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SPHERE VALUED NOISE STABILITY AND QUANTUM MAX-CUT HARDNESS

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We prove a vector-valued inequality for the Gaussian noise stability (i.e. we prove a vector-valued Borell inequality) for Euclidean functions taking values in the two-dimensional sphere, for all correlation parameters at most $1/10$ in absolute value. This inequality was conjectured (for all correlation parameters at most 1 in absolute value) by Hwang, Neeman, Parekh, Thompson and Wright. Such an inequality is needed to prove sharp computational hardness of the product state Quantum MAX-CUT problem, assuming the unique games conjecture. In fact, assuming the unique games conjecture, we show that the product state of Quantum MAX-CUT is NP-hard to approximate within a multiplicative factor of 0.9859. In contrast, a polynomial time algorithm is known with approximation factor 0.956 . . . .

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LARGE DEVIATIONS FOR THE q-DEFORMED POLYNUCLEAR GROWTH

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In this paper we study large time large deviations for the height function $\mathfrak{h}(x,t)$ of the q-deformed polynuclear growth introduced in (*Int. Math. Res. Not. IMRN* **7** (2023) 5728–5780). We show that the upper-tail deviations have speed t and derive an explicit formula for the rate function $\Phi_+(\mu)$. On the other hand, we show that the lower-tail deviations have speed t^2 and express the corresponding rate function $\Phi_-(\mu)$ in terms of a variational problem.

Our analysis relies on distributional identities between the height function $\mathfrak{h}$ and two important measures on the set of integer partitions: the Poissonized Plancherel measure and the cylindric Plancherel measure. Following a scheme developed in (*Ann. Inst. Henri Poincaré Probab. Stat.* **57** (2021) 778–799), we analyze a Fredholm determinant representation for the q-Laplace transform of $\mathfrak{h}(x,t)$ from which we extract exact Lyapunov exponents and through inversion the upper-tail rate function Φ_+. The proof of the lower-tail large deviation principle is more subtle and requires several novel ideas which combine classical asymptotic results for the Plancherel measure and log-concavity properties of Schur polynomials. The techniques we develop to characterize the lower-tail are rather flexible and have the potential to generalize to other solvable growth models.

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BOUNDARY HARNACK PRINCIPLE FOR NONLOCAL OPERATORS ON METRIC MEASURE SPACES

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In this paper a necessary and sufficient condition is obtained for the scale invariant boundary Harnack principle (BHP in abbreviation) for a large class of Hunt processes on metric measure spaces that are in weak duality with another Hunt process. We then apply it to a class of discontinuous subordinate Brownian motions with Gaussian components in $\mathbb{R}^d$ for which the Lévy density of the subordinators satisfies some mild comparability condition. We show that the scale invariant BHP holds for these subordinate Brownian motions in any Lipschitz domain that satisfies the interior cone condition with common angle $\theta \in (\cos^{-1}(1/\sqrt{d}), \pi)$ but fails in any truncated circular cone with angle $\theta \le \cos^{-1}(1/\sqrt{d})$, a Lipschitz domain whose Lipschitz constant is larger than or equal to $1/\sqrt{d-1}$.

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MULTISCALE GENESIS OF A TINY GIANT FOR PERCOLATION ON SCALE-FREE RANDOM GRAPHS

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We study the critical behavior for percolation on inhomogeneous random networks on n vertices, where the weights of the vertices follow a power-law distribution with exponent $\tau \in (2, 3)$. Such networks, often referred to as *scale-free networks*, exhibit critical behavior when the percolation probability tends to zero at an appropriate rate, as $n \to \infty$. We identify the critical window for several scale-free random graph models, such as the Norros–Reittu model, Chung–Lu model and generalized random graphs. Surprisingly, there exists a finite time inside the critical window, after which we see a sudden emergence of a "tiny" giant component. This is a novel behavior, which is in contrast with the critical behavior in other known universality classes with $\tau \in (3, 4)$ and $\tau > 4$.

Precisely, for edge-retention probabilities $\pi_n = \lambda n^{-(3-\tau)/2}$, there is an explicitly computable $\lambda_c > 0$ such that the critical window is of the form $\lambda \in (0, \lambda_c)$, where the largest clusters have size of order n^β with $\beta = (\tau^2 - 4\tau + 5)/[2(\tau - 1)] \in [\sqrt{2} - 1, \frac{1}{2})$ and have nondegenerate scaling limits, while in the supercritical regime $\lambda > \lambda_c$, a unique "tiny giant" component of size $\Theta(\sqrt{n})$ emerges, and its size concentrates. For $\lambda \in (0, \lambda_c)$, the scaling limit of the maximum component sizes can be described in terms of components of a one-dimensional inhomogeneous percolation model on $\mathbb{Z}_+$ studied in a seminal work by Durrett and Kesten (In *A Tribute to Paul Erdős* (1990) 161–176 Cambridge Univ. Press). For $\lambda > \lambda_c$, we use a relation to general inhomogeneous random graphs, as studied by Bollobás, Janson and Riordan (*Random Structures Algorithms* **31** (2007) 3–122), to prove that the sudden emergence of the tiny giant is caused by a phase transition inside a smaller core of vertices of weight of order at least $\sqrt{n}$.

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SPECTRAL ANALYSIS AND K-SPINE DECOMPOSITION OF INHOMOGENEOUS BRANCHING BROWNIAN MOTIONS. GENEALOGIES IN FULLY PUSHED FRONTS

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We consider a system of particles performing a one-dimensional dyadic branching Brownian motion with space-dependent branching rate, negative drift $-\mu$ and killed upon reaching 0. More precisely, the particles branch at rate $r(x) = (1 + W(x))/2$, where W is a compactly supported and nonnegative smooth function and the drift μ is chosen in such a way that the system is critical in some sense.

This particle system can be seen as an analytically tractable model for fluctuating fronts, describing the internal mechanisms driving the invasion of a habitat by a cooperating population. Recent studies from Birzu, Hallatschek and Korolev suggest the existence of three classes of fluctuating fronts: pulled, semipushed and fully pushed fronts. Here, we focus on the fully pushed regime. We establish a Yaglom law for this branching process and prove that the genealogy of the particles converges to a Brownian Coalescent Point Process using a method of moments.

In practice, the genealogy of the BBM is seen as a random marked metric measure space and we use spinal decomposition to prove its convergence in the Gromov-weak topology. We also carry out the spectral decomposition of a differential operator related to the BBM to determine the invariant measure of the spine as well as its mixing time.

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CONTOUR INTEGRAL FORMULAS FOR PUSHASEP ON THE RING

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We give contour integral formulas for the generating function of the joint distribution of the PushASEP on a ring. We obtained these formulas through a rigorous treatment of Bethe ansatz. The approach relies on residue computations and controlling the location of the Bethe roots, which we achieve by partially decoupling the Bethe equations and extending the system of equations. Moreover, we are able to use our formulas to compute the asymptotic fluctuations for the flat and step initial conditions at the relaxation time scale.

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TRANSIENCE OF VERTEX-REINFORCED JUMP PROCESSES WITH LONG-RANGE JUMPS

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We show that the vertex-reinforced jump process on the d-dimensional lattice with long-range jumps is transient in any dimension d as long as the initial weights do not decay too fast. The main ingredients in the proof are: an analysis of the corresponding random environment on finite boxes, a comparison with a hierarchical model, and the reduction of the hierarchical model to a nonhomogeneous effective one-dimensional model. For $d \geq 3$ we also prove transience of the vertex-reinforced jump process with possibly long-range jumps as long as nearest-neighbor weights are large enough.

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THE GAUSSIAN FREE-FIELD AS A STREAM FUNCTION: ASYMPTOTICS OF EFFECTIVE DIFFUSIVITY IN INFRA-RED CUT-OFF

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We analyze the large-time asymptotics of a passive tracer with drift equal to the curl of the Gaussian free field in two dimensions with ultra-violet cut-off at scale unity. We prove that the mean-squared displacement scales like $t\sqrt{\ln t}$, as predicted in the physics literature. This improves the asymptotics recently established in recent work of Cannizzaro, Haunschmidt-Sibitz, and Toninelli (*Ann. Probab.* **50** (2022) 2475–2498), which uses mathematical-physics type analysis in Fock space. Our approach involves studying the effective diffusivity λ_L of the process with an infra-red cut-off at scale L, and is based on techniques from stochastic homogenization.

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CLUSTER-SIZE DECAY IN SUPERCRITICAL KERNEL-BASED SPATIAL RANDOM GRAPHS

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We consider a large class of spatially-embedded random graphs that includes among others long-range percolation, continuum scale-free percolation and the age-dependent random connection model. We assume that the model is supercritical: there is an infinite component. We identify the stretch-exponent $\zeta \in (0,1)$ of the decay of the cluster-size distribution. That is, with $|\mathcal{C}(0)|$ denoting the number of vertices in the component of the vertex at $0 \in \mathbb{R}^d$, we prove

$$\mathbb{P}(k < |\mathcal{C}(0)| < \infty) = \exp(-\Theta(k^\zeta)) \quad \text{as } k \to \infty.$$

The value of ζ undergoes several phase transitions with respect to three main model parameters: the Euclidean dimension d, the power-law tail exponent τ of the degree distribution and a long-range parameter α governing the presence of long edges in Euclidean space.

In this paper we present the proof for the region in the phase diagram where the model is a generalization of continuum scale-free percolation and/or hyperbolic random graphs: ζ in this regime depends both on τ, α. We also prove that the second-largest component in a box of volume n is of size $\Theta((\log n)^{1/\zeta})$ with high probability. We develop a deterministic algorithm, the cover expansion, as new methodology. This algorithm enables us to prevent too large components that may be de-localized or locally dense in space.

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