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BULK UNIVERSALITY AND QUANTUM UNIQUE ERGODICITY FOR RANDOM BAND MATRICES IN HIGH DIMENSIONS

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We consider Hermitian random band matrices $H = (h_{xy})$ on the d-dimensional lattice $(\mathbb{Z}/L\mathbb{Z})^d$, where the entries $h_{xy} = \overline{h}_{yx}$ are independent centered complex Gaussian random variables with variances $s_{xy} = \mathbb{E}|h_{xy}|^2$. The variance matrix $S = (s_{xy})$ has a banded profile so that s_{xy} is negligible if $|x - y|$ exceeds the band width W. For dimensions $d \ge 7$, we prove the bulk eigenvalue universality of H under the condition $W \gg L^{95/(d+95)}$. Assuming that $W \ge L^{\varepsilon}$ for a small constant $\varepsilon > 0$, we also prove the quantum unique ergodicity for the bulk eigenvectors of H and a sharp local law for the Green's function $G(z) = (H - z)^{-1}$ up to $\operatorname{Im} z \gg W^{-5}L^{5-d}$. The local law implies that the bulk eigenvector entries of H are of order $\mathrm{O}(W^{-5/2}L^{-d/2+5/2})$ with high probability.

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PERCOLATION OF STRONGLY CORRELATED GAUSSIAN FIELDS II. SHARPNESS OF THE PHASE TRANSITION

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We establish the sharpness of the phase transition for a wide class of Gaussian percolation models, on $\mathbb{Z}^d$ or $\mathbb{R}^d$, $d \geq 2$, with correlations decaying at least algebraically with exponent $\alpha > 0$, including the discrete Gaussian free field ($d \geq 3$, $\alpha = d - 2$), the discrete Gaussian membrane model ($d \geq 5$, $\alpha = d - 4$), and many other examples both discrete and continuous. In particular, we do not assume positive correlations. This result is new for all strongly correlated models (i.e., $\alpha \in (0, d]$) in dimension $d \geq 3$ except the Gaussian free field, for which sharpness was proven in a recent breakthrough by Duminil-Copin et al. (*Duke Math. J.* **172** (2023) 839–913); even then, our proof is simpler and yields new near-critical information on the percolation density.

For planar fields which are continuous and positively correlated, we establish sharper bounds on the percolation density by exploiting a new 'weak mixing' property for strongly correlated Gaussian fields. As a byproduct, we establish the box-crossing property for the nodal set, of independent interest.

This is the second in a series of two papers studying level-set percolation of strongly correlated Gaussian fields, which can be read independently.

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A CENTRAL LIMIT THEOREM FOR THE NUMBER OF EXCURSION SET COMPONENTS OF GAUSSIAN FIELDS

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For a smooth stationary Gaussian field f on $\mathbb{R}^d$ and level $\ell \in \mathbb{R}$, we consider the number of connected components of the excursion set $\{f \geq \ell\}$ (or level set $\{f = \ell\}$) contained in large domains. The mean of this quantity is known to scale like the volume of the domain under general assumptions on the field. We prove that, assuming sufficient decay of correlations (e.g., the Bargmann–Fock field), a central limit theorem holds with volume-order scaling. Previously, such a result had only been established for "additive" geometric functionals of the excursion/level sets (e.g., the volume or Euler characteristic) using Hermite expansions. Our approach, based on a martingale analysis, is more robust and can be generalised to a wider class of topological functionals. A major ingredient in the proof is a third moment bound on critical points, which is of independent interest.

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SUDAKOV–FERNIQUE POST-AMP, AND A NEW PROOF OF THE LOCAL CONVEXITY OF THE TAP FREE ENERGY

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We develop an approach for studying the local convexity of a certain class of random objectives around the iterates of an AMP algorithm. Our approach involves applying the *Sudakov–Fernique inequality* conditionally on a long sequence of AMP iterates, and our main contribution is to demonstrate the way in which the resulting objective can be simplified and analyzed. As a consequence, we provide a new, and arguably simpler, proof of some of the results of Celentano, Fan and Mei (*Ann. Statist.* **51** (2023) 519–546), which establishes that the so-called TAP free energy in the $\mathbb{Z}_2$-synchronization problem is locally convex in the region to which AMP converges. We further prove a conjecture of Alaoui, Montanari and Sellke (In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science—FOCS 2022* (2022) 323–334 IEEE Computer Soc.) involving the local convexity of a related but distinct TAP free energy, which as a consequence, confirms that their algorithm efficiently samples from the Sherrington–Kirkpatrick Gibbs measure throughout the "easy" regime.

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THE EXTREMAL POINT PROCESS OF BRANCHING BROWNIAN MOTION IN $\mathbb{R}^d$

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We consider a branching Brownian motion in $\mathbb{R}^d$ with $d \geq 1$ in which the position $X_t^{(u)} \in \mathbb{R}^d$ of a particle u at time t can be encoded by its direction $\theta_t^{(u)} \in \mathbb{S}^{d-1}$ and its distance $R_t^{(u)}$ to 0. We prove that the *extremal point process* $\sum \delta_{(\theta_t^{(u)}, R_t^{(u)} - m_t^{(d)})}$ (where the sum is over all particles alive at time t and $m_t^{(d)}$ is an explicit centering term) converges in distribution to a randomly shifted, decorated Poisson point process on $\mathbb{S}^{d-1} \times \mathbb{R}$. More precisely, the so-called *clan-leaders* form a Cox process with intensity proportional to $D_\infty(\theta) e^{-\sqrt{2}r} \, \mathrm{d}r \, \mathrm{d}\theta$, where $D_\infty(\theta)$ is the limit of the derivative martingale in direction θ and the decorations are i.i.d. copies of the decoration process of the standard one-dimensional branching Brownian motion. This proves a conjecture of Stasiński, Berestycki and Mallein (*Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 1786–1810). The proof builds on that paper and on Kim, Lubetzky and Zeitouni (*Ann. Appl. Probab.* **33** (2023) 1315–1368).

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THE SPINE OF THE FLEMING–VIOT PROCESS DRIVEN BY BROWNIAN MOTION

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We show that the spine of the Fleming–Viot process driven by Brownian motion in a bounded Lipschitz domain with Lipschitz constant less than 1 converges to Brownian motion conditioned to stay in the domain forever.

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TWO-SIDED HEAT KERNEL ESTIMATES FOR SCHRÖDINGER OPERATORS WITH UNBOUNDED POTENTIALS

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Consider the Schrödinger operator $\mathcal{L}^V = -\Delta + V$ on $\mathbb{R}^d$, where $V : \mathbb{R}^d \to [0,\infty)$ is a nonnegative and locally bounded potential on $\mathbb{R}^d$ so that for all $x \in \mathbb{R}^d$ with $|x| \ge 1$, $c_1 g(|x|) \le V(x) \le c_2 g(|x|)$ with some constants $c_1, c_2 > 0$ and a nondecreasing and strictly positive function $g : [0,\infty) \to [1,+\infty)$ that satisfies $g(2r) \le c_0 g(r)$ for all $r > 0$ and $\lim_{r\to\infty} g(r) = \infty$. We establish global in time and qualitatively sharp bounds for the heat kernel of the associated Schrödinger semigroup by the probabilistic method. In particular, we can present global in space and time two-sided bounds of heat kernel even when the Schrödinger semigroup is not intrinsically ultracontractive. Furthermore, two-sided estimates for the corresponding Green's function are also obtained.

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CONDENSATION, BOUNDARY CONDITIONS, AND EFFECTS OF SLOW SITES IN ZERO-RANGE SYSTEMS

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We consider the space-time scaling limit of the particle mass in zero-range particle systems on a 1D discrete torus $\mathbb{Z}/N\mathbb{Z}$ with a finite number of defects. We focus on two classes of increasing jump rates g, when $g(n) \sim n^\alpha$, for $0 < \alpha \le 1$, and when g is a bounded function. In such a model, a particle at a regular site k jumps equally likely to a neighbor with rate $g(n)$, depending only on the number of particles n at k. At a defect site $k_{j,N}$, however, the jump rate is slowed down to $\lambda_j^{-1} N^{-\beta_j} g(n)$ when $g(n) \sim n^\alpha$, and to $\lambda_j^{-1} g(n)$ when g is bounded. Here N is a scaling parameter where the grid spacing is seen as $1/N$ and time is speeded up by N^2.

Starting from initial measures with $O(N)$ relative entropy with respect to an invariant measure, we show the hydrodynamic limit and characterize boundary behaviors at the macroscopic defect sites $x_j = \lim_{N\uparrow\infty} k_{j,N}/N$, for all defect strengths. For rates $g(n) \sim n^\alpha$, at critical or super-critical slow sites ($\beta_j = \alpha$ or $\beta_j > \alpha$), associated Dirichlet boundary conditions arise as a result of interactions with evolving atom masses or condensation at the defects. Differently, when g is bounded, at any slow site ($\lambda_j > 1$), we find the hydrodynamic density must be bounded above by a threshold value reflecting the strength of the defect. Moreover, the associated boundary conditions at slow sites change dynamically depending on the masses on the slow sites.

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LARGE DEVIATIONS FOR RANDOM HIVES AND THE SPECTRUM OF THE SUM OF TWO RANDOM MATRICES

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Suppose α, β are Lipschitz, strongly concave functions from $[0, 1]$ to $\mathbb{R}$ and γ is a concave function from $[0, 1]$ to $\mathbb{R}$ such that $\alpha(0) = \gamma(0) = 0$, $\alpha(1) = \beta(0) = 0$ and $\beta(1) = \gamma(1) = 0$. For an $n \times n$ Hermitian matrix W, let $\mathrm{spec}(W)$ denote the vector in $\mathbb{R}^n$ whose coordinates are the eigenvalues of W listed in nonincreasing order. Let $\lambda = \partial^-\alpha$, $\mu = \partial^-\beta$ on $(0, 1]$ and $\nu = \partial^-\gamma$, at all points of $(0, 1]$, where ∂^- is the left derivative. Let $\lambda_n(i) := n^2(\alpha(\frac{i}{n}) - \alpha(\frac{i-1}{n}))$, for $i \in [n]$, and similarly, $\mu_n(i) := n^2(\beta(\frac{i}{n}) - \beta(\frac{i-1}{n}))$ and $\nu_n(i) := n^2(\gamma(\frac{i}{n}) - \gamma(\frac{i-1}{n}))$.

Let X_n, Y_n be independent random Hermitian matrices from unitarily invariant distributions with spectra λ_n, μ_n, respectively. We define norm $\|\cdot\|_I$ to correspond in a certain way to the sup norm of an antiderivative. We prove that the following limit exists:

$$\lim_{n\to\infty} \frac{\log \mathbb{P}[\|\mathrm{spec}(X_n + Y_n) - \nu_n\|_I < n^2\epsilon]}{n^2}.$$

We interpret this limit in terms of the surface tension σ of continuum limits of the discrete hives defined by Knutson and Tao.

We provide matching large deviation upper and lower bounds for the spectrum of the sum of two random matrices X_n and Y_n, in terms of the surface tension σ mentioned above.

We also prove large deviation principles for random hives with α and β that are C^2, where the rate function can be interpreted in terms of the maximizer of a functional that is the sum of a term related to the free energy of hives associated with α, β and γ and a quantity related to logarithms of Vandermonde determinants associated with γ.

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SPECTRAL GAP AND CURVATURE OF MONOTONE MARKOV CHAINS

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We prove that the absolute spectral gap of any monotone Markov chain coincides with its optimal Ollivier–Ricci curvature, where the word *optimal* refers to the choice of the underlying metric. Moreover, we provide a new expression in terms of local variations of increasing functions, which has several practical advantages over the traditional variational formulation using the Dirichlet form. As an illustration, we explicitly determine the optimal curvature and spectral gap of the nonconservative exclusion process with heterogeneous reservoir densities on any network, despite the lack of reversibility.

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SUPERCONVERGENCE PHENOMENON IN WIENER CHAOSES

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We establish, in full generality, an unexpected phenomenon of strong regularization along normal convergence on Wiener chaoses. Namely, for every sequence of chaotic random variables, convergence in law to the Gaussian distribution is automatically upgraded to *superconvergence*: the regularity of the densities increases along the convergence, and all the derivatives converge uniformly on the real line. Our findings strikingly strengthen known results regarding modes of convergence for normal approximation on Wiener chaoses. Without additional assumptions, quantitative convergence in total variation is established by Nourdin and Peccati (*Probab. Theory Related Fields* **145** (2009) 75–118), and later on amplified to convergence in relative entropy by Nourdin, Peccati and Swan (*J. Funct. Anal.* **266** (2014) 3170–3207).

Our result is then extended to the multivariate setting and for polynomial mappings of a Gaussian field, provided the projection on the Wiener chaos of maximal degree admits a nondegenerate Gaussian limit. While our findings potentially apply to any context involving polynomial functionals of a Gaussian field, we emphasize, in this work, applications regarding: improved Carbery–Wright estimates near Gaussianity, normal convergence in entropy and in Fisher information, *superconvergence* for the spectral moments of Gaussian orthogonal ensembles, moments bounds for the inverse of strongly correlated Wishart-type matrices, and *superconvergence* in the Breuer–Major Theorem.

Our proofs leverage Malliavin's historical idea to establish smoothness of the density via the existence of negative moments of the Malliavin gradient, and we further develop a new paradigm to study this problem. Namely, we relate the existence of negative moments to some explicit spectral quantities associated with the Malliavin Hessian. This link relies on an adequate choice of the Malliavin gradient, which provides a novel decoupling procedure of independent interest. Previous attempts to establish convergence beyond entropy have imposed restrictive assumptions ensuring finiteness of negative moments for the Malliavin derivatives Our analysis renders these assumptions superfluous.

The terminology *superconvergence* was introduced by Bercovici and Voiculescu (*Probab. Theory Related Fields* **103** (1995) 215–222) for the central limit theorem in free probability.

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Future Issues