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BRANCHING RANDOM WALKS ON RELATIVELY HYPERBOLIC GROUPS

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Let Γ be a nonelementary, relatively hyperbolic group with a finite generating set. Consider a finitely supported admissible and symmetric probability measure μ on Γ and a probability measure ν on $\mathbb{N}$ with mean r. Let $\mathrm{BRW}(\Gamma, \nu, \mu)$ be the branching random walk on Γ with offspring distribution ν and base motion given by the random walk with step distribution μ. It is known that, for $1 < r \leq R$ with R the radius of convergence for the Green function of the random walk, the population of $\mathrm{BRW}(\Gamma, \nu, \mu)$ survives forever but eventually vacates every finite subset of Γ. We prove that in this regime, the growth rate of the trace of the branching random walk is equal to the growth rate $\omega_\Gamma(r)$ of the Green function of the underlying random walk. We also prove that the Hausdorff dimension of the limit set $\Lambda(r)$, which is the random subset of the Bowditch boundary consisting of all accumulation points of the trace of $\mathrm{BRW}(\Gamma, \nu, \mu)$, is equal to a constant times $\omega_\Gamma(r)$.

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CENTRAL LIMIT THEOREM FOR RÉNYI DIVERGENCE OF INFINITE ORDER

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For normalized sums Z_n of i.i.d. random variables, we explore necessary and sufficient conditions, which guarantee the normal approximation with respect to the Rényi divergence of infinite order. In terms of densities p_n of Z_n, this is a strengthened variant of the local limit theorem taking the form $\sup_x (p_n(x) - \varphi(x))/\varphi(x) \to 0$ as $n \to \infty$.

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MACROSCOPIC LOOPS IN THE LOOP $O(n)$ MODEL VIA THE XOR TRICK

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The loop $O(n)$ model is a family of probability measures on collections of nonintersecting loops on the hexagonal lattice, parameterized by a loop-weight n and an edge-weight x. Nienhuis predicts that, for $0 \le n \le 2$, the model exhibits two regimes separated by $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$: when $x < x_c(n)$, the loop lengths have exponential tails, while when $x \ge x_c(n)$, the loops are macroscopic.

In this paper, we prove three results regarding the existence of long loops in the loop $O(n)$ model:

– In the regime $(n, x) \in [1, 1+\delta) \times (1-\delta, 1]$ with $\delta > 0$ small, a configuration sampled from a translation-invariant Gibbs measure will either contain an infinite path or have infinitely many loops surrounding every face. In the subregime $n \in [1, 1+\delta)$ and $x \in (1-\delta, 1/\sqrt{n}]$, our results further imply Russo–Seymour–Welsh theory. This is the first proof of the existence of macroscopic loops in a positive area subset of the phase diagram.

– Existence of loops whose diameter is comparable to that of a finite domain whenever $n = 1$, $x \in (1, \sqrt{3}]$; this regime is equivalent to part of the antiferromagnetic regime of the Ising model on the triangular lattice.

– Existence of noncontractible loops on a torus when $n \in [1, 2]$, $x = 1$.

The main ingredients of the proof are: (i) the "XOR trick": if ω is a collection of short loops and Γ is a long loop, then the symmetric difference of ω and Γ necessarily includes a long loop as well; (ii) a reduction of the problem of finding long loops to proving that a percolation process on an auxiliary planar graph, built using the Chayes–Machta and Edwards–Sokal geometric expansions, has no infinite connected components and (iii) a recent result on the percolation threshold of Benjamini–Schramm limits of planar graphs.

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ON THE TIGHTNESS OF THE MAXIMUM OF BRANCHING BROWNIAN MOTION IN RANDOM ENVIRONMENT

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We consider one-dimensional branching Brownian motion in a spatially random branching environment (BBMRE) and show that for almost every realisation of the environment, the distributions of the maximal particle of the BBMRE recentred around its median are tight as time evolves. This result is in stark contrast to the fact that the transition fronts in the solution to the randomised Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation are, in general, not bounded uniformly in time. In particular, this highlights that—when compared to the settings of homogeneous branching Brownian motion and the F-KPP equation in a homogeneous environment—the introduction of a random environment leads to a much more intricate behaviour.

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ON THE INFLUENCE OF EDGES IN FIRST-PASSAGE PERCOLATION ON $\mathbb{Z}^d$

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We study first-passage percolation on $\mathbb{Z}^d$, $d \geq 2$, with independent weights whose common distribution is compactly supported in $(0, \infty)$ with a uniformly-positive density. Given $\epsilon > 0$ and $v \in \mathbb{Z}^d$, which edges have probability at least ϵ to lie on the geodesic between the origin and v? It is expected that all such edges lie at distance at most some $r(\epsilon)$ from either the origin or v, but this remains open in dimensions $d \geq 3$. We establish the closely-related fact that the *number* of such edges is at most some $C(\epsilon)$, uniformly in v. In addition, we prove a quantitative bound, allowing ϵ to tend to zero as $\|v\|$ tends to infinity, showing that there are at most $O(\epsilon^{-\frac{2d}{d-1}}(\log \|v\|)^C)$ such edges, uniformly in ϵ and v. The latter result addresses a problem raised by Benjamini–Kalai–Schramm (*Ann. Probab.* **31** (2003) 1970–1978).

Our technique further yields a strengthened version of a lower bound on transversal fluctuations due to Licea–Newman–Piza (*Probab. Theory Related Fields* **106** (1996) 559–591).

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FLUCTUATIONS OF PARTITION FUNCTIONS OF DIRECTED POLYMERS IN WEAK DISORDER BEYOND THE L^2-PHASE

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We study the directed polymer model in a bounded environment in weak disorder without L^2-boundedness, specifically the speed of homogenization for the field $(W_n^x)_{x\in\mathbb{Z}^d}$, where W_n^x denotes the associated martingale for the polymer starting from x. We show that a suitably recentered spatial average over a set of diameter $n^{1/2}$ convergence to zero at rate $n^{-\xi+o(1)}$, where the exponent is an explicit function of the inverse temperature β.

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APPROXIMATION METHOD TO METASTABILITY: AN APPLICATION TO NONREVERSIBLE, TWO-DIMENSIONAL ISING AND POTTS MODELS WITHOUT EXTERNAL FIELDS

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The main contribution of the current study is two-fold. First, we investigate the energy landscape of the Ising and Potts models on finite two-dimensional lattices without external fields in the low-temperature regime. The complete analysis of the energy landscape of these models was unknown because of its complicated *plateau* saddle structure between the ground states. We characterize this structure completely in terms of a random walk on the set of subtrees of a ladder graph. Second, we provide a considerable simplification of the well-known potential-theoretic approach to metastability. In particular, by replacing the role of variational principles, such as the Dirichlet and Thomson principles, with an H^1*-approximation of the equilibrium potential*, we develop a new method that can be applied to *nonreversible dynamics* as well in a simple manner. As an application of this method, we analyze metastable behavior of not only the reversible Metropolis–Hastings dynamics but also of several interesting nonreversible dynamics associated with the low-temperature Ising and Potts models explained above and derive the Eyring–Kramers law and the Markov chain model reduction of these models.

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THE CENTRAL LIMIT THEOREM ON NILPOTENT LIE GROUPS

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We formulate and establish the central limit theorem for products of i.i.d. random variables on arbitrary simply connected nilpotent Lie groups, allowing a possible bias. We find that some interesting new phenomena arise in the presence of a bias: the walk spreads out at a higher rate in the ambient group, while the limiting hypoelliptic diffusion process may not always have full support. We use elementary Fourier analysis to establish our results, which include Berry–Esseen bounds under optimal moment assumptions as well as an analogue of Donsker's invariance principle. Various examples of nilpotent Lie groups are treated in detail showing the variety of different behaviours. We also obtain a characterization of when the limiting distribution is an ordinary Gaussian and answer a question of Tutubalin regarding asymptotically close distributions on nilpotent Lie groups.

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DEGENERATE PROCESSES KILLED AT THE BOUNDARY OF A DOMAIN

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We investigate quasi-stationarity properties of Feller processes that are killed when exiting a relatively compact set. Our main result provides general conditions ensuring that such a process possesses a (possibly nonunique) quasi-stationary distribution. Conditions ensuring uniqueness and exponential convergence are discussed. Our conditions are well suited to the study of degenerate processes, such as nonelliptic diffusions or piecewise deterministic Markov processes (PDMP). The results are applied to stochastic differential equations, and we illustrate the application to PDMPs with an example.

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THE CONTACT PROCESS ON DYNAMIC REGULAR GRAPHS: SUBCRITICAL PHASE AND MONOTONICITY

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We study the contact process on a dynamic random d-regular graph with an edge-switching mechanism as well as an interacting particle system that arises from the local description of this process called the herds process. Both these processes were introduced in (da Silva, Oliveira and Valesin (2021)); there it was shown that the herds process has a phase transition with respect to the infectivity parameter λ, depending on the parameter v that governs the edge dynamics. Improving on a result of (da Silva, Oliveira and Valesin (2021)), we prove that the critical value of λ is strictly decreasing with v. We also prove that, in the subcritical regime, the extinction time of the herds process started from a single individual has an exponential tail. Finally, we apply these results to study the subcritical regime of the contact process on the dynamic d-regular graph. We show that, starting from all vertices infected, the infection goes extinct in a time that is logarithmic in the number of vertices of the graph, with high probability.

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