigenvalue comparison theorems and its geometric applications. *Math. Z.* **143** 289–297. MR0378001 https://doi.org/10.1007/BF01214381

[8] DOZIER, B. and SAPIR, J. (2022). Simple vs non-simple loops on random regular graphs. Available at arXiv:2209.11218.

[9] DUBASHINSKIĬ, M. B. (2019). On spectra of hyperbolic surfaces without thin handles. Available at arXiv:1901.01382.

[10] FRIEDMAN, J. (2008). A proof of Alon's second eigenvalue conjecture and related problems. *Mem. Amer. Math. Soc.* **195** viii+100. MR2437174 https://doi.org/10.1090/memo/0910

[11] HIDE, W. and MAGEE, M. (2023). Near optimal spectral gaps for hyperbolic surfaces. *Ann. of Math.* (2) **198** 791–824. MR4635304 https://doi.org/10.4007/annals.2023.198.2.6

[12] HUBER, H. (1974). Über den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen. *Comment. Math. Helv.* **49** 251–259. MR0365408 https://doi.org/10.1007/BF02566733

[13] MAGEE, M. and NAUD, F. (2020). Explicit spectral gaps for random covers of Riemann surfaces. *Publ. Math. Inst. Hautes Études Sci.* **132** 137–179. MR4179833 https://doi.org/10.1007/s10240-020-00118-w

[14] MAGEE, M., NAUD, F. and PUDER, D. (2022). A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \varepsilon$. *Geom. Funct. Anal.* **32** 595–661. MR4431124 https://doi.org/10.1007/s00039-022-00602-x

[15] MIRZAKHANI, M. (2007). Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.* **167** 179–222. MR2264808 https://doi.org/10.1007/s00222-006-0013-2

[16] MIRZAKHANI, M. (2010). On Weil–Petersson volumes and geometry of random hyperbolic surfaces. In *Proceedings of the International Congress of Mathematicians. Vol. II*. 1126–1145. Hindustan Book Agency, New Delhi. MR2827834

[17] MIRZAKHANI, M. (2013). Growth of Weil–Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.* **94** 267–300. MR3080483

[18] MIRZAKHANI, M. and PETRI, B. (2019). Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.* **94** 869–889. MR4046008 https://doi.org/10.4171/cmh/477

MSC2020 subject classifications. 22E60, 15A57, 17B20, 58C35.

[19] MIRZAKHANI, M. and ZOGRAF, P. (2015). Towards large genus asymptotics of intersection numbers on moduli spaces of curves. *Geom. Funct. Anal.* **25** 1258–1289. MR3385633 https://doi.org/10.1007/s00039-015-0336-5
[20] MONK, L. (2022). Benjamini–Schramm convergence and spectra of random hyperbolic surfaces of high genus. *Anal. PDE* **15** 727–752. MR4442839 https://doi.org/10.2140/apde.2022.15.727
[21] MONK, L. and THOMAS, J. (2022). The tangle-free hypothesis on random hyperbolic surfaces. *Int. Math. Res. Not. IMRN* 18154–18185. MR4514465 https://doi.org/10.1093/imrn/rnab160
[22] NIE, X., WU, Y. and XUE, Y. (2023). Large genus asymptotics for lengths of separating closed geodesics on random surfaces. *J. Topol.* **16** 106–175. MR4532491 https://doi.org/10.1112/topo.12276
[23] OTAL, J.-P. and ROSAS, E. (2009). Pour toute surface hyperbolique de genre g, $\lambda_{2g-2} > 1/4$. *Duke Math. J.* **150** 101–115. MR2560109 https://doi.org/10.1215/00127094-2009-048
[24] PARLIER, H., WU, Y. and XUE, Y. (2022). The simple separating systole for hyperbolic surfaces of large genus. *J. Inst. Math. Jussieu* **21** 2205–2214. MR4515293 https://doi.org/10.1017/S1474748021000190
[25] SARNAK, P. (1995). Arithmetic quantum chaos. In *The Schur Lectures* (1992) (*Tel Aviv*), *Israel Math. Conf. Proc.* **8** 183–236. Bar-Ilan Univ., Ramat Gan. MR1321639
[26] SELBERG, A. (1956). Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc.* (*N.S.*) **20** 47–87. MR0088511
[27] WRIGHT, A. (2020). A tour through Mirzakhani's work on moduli spaces of Riemann surfaces. *Bull. Amer. Math. Soc.* (*N.S.*) **57** 359–408. MR4108090 https://doi.org/10.1090/bull/1687
[28] WU, Y. and XUE, Y. (2022). Small eigenvalues of closed Riemann surfaces for large genus. *Trans. Amer. Math. Soc.* **375** 3641–3663. MR4402671 https://doi.org/10.1090/tran/8608
[29] WU, Y. and XUE, Y. (2022). Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16} - \epsilon$. *Geom. Funct. Anal.* **32** 340–410. MR4408435 https://doi.org/10.1007/s00039-022-00595-7
[30] WU, Y. and XUE, Y. (2022). Prime geodesic theorem and closed geodesics for large genus. Available at arXiv:2209.10415.

The Annals of Probability
2024, Vol. 52, No. 2, 576–657
https://doi.org/10.1214/23-AOP1666

SINGULAR KINETIC EQUATIONS AND APPLICATIONS

BY ZIMO HAO[1,a], XICHENG ZHANG[2,b], RONGCHAN ZHU[2,c] AND XIANGCHAN ZHU[3,d]

[1]*Fakultät für Mathematik, Universität Bielefeld,* [a]*zhao@math.uni-bielefeld.de*
[2]*Department of Mathematics, Beijing Institute of Technology,* [b]*xichengzhang@gmail.com,* [c]*zhurongchan@126.com*
[3]*Academy of Mathematics and Systems Science, Chinese Academy of Sciences,* [d]*zhuxiangchan@126.com*

In this paper we study singular kinetic equations on $\mathbb{R}^{2d}$ by the paracontrolled distribution method introduced in Gubinelli, Imkeller and Perkowski (*Forum Math. Pi* **3** (2015) e6–75). We first develop paracontrolled calculus in the kinetic setting and use it to establish the global well-posedness for the linear singular kinetic equations under the assumptions that the products of singular terms are well defined. We also demonstrate how the required products can be defined in the case that singular term is a Gaussian random field by probabilistic calculation. Interestingly, although the terms in the zeroth Wiener chaos of regularization approximation are not zero, they converge in suitable weighted Besov spaces, and no renormalization is required. As applications the global well-posedness for a nonlinear kinetic equation with singular coefficients is obtained by the entropy method. Moreover, we also solve the martingale problem for nonlinear kinetic distribution dependent stochastic differential equations with singular drifts.

REFERENCES

[1] ALEXANDRE, R. and VILLANI, C. (2004). On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** 61–95. MR2037247 https://doi.org/10.1016/S0294-1449(03)00030-1

[2] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7

[3] BENACHOUR, S., ROYNETTE, B., TALAY, D. and VALLOIS, P. (1998). Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.* **75** 173–201. MR1632193 https://doi.org/10.1016/S0304-4149(98)00018-0

[4] BONY, J.-M. (1981). Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. Éc. Norm. Supér.* (4) **14** 209–246. MR0631751

[5] BOSSY, M. and TALAY, D. (1997). A stochastic particle method for the McKean–Vlasov and the Burgers equation. *Math. Comp.* **66** 157–192. MR1370849 https://doi.org/10.1090/S0025-5718-97-00776-X

[6] BOUCHUT, F. (2002). Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl.* (9) **81** 1135–1159. MR1949176 https://doi.org/10.1016/S0021-7824(02)01264-3

[7] CANNIZZARO, G. and CHOUK, K. (2018). Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.* **46** 1710–1763. MR3785598 https://doi.org/10.1214/17-AOP1213

[8] CHANDRA, A., MOINAT, A. and WEBER, H. (2023). A priori bounds for the Φ^4 equation in the full sub-critical regime. *Arch. Ration. Mech. Anal.* **247** Paper No. 48. MR4586861 https://doi.org/10.1007/s00205-023-01876-7

[9] CHAUDRU DE RAYNAL, P. E. (2017). Strong existence and uniqueness for degenerate SDE with Hölder drift. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 259–286. MR3606742 https://doi.org/10.1214/15-AIHP716

[10] CHEN, Z.-Q. and ZHANG, X. (2018). L^p-maximal hypoelliptic regularity of nonlocal kinetic Fokker–Planck operators. *J. Math. Pures Appl.* (9) **116** 52–87. MR3826548 https://doi.org/10.1016/j.matpur.2017.10.003

MSC2020 subject classifications. Primary 60H15; secondary 35R60.
Key words and phrases. Paracontrolled calculus, kinetic equations, weighted anisotropic Besov spaces, second-order mean-field SDE, distributional drift.

[11] Coghi, M. and Flandoli, F. (2016). Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26** 1407–1442. MR3513594 https://doi.org/10.1214/15-AAP1120
[12] Crisan, D. and Xiong, J. (2010). Approximate McKean–Vlasov representations for a class of SPDEs. *Stochastics* **82** 53–68. MR2677539 https://doi.org/10.1080/17442500902723575
[13] Debussche, A. and Weber, H. (2018). The Schrödinger equation with spatial white noise potential. *Electron. J. Probab.* **23** Paper No. 28. MR3785398 https://doi.org/10.1214/18-EJP143
[14] Delarue, F. and Diel, R. (2016). Rough paths and 1d SDE with a time dependent distributional drift: Application to polymers. *Probab. Theory Related Fields* **165** 1–63. MR3500267 https://doi.org/10.1007/s00440-015-0626-8
[15] Delarue, F. and Menozzi, S. (2010). Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.* **259** 1577–1630. MR2659772 https://doi.org/10.1016/j.jfa.2010.05.002
[16] Dintelmann, P. (1996). Fourier multipliers between weighted anisotropic function spaces. I. Besov spaces. *Z. Anal. Anwend.* **15** 579–601. MR1406077 https://doi.org/10.4171/ZAA/717
[17] Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658
[18] Fournier, N., Hauray, M. and Mischler, S. (2014). Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* (*JEMS*) **16** 1423–1466. MR3254330 https://doi.org/10.4171/JEMS/465
[19] Funaki, T. (1984). A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete* **67** 331–348. MR0762085 https://doi.org/10.1007/BF00535008
[20] Gerencsér, M. and Hairer, M. (2019). A solution theory for quasilinear singular SPDEs. *Comm. Pure Appl. Math.* **72** 1983–2005. MR3987723 https://doi.org/10.1002/cpa.21816
[21] Gubinelli, M. and Hofmanová, M. (2019). Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. *Comm. Math. Phys.* **368** 1201–1266. MR3951704 https://doi.org/10.1007/s00220-019-03398-4
[22] Gubinelli, M., Imkeller, P. and Perkowski, N. (2015). Paracontrolled distributions and singular PDEs. *Forum Math. Pi* **3** e6–75. MR3406823 https://doi.org/10.1017/fmp.2015.2
[23] Gubinelli, M., Koch, H. and Oh, T. (2018). Renormalization of the two-dimensional stochastic nonlinear wave equations. *Trans. Amer. Math. Soc.* **370** 7335–7359. MR3841850 https://doi.org/10.1090/tran/7452
[24] Gubinelli, M., Koch, H. and Oh, T. (2018). Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity. *J. Eur. Math. Soc.*
[25] Hairer, M. (2014). A theory of regularity structures. *Invent. Math.* **198** 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4
[26] Hairer, M. and Labbé, C. (2018). Multiplicative stochastic heat equations on the whole space. *J. Eur. Math. Soc.* (*JEMS*) **20** 1005–1054. MR3779690 https://doi.org/10.4171/JEMS/781
[27] Hammersley, W. R. P., Šiška, D. and Szpruch, Ł. (2021). Weak existence and uniqueness for McKean–Vlasov SDEs with common noise. *Ann. Probab.* **49** 527–555. MR4255126 https://doi.org/10.1214/20-aop1454
[28] Hao, Z., Wu, M. and Zhang, X. (2020). Schauder estimates for nonlocal kinetic equations and applications. *J. Math. Pures Appl.* (9) **140** 139–184. MR4124429 https://doi.org/10.1016/j.matpur.2020.06.003
[29] Huang, L., Menozzi, S. and Priola, E. (2019). L^p estimates for degenerate non-local Kolmogorov operators. *J. Math. Pures Appl.* (9) **121** 162–215. MR3906169 https://doi.org/10.1016/j.matpur.2017.12.008
[30] Imbert, C. and Silvestre, L. (2021). The Schauder estimate for kinetic integral equations. *Anal. PDE* **14** 171–204. MR4229202 https://doi.org/10.2140/apde.2021.14.171
[31] Jabin, P.-E. and Wang, Z. (2016). Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J. Funct. Anal.* **271** 3588–3627. MR3558251 https://doi.org/10.1016/j.jfa.2016.09.014
[32] Jabin, P.-E. and Wang, Z. (2018). Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.* **214** 523–591. MR3858403 https://doi.org/10.1007/s00222-018-0808-y
[33] Kardar, M., Parisi, G. and Zhang, Y. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892.
[34] Kolmogorov, A. N. (1934). Zufällige Bewegungen. *Ann. Math.* **35** 116–117.
[35] Kremp, H. and Perkowski, N. (2022). Multidimensional SDE with distributional drift and Lévy noise. *Bernoulli* **28** 1757–1783. MR4411510 https://doi.org/10.3150/21-bej1394
[36] Lacker, D. (2023). Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions. *Probab. Math. Phys.* **4** 377–432. MR4595391 https://doi.org/10.2140/pmp.2023.4.377

[37] LANDAU, L. D. (1937). The kinetic equation in the case of Coulomb interaction. *Zh. Eksp. Teor. Fiz.* **7** 203–209. (Translated from German).

[38] LORENZI, L. (2005). Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^N$. *Differential Integral Equations* **18** 531–566. MR2136978

[39] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ.*, 1967). *Lecture Series in Differential Equations*, *Session* 7 41–57. Air Force Office of Scientific Research, Arlington, VA. MR0233437

[40] MISHURA, Y. and VERETENNIKOV, A. (2020). Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. *Theory Probab. Math. Statist.* **103** 59–101. MR4421344 https://doi.org/10.1090/tpms/1135

[41] MOURRAT, J.-C. and WEBER, H. (2017). Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.* **45** 2398–2476. MR3693966 https://doi.org/10.1214/16-AOP1116

[42] MOURRAT, J.-C. and WEBER, H. (2017). The dynamic Φ^4_3 model comes down from infinity. *Comm. Math. Phys.* **356** 673–753. MR3719541 https://doi.org/10.1007/s00220-017-2997-4

[43] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[44] OSADA, H. (1986). Propagation of chaos for the two-dimensional Navier–Stokes equation. *Proc. Japan Acad. Ser. A Math. Sci.* **62** 8–11. MR0839794

[45] OTTO, F., SAUER, J., SMITH, S. and WEBER, H. (2018). *Parabolic Equations with Rough Coefficients and Singular Forcing*. Available in arXiv:1803.07884.

[46] OTTO, F. and WEBER, H. (2019). Quasilinear SPDEs via rough paths. *Arch. Ration. Mech. Anal.* **232** 873–950. MR3925533 https://doi.org/10.1007/s00205-018-01335-8

[47] PERKOWSKI, N. and ROSATI, T. (2021). A rough super-Brownian motion. *Ann. Probab.* **49** 908–943. MR4255134 https://doi.org/10.1214/20-aop1464

[48] PERKOWSKI, N. and ROSATI, T. C. (2019). The KPZ equation on the real line. *Electron. J. Probab.* **24** Paper No. 117. MR4029420 https://doi.org/10.1214/19-ejp362

[49] POLIDORO, S. (2004). Recent results on Kolmogorov equations and applications. In *Proceedings of the Workshop on Second Order Subelliptic Equations and Applications* (I. Birindelli et al., eds.) 129–143. Università degli Studi della, Basilicata, Potenza.

[50] PRIOLA, E. (2009). Global Schauder estimates for a class of degenerate Kolmogorov equations. *Studia Math.* **194** 117–153. MR2534181 https://doi.org/10.4064/sm194-2-2

[51] RÖCKNER, M., XIE, L. and ZHANG, X. (2020). Superposition principle for non-local Fokker–Planck–Kolmogorov operators. *Probab. Theory Related Fields* **178** 699–733. MR4168386 https://doi.org/10.1007/s00440-020-00985-8

[52] RÖCKNER, M. and ZHANG, X. (2021). Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli* **27** 1131–1158. MR4255229 https://doi.org/10.3150/20-bej1268

[53] ROSENZWEIG, M. (2020). The mean-field limit of stochastic point vortex systems with multiplicative noise. Available in arXiv:2011.12180.

[54] SERFATY, S. (2020). Mean field limit for Coulomb-type flows. *Duke Math. J.* **169** 2887–2935. MR4158670 https://doi.org/10.1215/00127094-2020-0019

[55] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. *Princeton Mathematical Series*, *No*. 30. Princeton Univ. Press, Princeton, NJ. MR0290095

[56] SVANTE, J. (1997). *Gaussian Hilbert Spaces*. Cambridge Univ. Press, Cambridge.

[57] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[58] TRIEBEL, H. (2006). *Theory of Function Spaces*. *III*. *Monographs in Mathematics* **100**. Birkhäuser, Basel. MR2250142

[59] VLASOV, A. (1968). The vibrational properties of an electron gas. *Phys. Usp.* **10** 721–733.

[60] WANG, F.-Y. (2018). Distribution dependent SDEs for Landau type equations. *Stochastic Process. Appl.* **128** 595–621. MR3739509 https://doi.org/10.1016/j.spa.2017.05.006

[61] WANG, F.-Y. and ZHANG, X. (2016). Degenerate SDE with Hölder–Dini drift and non-Lipschitz noise coefficient. *SIAM J. Math. Anal.* **48** 2189–2226. MR3511355 https://doi.org/10.1137/15M1023671

[62] ZHANG, X. (2018). Stochastic Hamiltonian flows with singular coefficients. *Sci. China Math.* **61** 1353–1384. MR3833741 https://doi.org/10.1007/s11425-017-9127-0

[63] ZHANG, X. and ZHANG, X. (2021). Cauchy problem of stochastic kinetic equations. *Ann. Appl. Probab.* To appear. Available at arXiv:2103.02267. https://doi.org/10.1214/23-AAP1961

[64] ZHANG, X. and ZHAO, G. (2018). Singular Brownian diffusion processes. *Commun. Math. Stat.* **6** 533–581. MR3877717 https://doi.org/10.1007/s40304-018-0164-7

[65] ZHANG, X., ZHU, R. and ZHU, X. (2022). Singular HJB equations with applications to KPZ on the real line. *Probab. Theory Related Fields* **183** 789–869. MR4453317 https://doi.org/10.1007/s00440-022-01137-w

The Annals of Probability
2024, Vol. 52, No. 2, 658–712
https://doi.org/10.1214/23-AOP1667

THE MINKOWSKI CONTENT MEASURE FOR THE LIOUVILLE QUANTUM GRAVITY METRIC

BY EWAIN GWYNNE[a] AND JINWOO SUNG[b]

Department of Mathematics, University of Chicago,[a]*ewain@uchicago.edu,* [b]*jsung@math.uchicago.edu*

A Liouville quantum gravity (LQG) surface is a natural random two-dimensional surface, initially formulated as a random measure space and later as a random metric space. We show that the LQG measure can be recovered as the Minkowski measure with respect to the LQG metric, answering a question of Gwynne and Miller (*Invent. Math.* **223** (2021) 213–333). As a consequence, we prove that the metric structure of a γ-LQG surface determines its conformal structure for every $\gamma \in (0, 2)$. Our primary tool is the continuum mating-of-trees theory for space-filling SLE. In the course of our proof, we also establish a Hölder continuity result for space-filling SLE with respect to the LQG metric.

REFERENCES

[1] ABRAHAM, R., DELMAS, J.-F. and HOSCHEIT, P. (2013). A note on the Gromov–Hausdorff–Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* **18** no. 14, 21. MR3035742 https://doi.org/10.1214/EJP.v18-2116

[2] ALBERTS, T. and SHEFFIELD, S. (2011). The covariant measure of SLE on the boundary. *Probab. Theory Related Fields* **149** 331–371. MR2776618 https://doi.org/10.1007/s00440-009-0252-4

[3] ANG, M., FALCONET, H. and SUN, X. (2020). Volume of metric balls in Liouville quantum gravity. *Electron. J. Probab.* **25** Paper No. 160, 50. MR4193901 https://doi.org/10.1214/20-ejp564

[4] ARU, J. (2020). Gaussian multiplicative chaos through the lens of the 2D Gaussian free field. *Markov Process. Related Fields* **26** 17–56. MR4237161

[5] BENOIST, S. (2018). Natural parametrization of SLE: The Gaussian free field point of view. *Electron. J. Probab.* **23** Paper No. 103, 16. MR3870446 https://doi.org/10.1214/18-ejp232

[6] BERESTYCKI, N. (2017). An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.* **22** Paper No. 27, 12. MR3652040 https://doi.org/10.1214/17-ECP58

[7] BERESTYCKI, N. and POWELL, E. Gaussian free field and Liouville quantum gravity. To be published by Cambridge Univ. Press. Available at https://homepage.univie.ac.at/nathanael.berestycki/.

[8] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics: Probability and Statistics*. Wiley, New York. A Wiley-Interscience Publication. MR1700749 https://doi.org/10.1002/9780470316962

[9] BOGACHEV, V. I. (2007). *Measure Theory. Vol. I, II*. Springer, Berlin. MR2267655 https://doi.org/10.1007/978-3-540-34514-5

[10] BORGA, J. and GWYNNE, E. (2022). Permutons, meanders, and SLE-decorated Liouville quantum gravity. Available at arXiv:2207.02319.

[11] DEVLIN VI, C. (2023). Almost sure convergence of Liouville first passage percolation. Available at arXiv:2309.08001.

[12] DING, J., DUBÉDAT, J., DUNLAP, A. and FALCONET, H. (2020). Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *Publ. Math. Inst. Hautes Études Sci.* **132** 353–403. MR4179836 https://doi.org/10.1007/s10240-020-00121-1

[13] DING, J. and GWYNNE, E. (2020). The fractal dimension of Liouville quantum gravity: Universality, monotonicity, and bounds. *Comm. Math. Phys.* **374** 1877–1934. MR4076090 https://doi.org/10.1007/s00220-019-03487-4

[14] DING, J. and GWYNNE, E. (2021). The critical Liouville quantum gravity metric induces the Euclidean topology. Available at arXiv:2108.12067.

MSC2020 subject classifications. Primary 60D05, 60J67; secondary 60G57.

Key words and phrases. Liouville quantum gravity, LQG metric, Minkowski content, Schramm–Loewner evolution, mating of trees.

[15] DING, J., ZEITOUNI, O. and ZHANG, F. (2019). Heat kernel for Liouville Brownian motion and Liouville graph distance. *Comm. Math. Phys.* **371** 561–618. MR4019914 https://doi.org/10.1007/s00220-019-03467-8

[16] DUBÉDAT, J., FALCONET, H., GWYNNE, E., PFEFFER, J. and SUN, X. (2020). Weak LQG metrics and Liouville first passage percolation. *Probab. Theory Related Fields* **178** 369–436. MR4146541 https://doi.org/10.1007/s00440-020-00979-6

[17] DUFRESNE, D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.* **1–2** 39–79. MR1129194 https://doi.org/10.1080/03461238.1990.10413872

[18] DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2021). Liouville quantum gravity as a mating of trees. *Astérisque* **427** viii+258. MR4340069 https://doi.org/10.24033/ast

[19] DUPLANTIER, B. and SHEFFIELD, S. (2011). Liouville quantum gravity and KPZ. *Invent. Math.* **185** 333–393. MR2819163 https://doi.org/10.1007/s00222-010-0308-1

[20] GROMOV, M. (1999). *Metric Structures for Riemannian and Non-Riemannian Spaces*. *Progress in Mathematics* **152**. Birkhäuser, Boston. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR1699320

[21] GWYNNE, E. (2020). Random surfaces and Liouville quantum gravity. *Notices Amer. Math. Soc.* **67** 484–491. MR4186266 https://doi.org/10.1090/noti

[22] GWYNNE, E. (2021). Geodesic networks in Liouville quantum gravity surfaces. *Probab. Math. Phys.* **2** 643–684. MR4408022 https://doi.org/10.2140/pmp.2021.2.643

[23] GWYNNE, E., HOLDEN, N. and MILLER, J. (2020). An almost sure KPZ relation for SLE and Brownian motion. *Ann. Probab.* **48** 527–573. MR4089487 https://doi.org/10.1214/19-AOP1385

[24] GWYNNE, E., HOLDEN, N., MILLER, J. and SUN, X. (2017). Brownian motion correlation in the peanosphere for $\kappa > 8$. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1866–1889. MR3729638 https://doi.org/10.1214/16-AIHP774

[25] GWYNNE, E., HOLDEN, N. and SUN, X. (2020). A mating-of-trees approach for graph distances in random planar maps. *Probab. Theory Related Fields* **177** 1043–1102. MR4126936 https://doi.org/10.1007/s00440-020-00969-8

[26] GWYNNE, E., HOLDEN, N. and SUN, X. (2023). Mating of trees for random planar maps and Liouville quantum gravity: A survey. In *Topics in Statistical Mechanics*. *Panor. Synthèses* **59** 41–120. Soc. Math. France, Paris. MR4619311

[27] GWYNNE, E. and MILLER, J. (2017). Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov–Hausdorff–Prokhorov-uniform topology. *Electron. J. Probab.* **22** Paper No. 84, 47. MR3718712 https://doi.org/10.1214/17-EJP102

[28] GWYNNE, E. and MILLER, J. (2020). Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$. *Ann. Probab.* **48** 1861–1901. MR4124527 https://doi.org/10.1214/19-AOP1409

[29] GWYNNE, E. and MILLER, J. (2020). Local metrics of the Gaussian free field. *Ann. Inst. Fourier* (*Grenoble*) **70** 2049–2075. MR4245606

[30] GWYNNE, E. and MILLER, J. (2021). Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Invent. Math.* **223** 213–333. MR4199443 https://doi.org/10.1007/s00222-020-00991-6

[31] GWYNNE, E. and MILLER, J. (2021). Conformal covariance of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1016–1031. MR4260493 https://doi.org/10.1214/20-aihp1105

[32] GWYNNE, E. and MILLER, J. (2021). Characterizations of SLE_κ for $\kappa \in (4, 8)$ on Liouville quantum gravity. *Astérisque* **429** 129–242, 4390048.

[33] GWYNNE, E., MILLER, J. and SHEFFIELD, S. (2020). The Tutte embedding of the Poisson–Voronoi tessellation of the Brownian disk converges to $\sqrt{8/3}$-Liouville quantum gravity. *Comm. Math. Phys.* **374** 735–784. MR4072229 https://doi.org/10.1007/s00220-019-03610-5

[34] GWYNNE, E. and PFEFFER, J. (2022). KPZ formulas for the Liouville quantum gravity metric. *Trans. Amer. Math. Soc.* **375** 8297–8324. MR4504639 https://doi.org/10.1090/tran/8085

[35] HOLDEN, N., LAWLER, G. F., LI, X. and SUN, X. (2022). Minkowski content of Brownian cut points. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 455–488. MR4374682 https://doi.org/10.1214/21-aihp1159

[36] HUANG, L., RAO, H., WEN, Z. and XU, Y. (2021). On Minkowski measures of metric spaces. Available at arXiv:2111.00752.

[37] HUGHES, L. and MILLER, J. (2022). Equivalence of metric gluing and conformal welding in γ-Liouville quantum gravity for $\gamma \in (0, 2)$. arXiv e-prints. Available at arXiv:2212.00589.

[38] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[39] KECHRIS, A. S. (1995). *Classical Descriptive Set Theory*. *Graduate Texts in Mathematics* **156**. Springer, New York. MR1321597 https://doi.org/10.1007/978-1-4612-4190-4

[40] LAWLER, G. F. (2015). Minkowski content of the intersection of a Schramm–Loewner evolution (SLE) curve with the real line. *J. Math. Soc. Japan* **67** 1631–1669. MR3417507 https://doi.org/10.2969/jmsj/06741631

[41] LAWLER, G. F. and REZAEI, M. A. (2015). Minkowski content and natural parameterization for the Schramm–Loewner evolution. *Ann. Probab.* **43** 1082–1120. MR3342659 https://doi.org/10.1214/13-AOP874

[42] LAWLER, G. F. and SHEFFIELD, S. (2011). A natural parametrization for the Schramm–Loewner evolution. *Ann. Probab.* **39** 1896–1937. MR2884877 https://doi.org/10.1214/10-AOP560

[43] LE GALL, J.-F. (2022). The volume measure of the Brownian sphere is a Hausdorff measure. *Electron. J. Probab.* **27** Paper No. 113, 28. MR4474537 https://doi.org/10.1214/22-ejp837

[44] MILLER, J. (2018). Liouville quantum gravity as a metric space and a scaling limit. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro* 2018. *Invited Lectures*. **IV** 2945–2971. World Sci. Publ., Hackensack. MR3966518

[45] MILLER, J. and SHEFFIELD, S. (2016). Imaginary geometry I: Interacting SLEs. *Probab. Theory Related Fields* **164** 553–705. MR3477777 https://doi.org/10.1007/s00440-016-0698-0

[46] MILLER, J. and SHEFFIELD, S. (2017). Imaginary geometry IV: Interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields* **169** 729–869. MR3719057 https://doi.org/10.1007/s00440-017-0780-2

[47] MILLER, J. and SHEFFIELD, S. (2020). Liouville quantum gravity and the Brownian map I: The QLE$(8/3, 0)$ metric. *Invent. Math.* **219** 75–152. MR4050102 https://doi.org/10.1007/s00222-019-00905-1

[48] MILLER, J. and SHEFFIELD, S. (2021). Liouville quantum gravity and the Brownian map II: Geodesics and continuity of the embedding. *Ann. Probab.* **49** 2732–2829. MR4348679 https://doi.org/10.1214/21-aop1506

[49] MILLER, J. and SHEFFIELD, S. (2021). Liouville quantum gravity and the Brownian map III: The conformal structure is determined. *Probab. Theory Related Fields* **179** 1183–1211. MR4242633 https://doi.org/10.1007/s00440-021-01026-8

[50] MÖRTERS, P. and PERES, Y. (2010). *Brownian Motion*. *Cambridge Series in Statistical and Probabilistic Mathematics* **30**. Cambridge Univ. Press, Cambridge. With an appendix by Oded Schramm and Wendelin Werner. MR2604525 https://doi.org/10.1017/CBO9780511750489

[51] POLYAKOV, A. M. (1981). Quantum geometry of bosonic strings. *Phys. Lett. B* **103** 207–210. MR0623209 https://doi.org/10.1016/0370-2693(81)90743-7

[52] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356 https://doi.org/10.1214/13-PS218

[53] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706 https://doi.org/10.1007/s00440-006-0050-1

[54] SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

[55] SHEFFIELD, S. (2022). *What is a random surface? Proc. Int. Cong. Math.* 2022 **2**. EMS Press, Berlin. https://doi.org/10.1017/ICM2022/187

[56] SHEFFIELD, S. and WANG, M. (2016). Field-measure correspondence in Liouville quantum gravity almost surely commutes with all conformal maps simultaneously. Available at arXiv:1605.06171.

[57] VARGAS, V. (2017). Lecture notes on Liouville theory and the DOZZ formula. Available at arXiv:1712.00829.

[58] WERNER, W. and POWELL, E. (2021). *Lecture Notes on the Gaussian Free Field*. *Cours Spécialisés* [*Specialized Courses*] **28**. Société Mathématique de France, Paris. MR4466634

[59] YOR, M. (1992). Sur certaines fonctionnelles exponentielles du mouvement brownien réel. *J. Appl. Probab.* **29** 202–208. MR1147781 https://doi.org/10.1017/s002190020010676x

[60] ZHAN, D. (2022). Boundary Green's functions and Minkowski content measure of multi-force-point $\mathrm{SLE}_\kappa(\underline{\rho})$. *Stochastic Process. Appl.* **151** 265–306. MR4445708 https://doi.org/10.1016/j.spa.2022.06.016

The Annals of Probability
2024, Vol. 52, No. 2, 713–736
https://doi.org/10.1214/23-AOP1668

CUTOFF PROFILE OF THE METROPOLIS BIASED CARD SHUFFLING

BY LINGFU ZHANG[a]

Department of Statistics, University of California, [a]*lfzhang@berkeley.edu*

We consider the Metropolis biased card shuffling (also called the multi-species ASEP on a finite interval or the random Metropolis scan). Its convergence to stationarity was believed to exhibit a total-variation cutoff, and that was proved a few years ago by Labbé and Lacoin (*Ann. Probab.* **47** (2019) 1541–1586). In this paper, we prove that (for N cards) the cutoff window is in the order of $N^{1/3}$, and the cutoff profile is given by the Tracy–Widom GOE distribution function. This confirms a conjecture by Bufetov and Nejjar (*Probab. Theory Related Fields* **83** (2022) 229–253). Our approach is different from (*Ann. Probab.* **47** (2019) 1541–1586), by comparing the card shuffling with the multispecies ASEP on $\mathbb{Z}$, and using Hecke algebra and recent ASEP shift-invariance and convergence results. Our result can also be viewed as a generalization of the Oriented Swap Process finishing time convergence (*Ann. Appl. Probab.* **32** (2022) 753–763), which is the TASEP version (of our result).

REFERENCES

[1] AGGARWAL, A. (2017). Convergence of the stochastic six-vertex model to the ASEP: Stochastic six-vertex model and ASEP. *Math. Phys. Anal. Geom.* **20** Paper No. 3. MR3589552 https://doi.org/10.1007/s11040-016-9235-8

[2] AGGARWAL, A., CORWIN, I. and GHOSAL, P. (2023). The ASEP speed process. *Adv. Math.* **422** Paper No. 109004, 57. MR4576839 https://doi.org/10.1016/j.aim.2023.109004

[3] ALDOUS, D. (1983). Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on Probability, XVII. Lecture Notes in Math.* **986** 243–297. Springer, Berlin. MR0770418 https://doi.org/10.1007/BFb0068322

[4] ALDOUS, D. and DIACONIS, P. (1986). Shuffling cards and stopping times. *Amer. Math. Monthly* **93** 333–348. MR0841111 https://doi.org/10.2307/2323590

[5] AMIR, G., ANGEL, O. and VALKÓ, B. (2011). The TASEP speed process. *Ann. Probab.* **39** 1205–1242. MR2857238 https://doi.org/10.1214/10-AOP561

[6] ANGEL, O., HOLROYD, A. and ROMIK, D. (2009). The oriented swap process. *Ann. Probab.* **37** 1970–1998. MR2561438 https://doi.org/10.1214/09-AOP456

[7] BAYER, D. and DIACONIS, P. (1992). Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.* **2** 294–313. MR1161056

[8] BEN-HAMOU, A. and SALEZ, J. (2017). Cutoff for nonbacktracking random walks on sparse random graphs. *Ann. Probab.* **45** 1752–1770. MR3650414 https://doi.org/10.1214/16-AOP1100

[9] BENJAMINI, I., BERGER, N., HOFFMAN, C. and MOSSEL, E. (2005). Mixing times of the biased card shuffling and the asymmetric exclusion process. *Trans. Amer. Math. Soc.* **357** 3013–3029. MR2135733 https://doi.org/10.1090/S0002-9947-05-03610-X

[10] BERTINI, L. and GIACOMIN, G. (1997). Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183** 571–607. MR1462228 https://doi.org/10.1007/s002200050044

[11] BORODIN, A. and BUFETOV, A. (2021). Color-position symmetry in interacting particle systems. *Ann. Probab.* **49** 1607–1632. MR4260463 https://doi.org/10.1214/20-aop1463

[12] BORODIN, A., CORWIN, I. and GORIN, V. (2016). Stochastic six-vertex model. *Duke Math. J.* **165** 563–624. MR3466163 https://doi.org/10.1215/00127094-3166843

[13] BORODIN, A., GORIN, V. and WHEELER, M. (2022). Shift-invariance for vertex models and polymers. *Proc. Lond. Math. Soc.* (3) **124** 182–299. MR4400934 https://doi.org/10.1112/plms.12427

MSC2020 subject classifications. Primary 60K35; secondary 60C05, 80C22.

Key words and phrases. Cutoff profile, biased card shuffling, Markov chain mixing, simple exclusion process, Tracy–Widom distribution, KPZ universality, Hecke algebra.

[14] BUFETOV, A., GORIN, V. and ROMIK, D. (2022). Absorbing time asymptotics in the oriented swap process. *Ann. Appl. Probab.* **32** 753–763. MR4414694 https://doi.org/10.1214/21-aap1695
[15] BUFETOV, A. and NEJJAR, P. (2022). Cutoff profile of ASEP on a segment. *Probab. Theory Related Fields* **183** 229–253. MR4421174 https://doi.org/10.1007/s00440-021-01104-x
[16] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2020). Mixing time of the adjacent walk on the simplex. *Ann. Probab.* **48** 2449–2493. MR4152648 https://doi.org/10.1214/20-AOP1428
[17] CHATTERJEE, S., DIACONIS, P., SLY, A. and ZHANG, L. (2022). A phase transition for repeated averages. *Ann. Probab.* **50** 1–17. MR4385355 https://doi.org/10.1214/21-AOP1526
[18] DIACONIS, P. (1996). The cutoff phenomenon in finite Markov chains. *Proc. Natl. Acad. Sci. USA* **93** 1659–1664. MR1374011 https://doi.org/10.1073/pnas.93.4.1659
[19] DIACONIS, P. and RAM, A. (2000). Analysis of systematic scan Metropolis algorithms using Iwahori–Hecke algebra techniques *Michigan Math. J.* **48** 157–190. MR1786485 https://doi.org/10.1307/mmj/1030132713
[20] DIACONIS, P. and SHAHSHAHANI, M. (1981). Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** 159–179. MR0626813 https://doi.org/10.1007/BF00535487
[21] DING, J., LUBETZKY, E. and PERES, Y. (2010). Total variation cutoff in birth-and-death chains. *Probab. Theory Related Fields* **146** 61–85. MR2550359 https://doi.org/10.1007/s00440-008-0185-3
[22] GALASHIN, P. (2021). Symmetries of stochastic colored vertex models. *Ann. Probab.* **49** 2175–2219. MR4317703 https://doi.org/10.1214/20-aop1502
[23] GANTERT, N., NESTORIDI, E. and SCHMID, D. (2023). Mixing times for the simple exclusion process with open boundaries. *Ann. Appl. Probab.* **33** 1172–1212. MR4564424 https://doi.org/10.1214/22-aap1839
[24] GONÇALVES, P., JARA, M., MARINHO, R. and MENEZES, O. (2021). Sharp convergence to equilibrium for the SSEP with reservoirs. Available at arXiv:2110.06353.
[25] GREENBERG, S., PASCOE, A. and RANDALL, D. (2009). Sampling biased lattice configurations using exponential metrics. In *Proceedings of the Twentieth Annual ACM–SIAM Symposium on Discrete Algorithms* 76–85. SIAM, Philadelphia, PA. MR2809307
[26] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** 437–476. MR1737991 https://doi.org/10.1007/s002200050027
[27] LABBÉ, C. and LACOIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 https://doi.org/10.1214/18-AOP1290
[28] LABBÉ, C. and PETIT, E. (2022). Hydrodynamic limit and cutoff for the biased adjacent walk on the simplex. Available at arXiv:2203.04008.
[29] LACOIN, H. (2021). Mixing time and cutoff for one dimensional particle systems. In *Proceedings of the ICM 2022*. To appear. Available at arXiv:2111.06436.
[30] LACOIN, H. (2016). The cutoff profile for the simple exclusion process on the circle. *Ann. Probab.* **44** 3399–3430. MR3551201 https://doi.org/10.1214/15-AOP1053
[31] LACOIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 https://doi.org/10.1214/15-AOP1004
[32] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*: *Second Edition*. Amer. Math. Soc., Providence, RI. MR3726904 https://doi.org/10.1090/mbk/107
[33] LIGGETT, T. M. (1985). *Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4
[34] LUBETZKY, E. and PERES, Y. (2016). Cutoff on all Ramanujan graphs. *Geom. Funct. Anal.* **26** 1190–1216. MR3558308 https://doi.org/10.1007/s00039-016-0382-7
[35] LUBETZKY, E. and SLY, A. (2010). Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.* **153** 475–510. MR2667423 https://doi.org/10.1215/00127094-2010-029
[36] LUBETZKY, E. and SLY, A. (2013). Cutoff for the Ising model on the lattice. *Invent. Math.* **191** 719–755. MR3020173 https://doi.org/10.1007/s00222-012-0404-5
[37] MATETSKI, K., QUASTEL, J. and REMENIK, D. (2021). The KPZ fixed point. *Acta Math.* **227** 115–203. MR4346267 https://doi.org/10.4310/acta.2021.v227.n1.a3
[38] NESTORIDI, E. and OLESKER-TAYLOR, S. (2022). Limit profiles for reversible Markov chains. *Probab. Theory Related Fields* **182** 157–188. MR4367947 https://doi.org/10.1007/s00440-021-01061-5
[39] ORTMANN, J., QUASTEL, J. and REMENIK, D. (2017). A Pfaffian representation for flat ASEP. *Comm. Pure Appl. Math.* **70** 3–89. MR3581823 https://doi.org/10.1002/cpa.21644
[40] QUASTEL, J. and REMENIK, D. (2014). Airy processes and variational problems. In *Topics in Percolative and Disordered Systems*. *Springer Proc. Math. Stat.* **69** 121–171. Springer, New York. MR3229288 https://doi.org/10.1007/978-1-4939-0339-9_5

[41] QUASTEL, J. and SARKAR, S. (2023). Convergence of exclusion processes and the KPZ equation to the KPZ fixed point. *J. Amer. Math. Soc.* **36** 251–289. MR4495842 https://doi.org/10.1090/jams/999
[42] SALEZ, J. (2023). Cutoff for non-negatively curved Markov chains. *J. Eur. Math. Soc.* Published online first. https://doi.org/10.4171/JEMS/1348
[43] SALEZ, J. (2023). Universality of cutoff for exclusion with reservoirs. *Ann. Probab.* **51** 478–494. MR4546624 https://doi.org/10.1214/22-aop1600
[44] SALOFF-COSTE, L. (2004). Random walks on finite groups. In *Probability on Discrete Structures*. *Encyclopaedia Math. Sci.* **110** 263–346. Springer, Berlin. MR2023654 https://doi.org/10.1007/978-3-662-09444-0_5
[45] TEYSSIER, L. (2020). Limit profile for random transpositions. *Ann. Probab.* **48** 2323–2343. MR4152644 https://doi.org/10.1214/20-AOP1424
[46] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246
[47] TRACY, C. A. and WIDOM, H. (1996). On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* **177** 727–754. MR1385083
[48] TRACY, C. A. and WIDOM, H. (2009). Asymptotics in ASEP with step initial condition. *Comm. Math. Phys.* **290** 129–154. MR2520510 https://doi.org/10.1007/s00220-009-0761-0
[49] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. MR2023023 https://doi.org/10.1214/aoap/1075828054
[50] ZHANG, L. (2023). Shift-invariance of the colored TASEP and finishing times of the oriented swap process. *Adv. Math.* **415** Paper No. 108884, 60. MR4541342 https://doi.org/10.1016/j.aim.2023.108884

The Annals of Probability
2024, Vol. 52, No. 2, 737–763
https://doi.org/10.1214/23-AOP1669
This research was funded, in whole or in part, by Fonds National de la recherche - FNR, 022/17372844/FraMSLA.

SPECTRAL CENTRAL LIMIT THEOREM FOR ADDITIVE FUNCTIONALS OF ISOTROPIC AND STATIONARY GAUSSIAN FIELDS

BY LEONARDO MAINI[a] AND IVAN NOURDIN[b]

Department of Mathematics, University of Luxembourg, [a]*leonardo.maini@uni.lu,* [b]*ivan.nourdin@uni.lu*

Let $B = (B_x)_{x\in\mathbb{R}^d}$ be a collection of $N(0,1)$ random variables forming a real-valued continuous stationary Gaussian field on $\mathbb{R}^d$, and set $C(x-y) = \mathbb{E}[B_x B_y]$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[\varphi(N)^2] < \infty$ with $N \sim N(0,1)$, let R be the Hermite rank of φ, and consider $Y_t = \int_{tD} \varphi(B_x)\,dx$, $t > 0$ with $D \subset \mathbb{R}^d$ compact.

Since the pioneering works from the 1980s by Breuer, Dobrushin, Major, Rosenblatt, Taqqu and others, central and noncentral limit theorems for Y_t have been constantly refined, extended and applied to an increasing number of diverse situations, to such an extent that it has become a field of research in its own right.

The common belief, representing the intuition that specialists in the subject have developed over the last four decades, is that as $t \to \infty$ the fluctuations of Y_t around its mean are, in general (i.e., except possibly in very special cases), Gaussian when B has short memory, and non-Gaussian when B has long memory and $R \geq 2$.

We show in this paper that this intuition forged over the last 40 years can be wrong, and not only marginally or in critical cases. We will indeed bring to light a variety of situations where Y_t admits Gaussian fluctuations in a long memory context.

To achieve this goal, we state and prove a spectral central limit theorem, which extends the conclusion of the celebrated Breuer–Major theorem to situations where $C \notin L^R(\mathbb{R}^d)$. Our main mathematical tools are the Malliavin–Stein method and Fourier analysis techniques.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[2] BERRY, M. V. (1977). Regular and irregular semiclassical wavefunctions. *J. Phys. A* **10** 2083–2091. MR0489542

[3] BRANDOLINI, L., HOFMANN, S. and IOSEVICH, A. (2003). Sharp rate of average decay of the Fourier transform of a bounded set. *Geom. Funct. Anal.* **13** 671–680. MR2006553 https://doi.org/10.1007/s00039-003-0426-7

[4] BREUER, P. and MAJOR, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. MR0716933 https://doi.org/10.1016/0047-259X(83)90019-2

[5] DIERICKX, G., NOURDIN, I., PECCATI, G. and ROSSI, M. (2023). Small scale CLTs for the nodal length of monochromatic waves. *Comm. Math. Phys.* **397** 1–36. MR4538280 https://doi.org/10.1007/s00220-022-04422-w

[6] DOBRUSHIN, R. L. and MAJOR, P. (1979). Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27–52. MR0550122 https://doi.org/10.1007/BF00535673

[7] FLORET, K. (1997). Natural norms on symmetric tensor products of normed spaces. *Note Mat.* **17** 153–188. MR1749787

[8] GALERNE, B. (2011). Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. *Image Anal. Stereol.* **30** 39–51. MR2816305 https://doi.org/10.5566/ias.v30.p39-51

MSC2020 subject classifications. 60F05, 60H07, 60G15.

Key words and phrases. Spectral central limit theorem, stationary Gaussian fields, isotropic Gaussian fields, long memory, short memory, Malliavin–Stein method, Hermite rank, Fourier analysis.

[9] GOLINSKII, L., MALAMUD, M. and ORIDOROGA, L. (2018). Radial positive definite functions and Schoenberg matrices with negative eigenvalues. *Trans. Amer. Math. Soc.* **370** 1–25. MR3717972 https://doi.org/10.1090/tran/6876

[10] GORBACHEV, D. and TIKHONOV, S. (2019). Doubling condition at the origin for non-negative positive definite functions. *Proc. Amer. Math. Soc.* **147** 609–618. MR3894899 https://doi.org/10.1090/proc/14191

[11] KRASIKOV, I. (2014). Approximations for the Bessel and Airy functions with an explicit error term. *LMS J. Comput. Math.* **17** 209–225. MR3230865 https://doi.org/10.1112/S1461157013000351

[12] LEONENKO, N. (1999). *Limit Theorems for Random Fields with Singular Spectrum. Mathematics and Its Applications* **465**. Kluwer Academic, Dordrecht. MR1687092 https://doi.org/10.1007/978-94-011-4607-4

[13] MAINI, L. (2024). Asymptotic covariances for functionals of weakly stationary random fields. *Stochastic Process. Appl.* **170** 104297. MR4689941 https://doi.org/10.1016/j.spa.2024.104297

[14] MARINUCCI, D. and WIGMAN, I. (2014). On nonlinear functionals of random spherical eigenfunctions. *Comm. Math. Phys.* **327** 849–872. MR3192051 https://doi.org/10.1007/s00220-014-1939-7

[15] NOTARNICOLA, M. (2021). Probabilistic limit theorems and the geometry of random fields. Ph.D. thesis, Univ. Luxembourg.

[16] NOTARNICOLA, M., PECCATI, G. and VIDOTTO, A. (2023). Functional convergence of Berry's nodal lengths: Approximate tightness and total disorder. *J. Stat. Phys.* **190** Paper No. 97, 41. MR4587627 https://doi.org/10.1007/s10955-023-03111-9

[17] NOURDIN, I. and NUALART, D. (2020). The functional Breuer–Major theorem. *Probab. Theory Related Fields* **176** 203–218. MR4055189 https://doi.org/10.1007/s00440-019-00917-1

[18] NOURDIN, I., NUALART, D. and TUDOR, C. A. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1055–1079. MR2744886 https://doi.org/10.1214/09-AIHP342

[19] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality. Cambridge Tracts in Mathematics* **192**. Cambridge Univ. Press, Cambridge. MR2962301 https://doi.org/10.1017/CBO9781139084659

[20] NOURDIN, I., PECCATI, G. and PODOLSKIJ, M. (2011). Quantitative Breuer–Major theorems. *Stochastic Process. Appl.* **121** 793–812. MR2770907 https://doi.org/10.1016/j.spa.2010.12.006

[21] NOURDIN, I., PECCATI, G. and ROSSI, M. (2019). Nodal statistics of planar random waves. *Comm. Math. Phys.* **369** 99–151. MR3959555 https://doi.org/10.1007/s00220-019-03432-5

[22] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[23] NUALART, D. and PECCATI, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863 https://doi.org/10.1214/009117904000000621

[24] PECCATI, G. and VIDOTTO, A. (2020). Gaussian random measures generated by Berry's nodal sets. *J. Stat. Phys.* **178** 996–1027. MR4064212 https://doi.org/10.1007/s10955-019-02477-z

[25] ROSENBLATT, M. (1960). Independence and dependence. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II* 431–443. Univ. California Press, Berkeley–Los Angeles, Calif. MR0133863

[26] SCHOENBERG, I. J. (1938). Metric spaces and completely monotone functions. *Ann. of Math.* (2) **39** 811–841. MR1503439 https://doi.org/10.2307/1968466

[27] TAQQU, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* **50** 53–83. MR0550123 https://doi.org/10.1007/BF00535674

[28] TUDOR, C. A. (2013). *Analysis of Variations for Self-Similar Processes: A Stochastic Calculus Approach. Probability and Its Applications* (*New York*). Springer, Cham. MR3112799 https://doi.org/10.1007/978-3-319-00936-0

The Annals of Probability

Future Issues

Articles

Bulk universality and quantum unique ergodicity for random band matrices in high dimensions
CHANGJI XU, FAN YANG, HORNG-TZER YAU AND JUN YIN

Percolation of strongly correlated Gaussian fields II. Sharpness of the phase transition
STEPHEN MUIRHEAD

Regularized modified log-Sobolev inequalities and comparison of Markov chains
KONSTANTIN TIKHOMIROV AND PIERRE YOUSSEF

Random partitions under the Plancherel–Hurwitz measure, high-genus Hurwitz numbers and maps
GUILLAUME CHAPUY, BAPTISTE LOUF AND HARRIET WALSH

The discrete Gaussian model, I. Renormalisation group flow at high temperature
ROLAND BAUERSCHMIDT, JIWOON PARK AND PIERRE-FRANÇOIS RODRIGUEZ

The discrete Gaussian model, II. Infinite-volume scaling limit at high temperature
ROLAND BAUERSCHMIDT, JIWOON PARK AND PIERRE-FRANÇOIS RODRIGUEZ

An invariance principle for the 1D KPZ equation
ARKA ADHIKARI AND SOURAV CHATTERJEE

A central limit theorem for the number of excursion set components of Gaussian fields
DMITRY BELIAEV, MICHAEL MCAULEY AND STEPHEN MUIRHEAD

Sudakov–Fernique post-AMP, and a new proof of the local convexity of the TAP free energy
MICHAEL CELENTANO

The Critical Density for Activated Random Walks is always less than 1
AMINE ASSELAH, NICOLAS FORIEN AND ALEXANDRE GAUDILLIERE

Strict monotonicity for first passage percolation on graphs of polynomial growth and quasi-trees
CHRISTIAN GORSKI

The extremal point process of branching Brownian motion in $\mathbb{R}^d$
JULIEN BERESTYCKI, YUJIN HONG KIM, EYAL LUBETZKY, BASTIEN MALLEIN AND OFER ZEITOUNI

The Spine of the Fleming–Viot process driven by Brownian motion
KRZYSZTOF BURDZY AND JANOS ENGLANDER

Biased random walk on dynamical percolation
SEBASTIAN ANDRES, NINA GANTERT, DOMINIK SCHMID AND PERLA SOUSI

Two-sided heat kernel estimates for Schrödinger operators with unbounded potentials
XIN CHEN AND JIAN WANG

Diffusive scaling limit of the Busemann process in Last Passage Percolation
OFER BUSANI

The number of ends in the uniform spanning tree for recurrent unimodular random graphs
DIEDERIK VAN ENGELENBURG AND TOM HUTCHCROFT

Fractional diffusion limit for a kinetic Fokker–Planck equation with diffusive boundary conditions in the half-line
LOÏC BÉTHENCOURT

Condensation, boundary conditions, and effects of slow sites in zero-range systems
SUNDER SETHURAMAN AND JIANFEI XUE

Dynamical loop equation
VADIM GORIN AND JIAOYANG HUANG

Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations
KONSTANTINOS DAREIOTIS AND MÁTÉ GERENCSÉR

Large deviations for random hives and the spectrum of the sum of two random matrices
HARIHARAN NARAYANAN AND SCOTT SHEFFIELD

Spectral gap and curvature of monotone Markov chains
JUSTIN SALEZ