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FINITE RANGE INTERLACEMENTS AND COUPLINGS

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In this article we consider the interlacement set $\mathcal{I}^u$ at level $u > 0$ on $\mathbb{Z}^d$, $d \geq 3$, and its finite range version $\mathcal{I}^{u,L}$ for $L > 0$, given by the union of the ranges of a Poisson cloud of random walks on $\mathbb{Z}^d$ having intensity u/L and killed after L steps. As $L \to \infty$, the random set $\mathcal{I}^{u,L}$ has a nontrivial (local) limit, which is precisely $\mathcal{I}^u$. A natural question is to understand how the sets $\mathcal{I}^{u,L}$ and $\mathcal{I}^u$ can be related, if at all, in such a way that their intersections with a box of large radius R almost coincide. We address this question, which depends sensitively on R, by developing couplings allowing for a similar comparison to hold with very high probability for $\mathcal{I}^{u,L}$ and $\mathcal{I}^{u',2L}$, with $u' \approx u$. In particular, for the vacant set $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ with values of u near the critical percolation threshold, our couplings remain effective at scales $R \gg \sqrt{L}$, which corresponds to a natural barrier across which the walks of length L, comprised in $\mathcal{I}^{u,L}$, *de-solidify* inside a box of radius R, that is, lose their intrinsic long-range structure to become increasingly "dust-like." These mechanisms are complementary to the *solidification* effects recently exhibited in (*J. Eur. Math. Soc. (JEMS)* **22** (2020) 2629–2672). By iterating the resulting couplings over dyadic scales L, the models $\mathcal{I}^{u,L}$ are seen to constitute a stationary finite range approximation of $\mathcal{I}^u$ at large spatial scales near the critical point u_*. Among others, these couplings are important ingredients for the characterization of the phase transition for percolation of the vacant sets of random walk and random interlacements in the companion articles (*Preprint*, arXiv:2308.07919) and (*Proc. Lond. Math. Soc. (3)* **129** (2024) 1–49).

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LIOUVILLE CONFORMAL FIELD THEORY AND THE QUANTUM ZIPPER

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Sheffield showed that conformally welding a γ-Liouville quantum gravity (LQG) surface to itself gives a Schramm–Loewner evolution (SLE) curve with parameter $\kappa = \gamma^2$ as the interface, and Duplantier–Miller–Sheffield proved similar results for $\kappa = \frac{16}{\gamma^2}$ for γ-LQG surfaces with boundaries decorated by looptrees of disks or by continuum random trees. We study these dynamics for LQG surfaces coming from Liouville conformal field theory (LCFT). At stopping times depending only on the curve, we give an explicit description of the surface and curve in terms of LCFT and SLE. This has applications to both LCFT and SLE. We prove the boundary BPZ equations for LCFT, a crucial input for subsequent work with Remy, Sun and Zhu deriving the structure constants of boundary LCFT. With Yu we prove the reversibility of whole-plane SLE_κ for $\kappa > 8$ via a novel radial mating-of-trees and will show the space of LCFT surfaces is closed under conformal welding.

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LOOP SOUP REPRESENTATION FOR ZETA-REGULARISED DETERMINANTS OF TWISTED LAPLACIANS AND COVARIANT SYMANZIK IDENTITIES

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We derive a stochastic representation for determinants of Laplace-type operators on vector bundles over manifolds. Namely, inverse powers of those determinants are written as the expectation of a product of holonomies defined over Brownian loop soups. Our results hold over compact manifolds of dimension $d \in \{2, 3\}$, in the presence of a mass or a boundary. We derive a few consequences, including some continuity of these determinants as a function of the operator, and the conformal invariance of the determinant on surfaces.

This expression allows us to construct the scalar field minimally coupled to a prescribed random smooth gauge field, which we prove obeys the so-called Symanzik identities. Some of these results are continuous analogues of the work of A. Kassel and T. Lévy in the discrete.

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HYPERBOLIC ANDERSON EQUATIONS WITH GENERAL TIME-INDEPENDENT GAUSSIAN NOISE: STRATONOVICH REGIME

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In this paper, we investigate the hyperbolic Anderson equation generated by a time-independent Gaussian noise with two objectives: the solvability and intermittency. First, we prove that Dalang's condition is necessary and sufficient for the existence of the solution. Second, we establish the precise long time and high moment asymptotics for the solution under the usual homogeneity assumption of the covariance of the Gaussian noise. Our approach is fundamentally different from the ones existing in literature. The main contributions in our approach include the representation of Stratonovich moment under Laplace transform via the moments of the Brownian motions in Gaussian potentials and some large deviation skills developed in dealing effectively with the Stratonovich chaos expansion.

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CHARACTERIZING GIBBS STATES FOR AREA-TILTED BROWNIAN LINES

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Gibbsian line ensembles are families of Brownian lines arising in many natural contexts such as the level curves of three dimensional Ising interfaces, the solid-on-solid model, multilayered polynuclear growth, trajectories of eigenvalues as the entries of the corresponding matrix perform diffusions, to name a few. In particular, line ensembles with area tilt potentials play a significant role in the study of wetting and entropic repulsion phenomena. An important example is a class of nonintersecting Brownian lines above a hard wall, which are subject to geometrically growing area tilt potentials, which we call the λ-*tilted line ensemble*, where $\lambda > 1$ is the parameter governing the geometric growth. The model was introduced in (*Electron. J. Probab.* **24** (2019) 37) as a putative scaling limit of the level lines of entropically repulsed solid-on-solid interfaces. While this model has infinitely many lines and is nonintegrable, the case of the single line, known as the Ferrari–Spohn diffusion, is well studied (*Ann. Probab.* **33** (2005) 1302–1325). In this article we address the problem of classifying *all* Gibbs measures for λ-tilted line ensembles. A stationary infinite volume Gibbs measure was already constructed in (*Electron. J. Probab.* **24** (2019) 37; In *Statistical Mechanics of Classical and Disordered Systems* (2019) 241–266 Springer), and the uniqueness of this translation invariant Gibbs measure was established in (*Probab. Math. Phys.* **6** (2025) 195–239). Our main result here is a strong characterization for Gibbs measures of λ-tilted line ensembles in terms of a two parameter family. Namely, we show that the extremal Gibbs measures are completely characterized by the behavior of the top line X^1 at positive and negative infinity, which must satisfy the parabolic growth

$$X^1(t) = t^2 + L|t|\mathbf{1}_{t<0} + R|t|\mathbf{1}_{t>0} + o(|t|) \quad \text{as } |t| \to \infty,$$

where L, R are real parameters, including $-\infty$, with $L + R < 0$, while the lower lying lines remain uniformly confined. The case $L = R = -\infty$ corresponds to the unique translation invariant Gibbs measure. The result bears some analogy to the Airy wanderers, an integrable model introduced and studied in (*Ann. Probab.* **38** (2010) 714–769) in the context of the Airy line ensemble, except in this case only the top line can wander, owing to the geometrically increasing nature of the area tilting factor. A crucial step in our proof, which holds independent significance, is a complete characterization of the extremal Gibbs states associated to a single area-tilted Brownian excursion, describing the equilibrium states of the top line when the lower lines are absent, and which can be interpreted as nontranslation invariant versions of the Ferrari–Spohn diffusion.

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ON THE SPECTRAL EDGE OF NON-HERMITIAN RANDOM MATRICES

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For general non-Hermitian large random matrices X and deterministic deformation matrices A, we prove that the local eigenvalue statistics of $A + X$ close to the typical edge points of its spectrum are universal. Furthermore, we show that, under natural assumptions, on A the spectrum of $A + X$ does not have outliers at a distance larger than the natural fluctuation scale of the eigenvalues. As a consequence, the number of eigenvalues in each component of $\mathrm{Spec}(A + X)$ is deterministic.

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CROSSING EXPONENT IN THE BROWNIAN LOOP SOUP

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We study the clusters of loops in a Brownian loop soup in some bounded two-dimensional domain with subcritical intensity $\theta \in (0, 1/2]$. We show that a cluster which is targeting a given interior point forms a dynamical system, which satisfies asymptotically a fixed point equation. As a consequence, we obtain an exact expression for the asymptotic probability of the existence of a cluster crossing a given annulus of radii r and r^s as $r \to 0$ ($s > 1$ fixed). Relying on this result, we then show that the probability for a macroscopic cluster to hit a given disc of radius r decays like $|\log r|^{-1+\theta+o(1)}$ as $r \to 0$. Finally, we characterise the polar sets of clusters, that is, sets that are not hit by the closure of any cluster, in terms of $\log^\alpha$-capacity.

This paper reveals a connection between the 1D and 2D Brownian loop soups. This connection in turn implies the existence of a second critical intensity $\theta = 1$ that describes a phase transition in the percolative behaviour of large loops on a logarithmic scale targeting an interior point of the domain.

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LEVEL CROSSINGS OF FRACTIONAL BROWNIAN MOTION

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Since the classical work of Lévy, it is known that the local time of Brownian motion can be characterized through the limit of level crossings. While subsequent extensions of this characterization have primarily focused on Markovian or martingale settings, this work presents a highly anticipated extension to fractional Brownian motion—a prominent non-Markovian and non-martingale process. Our result is viewed as a fractional analogue of Chacon et al. (*Z. Wahrsch. Verw. Gebiete* **57** (1981) 197–211). Consequently, it provides a global path-by-path construction of fractional Brownian local time.

Due to the absence of conventional probabilistic tools in the fractional setting, our approach utilizes a completely different argument with a flavor of the subadditive ergodic theorem, combined with the shifted stochastic sewing lemma recently obtained in (*Forum Math. Sigma* **12** (2024) 52).

Furthermore, we prove an almost-sure convergence of the $(1/H)$-th variation of fractional Brownian motion with the Hurst parameter H, along random partitions defined by level crossings, called Lebesgue partitions. This result raises an interesting conjecture on the limit, which seems to capture non-Markovian nature of fractional Brownian motion.

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2025

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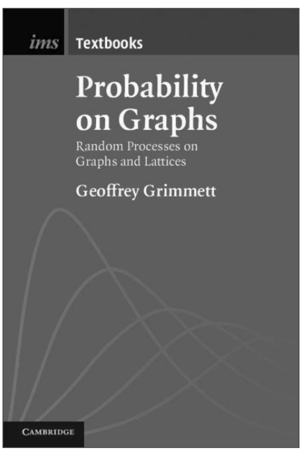

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