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SECOND-ORDER FRACTIONAL MEAN-FIELD SDES WITH SINGULAR KERNELS AND MEASURE INITIAL DATA

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In this paper we establish the local and global well-posedness of weak and strong solutions to second-order fractional mean-field SDEs with singular/distribution interaction kernels and measure initial value, where the kernel can be the Newton or Coulomb potential, Riesz potential, Biot–Savart law, etc. Moreover, we also show the stability, smoothness, and the short time singularity and large time decay estimates of the distribution density. Our results reveal a phenomenon that for *nonlinear* mean-field equations, the regularity of the initial distribution could balance the singularity of the kernel. The precise relationship between the singularity of kernels and the regularity of initial values are calculated, which belongs to the subcritical regime in the scaling sense. In particular, our results provide a microscopic probabilistic explanation and establish a unified treatment for many physical models, such as the fractional Vlasov–Poisson–Fokker–Planck system, the vorticity formulation of 2D-fractal Navier–Stokes equations, surface quasi-geostrophic models, etc.

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DIRECTED SPATIAL PERMUTATIONS ON ASYMMETRIC TORI

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We investigate a model of random spatial permutations on two-dimensional tori and establish that the joint distribution of large cycles is asymptotically given by the Poisson–Dirichlet distribution with parameter one. The asymmetry of the tori we consider leads to a spatial bias in the permutations, and this allows for a simple argument to deduce the existence of mesoscopic cycles. The main challenge is to leverage this mesoscopic structure to establish the existence and distribution of macroscopic cycles. We achieve this by a dynamical resampling argument in conjunction with a method developed by Schramm for the study of random transpositions on the complete graph. Our dynamical analysis implements generic heuristics for the occurrence of the Poisson–Dirichlet distribution in random spatial permutations and hence may be of more general interest.

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FAST RELAXATION OF THE RANDOM FIELD ISING DYNAMICS

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We study the convergence properties of Glauber dynamics for the random field Ising model (RFIM) with ferromagnetic interactions on finite domains of $\mathbb{Z}^d$, $d \geq 2$. Of particular interest is the *Griffiths phase* where correlations decay exponentially fast in expectation over the quenched disorder, but there exist arbitrarily large islands of weak fields where low-temperature behavior is observed. Our results are twofold:

1. Under *weak spatial mixing* (boundary-to-bulk exponential decay of correlations) in expectation, we show that the dynamics satisfy a *weak* Poincaré inequality, implying polynomial relaxation to equilibrium over timescales polynomial in the volume N of the domain and polynomial time mixing from a warm start. From this we construct a polynomial-time approximate sampling algorithm, based on running Glauber dynamics, over an increasing sequence of approximations of the domain.

2. Under *strong spatial mixing* (exponential decay of correlations even near boundary pinnings) in expectation, we prove a *full* Poincaré inequality, implying exponential relaxation to equilibrium and $N^{o(1)}$-mixing time. Both weak and strong spatial mixing hold at any temperature, provided the external fields are strong enough.

Our proofs combine a stochastic localization technique, which has the effect of increasing the variance of the field, with a field-dependent coarse graining which controls the subcritical percolation process of sites with weak fields.

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NEAR CRITICAL SCALING RELATIONS FOR PLANAR BERNOULLI PERCOLATION WITHOUT DIFFERENTIAL INEQUALITIES

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We provide a new proof of the near-critical scaling relation $\beta = \xi_1 \nu$ for Bernoulli percolation on the square lattice already proved by Kesten in 1987. We rely on a novel approach that does not invoke Russo's formula but rather relates differences in crossing probabilities at different scales. The argument is shorter and more robust than previous ones and is more likely to be adapted to other models. The same approach may be used to prove the other scaling relations appearing in Kesten's work.

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DENSITY FLUCTUATIONS IN WEAKLY INTERACTING PARTICLE SYSTEMS VIA THE DEAN–KAWASAKI EQUATION

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The Dean–Kawasaki equation—one of the most fundamental SPDEs of fluctuating hydrodynamics—has been proposed as a model for density fluctuations in weakly interacting particle systems. In its original form, it is highly singular and fails to be renormalizable, even by approaches such as regularity structures and paracontrolled distributions, hindering mathematical approaches to its rigorous justification. It has been understood recently that it is natural to introduce a suitable regularization, for example, by applying a formal spatial discretization or by truncating high-frequency noise: This yields well-posed equations that should still precisely approximate the law of the particle density fluctuations.

In the present work, we prove that a regularization in the form of a formal discretization of the Dean–Kawasaki equation indeed accurately describes density fluctuations in systems of weakly interacting diffusing particles: We show that, in suitable weak metrics, the law of fluctuations as predicted by the discretized Dean–Kawasaki SPDE approximates the law of fluctuations of the original particle system, up to an error that is of arbitrarily high order in the inverse particle number and a discretization error. In particular, the Dean–Kawasaki equation provides a means for efficient and accurate simulations of density fluctuations in weakly interacting particle systems.

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MULTIPLE POINTS ON THE BOUNDARIES OF BROWNIAN LOOP-SOUP CLUSTERS

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For a Brownian loop soup with intensity $c \in (0, 1]$ in the unit disk, we show that almost surely, the set of simple (resp., double) points on any portion of boundary of any of its clusters has Hausdorff dimension $2 - \xi_c(2)$ (resp., $2 - \xi_c(4)$), where $\xi_c(k)$ is the generalized disconnection exponent computed in (*Probab. Theory Related Fields* **179** (2021) 117–164). As a consequence, when the dimension is positive, such points are a.s. dense on every boundary of every cluster. There are a.s. no triple points on the cluster boundaries.

As an intermediate result, we establish a *separation lemma* for Brownian loop soups, which is a powerful tool for obtaining sharp estimates on nonintersection and nondisconnection probabilities in the setting of loop soups. In particular, it allows us to define a family of *generalized intersection exponents* $\xi_c(k, \lambda)$, and show that $\xi_c(k)$ is the limit as $\lambda \searrow 0$ of $\xi_c(k, \lambda)$.

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THE NUMBER OF LIMIT CYCLES BIFURCATING FROM A RANDOMLY PERTURBED CENTER

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We consider the average number of limit cycles that bifurcate from a randomly perturbed linear center where the perturbation consists of random (bivariate) polynomials with independent coefficients. This problem reduces, by way of classical perturbation theory of the Poincaré first return map, to a problem on the real zeros of a random *univariate* polynomial $f_n(x) = \sum_{m=0}^{n} c_m \xi_m x^m$ with independent coefficients ξ_m having mean zero, variance 1 and $c_m \sim m^{-1/2}$. This polynomial belongs to the class of *generalized Kac polynomials* at the critical regime. We provide asymptotics for the average number of real zeros and answer the question on bifurcating limit cycles. Additionally, we provide the correct order of the mean number of real roots in the subcritical regime.

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SOLVABLE MODELS IN THE KPZ CLASS: APPROACH THROUGH PERIODIC AND FREE BOUNDARY SCHUR MEASURES

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We explore probabilistic consequences of correspondences between q-Whittaker measures and periodic and free boundary Schur measures established by the authors in the recent paper, *Forum of Mathematics, Pi, 11:e27, 2023*. The result is a comprehensive theory of solvability of stochastic models in the KPZ class where exact formulas descend from mapping to explicit determinantal and Pfaffian point processes. We discover new variants of known results as determinantal formulas for the current distribution of the ASEP on the line and new results such as Fredholm–Pfaffian formulas for the distribution of the point-to-point partition function of the Log Gamma polymer model in half space. In the latter case, scaling limits and asymptotic analysis allow to establish Baik–Rains phase transition for the height function of the KPZ equation on the half line at the origin.

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REGULARITY OF THE SCHRAMM–LOEWNER EVOLUTION: UP-TO-CONSTANT VARIATION AND MODULUS OF CONTINUITY

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We find optimal (up to constant) bounds for the following measures for the regularity of the Schramm–Loewner evolution (SLE): variation regularity, modulus of continuity and law of the iterated logarithm. For the latter two, we consider the SLE with its natural parametrization. More precisely, denoting by $d \in (0,2]$ the dimension of the curve, we show the following:

1. The optimal ψ-variation is $\psi(x) = x^d(\log\log x^{-1})^{-(d-1)}$ in the sense that η is a.s. of finite ψ-variation for this ψ and not for any function decaying more slowly as $x \downarrow 0$.

2. The optimal modulus of continuity is $\omega(s) = c\,s^{1/d}(\log s^{-1})^{1-1/d}$, that is, for some random $c > 0$ we have $|\eta(t) - \eta(s)| \le \omega(t-s)$ a.s., while this does not hold for any function ω decaying faster as $s \downarrow 0$.

3. $\limsup_{t\downarrow 0} |\eta(t)|\,(t^{1/d}(\log\log t^{-1})^{1-1/d})^{-1}$ is a.s. equal to a deterministic constant in $(0,\infty)$.

We also show that the natural parametrization of SLE is given by the fine mesh limit of the ψ-variation. As part of our proof, we show that every stochastic process whose increments satisfy a particular moment condition attains a certain variation regularity.

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UNIVERSALITY OF GLOBAL ASYMPTOTICS OF JACK-DEFORMED RANDOM YOUNG DIAGRAMS AT VARYING TEMPERATURES

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This paper establishes universal formulas describing the global asymptotics of two distinct discrete versions of β-ensembles in the high, low and fixed temperature regimes. Our results affirmatively answer a question posed by the second author and Śniady.

We first introduce a special class of Jack measures on Young diagrams of arbitrary size, called the "Jack–Thoma measures," and prove the LLN and CLT in the three aforementioned limit regimes. In each case, we provide explicit formulas for polynomial observables of the limit shape and Gaussian fluctuations around the limit shape. These formulas have surprising positivity properties and are expressed as sums of weighted lattice paths. Second, we show that the previous formulas are universal: they also describe the limit shape and Gaussian fluctuations for the model of random Young diagrams of a fixed size derived from Jack characters with the approximate factorization property. Finally, in stark contrast with continuous β-ensembles, we show that the limit shapes at high and low temperatures of our random Young diagrams are one-sided infinite staircase shapes. For the Jack–Plancherel measure, we describe this shape explicitly by relating its local minima with the zeroes of Bessel functions.

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LARGE DEVIATIONS OF THE LARGEST EIGENVALUE OF SUPERCRITICAL SPARSE WIGNER MATRICES

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Consider a random symmetric matrix with i.i.d. entries on and above its diagonal that are products of Bernoulli random variables and random variables with sub-Gaussian tails. Such a matrix will be called a sparse Wigner matrix and can be viewed as the adjacency matrix of a random network with sub-Gaussian weights on its edges. In the regime where the mean degree is at least logarithmic in dimension, the edge eigenvalues of an appropriately scaled sparse Wigner matrix stick to the edges of the support of the semicircle law. We show that, in this sparsity regime, the large deviations upper tail event of the largest eigenvalue of a sparse Wigner matrix with sub-Gaussian entries is generated by either the emergence of a high degree vertex with a large vertex weight or that of a clique with large edge weights. Interestingly, the rate function obtained is discontinuous at the typical value of the largest eigenvalue, which accounts for the fact that its large deviation behaviour is generated by finite rank perturbations. This complements the results of Ganguly and Nam (*Probab. Theory Related Fields* **184** (2022) 613–679), and Ganguly, Hiesmayr, and Nam (*J. Lond. Math. Soc. (2)* **110** (2024) Paper No. e12954, 64), which considered the case where the mean degree is constant.

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