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BOUNDARY TOUCHING PROBABILITY AND NESTED-PATH EXPONENT FOR NONSIMPLE CLE

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The conformal loop ensemble (CLE) has two phases: for $\kappa \in (8/3, 4]$, the loops are simple and do not touch each other or the boundary; for $\kappa \in (4, 8)$, the loops are nonsimple and may touch each other and the boundary. For $\kappa \in (4, 8)$, we derive the probability that the loop surrounding a given point touches the domain boundary. We also obtain the law of the conformal radius of this loop seen from the given point conditioned on the loop touching the boundary or not, refining a result of Schramm–Sheffield–Wilson (*Comm. Math. Phys.* (2009) **288** 43–53). As an application, we exactly evaluate the CLE counterpart of the nested-path exponent for the Fortuin–Kasteleyn (FK) random cluster model recently introduced by Song–Tan–Zhang–Jacobsen–Nienhuis–Deng (*J. Phys. A* **55** (2022) Paper No. 204002). This exponent describes the asymptotic behavior of the number of nested open paths in the open cluster containing the origin when the cluster is large. For Bernoulli percolation, which corresponds to $\kappa = 6$, the exponent was derived recently in Song–Jacobsen–Nienhuis–Sportiello–Deng (2023) by a color switching argument. For $\kappa \neq 6$ and, in particular, for the FK-Ising case, our formula appears to be new. Our derivation begins with Sheffield's construction of CLE from which the quantities of interest can be expressed by radial SLE. We solve the radial SLE problem using the coupling between SLE and Liouville quantum gravity, along with the exact solvability of Liouville conformal field theory.

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THE THREE-DIMENSIONAL STOCHASTIC ZAKHAROV SYSTEM

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We study the three-dimensional stochastic Zakharov system in the energy space, where the Schrödinger equation is driven by linear multiplicative noise and the wave equation is driven by additive noise. We prove the well-posedness of the system up to the maximal existence time and provide a blow-up alternative. We further show that the solution exists at least as long as it remains below the ground state. Two main ingredients of our proof are refined rescaling transformations and the normal form method. Moreover, in contrast to the deterministic setting, our functional framework also incorporates the local smoothing estimate for the Schrödinger equation in order to control lower-order perturbations arising from the noise. Finally, we prove a regularization by noise result, which states that finite time blowup before any given time can be prevented with high probability by adding sufficiently large nonconservative noise. The key point of its proof is an estimate in Strichartz spaces for solutions of a Schrödinger type equation with a nonlocal potential involving the free wave.

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A STOCHASTIC ANALYSIS OF SUBCRITICAL EUCLIDEAN FERMIONIC FIELD THEORIES

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Building on previous work on the stochastic analysis for Grassmann random variables, we introduce a forward–backward stochastic differential equation (FBSDE), which provides a stochastic quantisation of Grassmann measures. Our method is inspired by the so-called continuous renormalisation group, but avoids the technical difficulties encountered in the direct study of the flow equation for the effective potentials. As an application, we construct a family of weakly coupled subcritical Euclidean fermionic field theories and prove exponential decay of correlations.

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EXACT PHASE TRANSITIONS FOR STOCHASTIC BLOCK MODELS AND RECONSTRUCTION ON TREES

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In this paper we continue to rigorously establish the predictions in ground breaking work in statistical physics by Decelle, Krzakala, Moore, Zdeborová (*Phys. Rev. E* **84** (2011) 066106) regarding the block model, in particular, in the case of $q = 3$ and $q = 4$ communities.

We prove that, for $q = 3$ and $q = 4$, there is no computational-statistical gap if the average degree is above some constant by showing it is information theoretically impossible to detect below the Kesten–Stigum bound.

The proof is based on showing that, for the broadcast process on Galton–Watson trees, reconstruction is impossible for $q = 3$ and $q = 4$ if the average degree is sufficiently large. This improves on the result of Sly (*Ann. Probab.* **39** (2011) 1365–1406), who proved similar results for regular trees for $q = 3$. Our analysis of the critical case $q = 4$ provides a detailed picture showing that the tightness of the Kesten–Stigum bound in the antiferromagnetic case depends on the average degree of the tree.

Our results prove conjectures of Decelle, Krzakala, Moore, Zdeborová (*Phys. Rev. E* **84** (2011) 066106), Moore (*Bull. Eur. Assoc. Theor. Comput. Sci. EATCS* **121** (2017) 26–61), Abbe and Sandon (*Comm. Pure Appl. Math.* **71** (2018) 1334–1406) and Ricci-Tersenghi, Semerjian, and Zdeborová (*Phys. Rev. E* **99** (2019) 042109). Our proofs are based on a new general coupling of the tree and graph processes and on a refined analysis of the broadcast process on the tree.

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UNIVERSALITY OF DIRECTED POLYMERS IN THE INTERMEDIATE DISORDER REGIME

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Krishnan and Quastel (*Ann. Appl. Probab.* **28** (2018) 3736–3764) showed that the fluctuations of Seppäläinen's log-gamma polymer converge in law to the Tracy–Widom GUE distribution in the intermediate disorder regime. This regime corresponds to taking the inverse temperature β to depend on the length of the polymer $2n$, with $\beta = n^{-\alpha}$ for some $\alpha < 1/4$. They also conjectured that this should hold for directed polymers with general i.i.d weights. We prove that their conjecture is true for any $1/5 < \alpha < 1/4$.

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RECURRENCE AND TRANSIENCE OF MULTIDIMENSIONAL ELEPHANT RANDOM WALKS

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We prove a conjecture by Bertoin that the multidimensional elephant random walk on $\mathbb{Z}^d$ is transient in dimensions $d \geq 3$. We show that it undergoes a phase transition in dimensions $d = 1, 2$ between recurrence and transience at $p = (2d + 1)/(4d)$.

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FLUCTUATIONS AND CORRELATIONS IN WEAKLY ASYMMETRIC SIMPLE EXCLUSION ON A RING SUBJECT TO AN ATYPICAL CURRENT

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We consider the weakly asymmetric simple exclusion process on a ring, driven out of equilibrium by tilting the dynamics so as to enforce a macroscopic current of particles on a large time interval. In this current-biased dynamics, the tilt by the current makes the dynamics nonlocal, nonhomogeneous and induces long-range correlations. We compute the correlation structure in the large time, large size limit for a certain range of asymmetry and current strength, recovering heuristic results of Bodineau et al. (*J. Stat. Phys.* **133** (2008) 1013–1031). In addition, in this range of parameters, we characterise the full dynamics of fluctuations around the optimal density profile in the current-biased dynamics. The key ingredient at the microscopic scale is a precise relative entropy estimate at the level of correlations. We also discuss how to remove the (technical) restriction on the range of parameters.

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SCALING LIMITS OF PLANAR MAPS UNDER THE SMITH EMBEDDING

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The Smith embedding of a finite planar map with two marked vertices, possibly with conductances on the edges, is a way of representing the map as a tiling of a finite cylinder by rectangles. In this embedding, each edge of the planar map corresponds to a rectangle, and each vertex corresponds to a horizontal segment. Given a sequence of finite planar maps embedded in an infinite cylinder such that the random walk on both the map and its planar dual converges to Brownian motion modulo time change, we prove that the a priori embedding is close to an affine transformation of the Smith embedding at large scales. By applying this result, we prove that the Smith embeddings of mated-CRT maps with the sphere topology converge to γ-Liouville quantum gravity (γ-LQG).

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