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THE GEOMETRY OF COALESCING RANDOM WALKS, THE BROWNIAN WEB DISTANCE AND KPZ UNIVERSALITY

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Coalescing simple random walks in the plane form an infinite tree. A natural directed distance on this tree is given by the number of jumps between branches when one is only allowed to move in one direction. The Brownian web distance is the scale-invariant limit of this directed metric. It is integer-valued and has scaling exponents $0:1:2$ as compared to $1:2:3$ in the KPZ world. However, we show that the shear limit of the Brownian web distance is still given by the Airy process. We conjecture that our limit theorem can be extended to the full directed landscape.

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ASYMPTOTICS OF THE FIRST-PASSAGE FUNCTION ON FREE AND FUCHSIAN GROUPS

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We derive explicit estimates for the asymptotics of the first-passage function for random walks on free groups supported on the powers of standard generators, and use them to prove the singularity of the hitting measure for a similarly defined class of random walks on Fuchsian groups.

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WASSERSTEIN DIFFUSION ON MULTIDIMENSIONAL SPACES

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Given any closed Riemannian manifold M, we construct a stochastic perturbation of the heat flow as a continuous Markov process on the space $\mathcal{P}(\mathsf{M})$ of probability measures on M, that is:

(1) reversible w.r.t. the entropic measure $\mathbb{P}^\beta$ on $\mathcal{P}(\mathsf{M})$, heuristically given as

$$d\mathbb{P}^\beta(\mu) = \frac{1}{Z} e^{-\beta \operatorname{Ent}(\mu|\mathfrak{m})}\, d\mathbb{P}^*(\mu);$$

(2) associated with a regular Dirichlet form with carré du champ derived from the Wasserstein gradient in the sense of Otto calculus

$$\mathcal{E}_{\mathsf{W}}(f) = \liminf_{g\to f} \frac{1}{2}\int_{\mathcal{P}(\mathsf{M})} \|\nabla_{\mathsf{W}} g\|^2(\mu)\, d\mathbb{P}^\beta(\mu);$$

(3) nondegenerate, at least in the case of the n-sphere and the n-torus.

And yet it moves.
Galileo Galilei, 1633

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ARM EXPONENT FOR THE GAUSSIAN FREE FIELD ON METRIC GRAPHS IN INTERMEDIATE DIMENSIONS

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We investigate the bond percolation model on transient weighted graphs G induced by the excursion sets of the Gaussian free field on the corresponding metric graph. We assume that balls in G have polynomial volume growth with growth exponent α and that the Green's function for the random walk on G exhibits a power law decay with exponent ν, in the regime $1 \leq \nu \leq \frac{\alpha}{2}$. In particular, this includes the cases of $G = \mathbb{Z}^3$ for which $\nu = 1$, and $G = \mathbb{Z}^4$ for which $\nu = \frac{\alpha}{2} = 2$. For all such graphs, we determine the leading-order asymptotic behavior for the critical one-arm probability, which we prove decays with distance R, like $R^{-\frac{\nu}{2}+o(1)}$. Our results are, in fact, more precise and yield logarithmic corrections when $\nu > 1$ as well as corrections of order $\log\log R$ when $\nu = 1$. We further obtain very sharp upper bounds on truncated two-point functions close to criticality, which are new when $\nu > 1$ and essentially optimal when $\nu = 1$. This extends previous results from (*Invent. Math.* **232** (2023) 229–299; *Ann. Probab.* **48** (2020) 1411–1435).

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EXTREMAL RANDOM MATRICES WITH INDEPENDENT ENTRIES AND MATRIX SUPERCONCENTRATION INEQUALITIES

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We prove nonasymptotic matrix concentration inequalities for the spectral norm of (sub)Gaussian random matrices with centered independent entries that capture fluctuations at the Tracy–Widom scale. This considerably improves previous bounds in this setting, due to Bandeira and Van Handel, and establishes the best possible tail behavior for random matrices with an arbitrary variance pattern. These bounds arise from an extremum problem for nonhomogeneous random matrices: among all variance patterns with a given sparsity parameter, the random matrix moments are maximized by block-diagonal matrices with i.i.d. entries in each block. As part of the proof, we obtain sharp bounds on large moments of Gaussian Wishart matrices.

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LARGE DEVIATIONS FOR 2D NAVIER–STOKES EQUATIONS DRIVEN BY A PERIODIC FORCE AND A DEGENERATE NOISE

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We consider the incompressible 2D Navier–Stokes equations on the torus, driven by a deterministic time periodic force and a noise that is white in time and degenerate in Fourier space. The main result is twofold.

First, we establish a Ruelle–Perron–Frobenius type theorem for the time inhomogeneous Feynman–Kac evolution operators with regular potentials associated with the stochastic Navier–Stokes system. The theorem characterizes asymptotic behaviors of the Feynman–Kac operators in terms of the periodic family of principal eigenvalues and corresponding unique eigenvectors. The proof involves a time inhomogeneous version of Ruelle's lower bound technique.

Second, utilizing this Ruelle–Perron–Frobenius type theorem and a Kifer's criterion, we establish a Donsker–Varadhan type large deviation principle with a nontrivial good rate function for the occupation measures of the time inhomogeneous solution processes.

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LOCALITY FOR SINGULAR STOCHASTIC PDES

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This work deals with singular stochastic PDEs driven by nontranslation invariant differential operators. We describe the renormalized equation for a very large class of spacetime dependent renormalization schemes. Our approach bypasses in particular the use of decorated trees with extended decorations.

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FRACTAL GEOMETRY OF THE PARABOLIC ANDERSON MODEL IN 2D AND 3D WITH WHITE NOISE POTENTIAL

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We study the parabolic Anderson model (PAM)

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\xi(x) & t > 0, x \in \mathbb{R}^d, \\ u(0,x) \equiv 1 & x \in \mathbb{R}^d, \end{cases}$$

where ξ is spatial white noise on $\mathbb{R}^d$ with $d \in \{2, 3\}$. We show that the peaks of the PAM are macroscopically multifractal. More precisely, we prove that the spatial peaks of the PAM have infinitely many distinct values, and we compute the macroscopic Hausdorff dimension (introduced by Barlow and Taylor (*J. Phys. A: Math. Gen.* **22** (1989) 2621–2628; *Proc. Lond. Math. Soc. (3)* **64** (1992) 125–152)) of those peaks. As a byproduct, we obtain the exact spatial asymptotics of the solution of the PAM at any fixed, sufficiently large time. We also study the spatiotemporal peaks of the PAM and show their macroscopic multifractality. Some of the major tools used in our proof techniques include paracontrolled calculus and tail probabilities of the largest point in the spectrum of the *Anderson Hamiltonian*.

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IDEAL POISSON–VORONOI TESSELLATIONS ON HYPERBOLIC SPACES

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We study the limit in low intensity of Poisson–Voronoi tessellations in hyperbolic spaces $\mathbb{H}_d$ for $d \geq 2$. In contrast to the Euclidean setting, a limiting nontrivial ideal tessellation $\mathcal{V}_d$ appears as the intensity tends to 0. The tessellation $\mathcal{V}_d$ is a natural, isometry-invariant decomposition of $\mathbb{H}_d$ into countably many unbounded polytopes, each with a unique end. We study its basic properties, in particular, the geometric features of its cells.

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TRIVIALITY OF THE SCALING LIMITS OF CRITICAL ISING AND φ^4 MODELS WITH EFFECTIVE DIMENSION AT LEAST FOUR

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We prove that any scaling limit of a critical reflection positive Ising or φ^4 model of effective dimension d_{eff} at least four is Gaussian. This extends the recent breakthrough work of Aizenman and Duminil-Copin (*Ann. of Math. (2)* **194** (2021) 163–235)—which demonstrates the corresponding result in the setup of nearest-neighbour interactions in dimension four—to the case of long-range reflection positive interactions satisfying $d_{\text{eff}} = 4$. The proof relies on the random current representation which provides a geometric interpretation of the deviation of the models' correlation functions from Wick's law. When $d = 4$, long-range interactions are handled with the derivation of a criterion that relates the speed of decay of the interaction to two different mechanisms that entail Gaussianity: interactions with a sufficiently slow decay induce a faster decay at the level of the model's two-point function, while sufficiently fast decaying interactions force a simpler geometry on the currents which allows to extend nearest-neighbour arguments. When $1 \le d \le 3$ and $d_{\text{eff}} = 4$, the phenomenology is different, as long-range effects play a prominent role.

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KRONECKER-PRODUCT RANDOM MATRICES AND A MATRIX LEAST SQUARES PROBLEM

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We study the eigenvalue distribution and resolvent of a Kronecker-product random matrix model $A \otimes I_{n\times n} + I_{n\times n} \otimes B + \Theta \otimes \Xi \in \mathbb{C}^{n^2\times n^2}$, where A, B are independent Wigner matrices and Θ, Ξ are deterministic and diagonal. For fixed spectral arguments, we establish a quantitative approximation for the Stieltjes transform by that of an approximating free operator and a diagonal deterministic equivalent approximation for the resolvent. We further obtain sharp estimates in operator norm for the $n \times n$ resolvent blocks and show that off-diagonal resolvent entries fall on two differing scales of $n^{-1/2}$ and n^{-1}, depending on their locations in the Kronecker structure.

Our study is motivated by consideration of a matrix-valued least-squares optimization problem $\min_{X\in\mathbb{R}^{n\times n}} \frac{1}{2}\|XA + BX\|_F^2 + \frac{1}{2}\sum_{ij} \xi_i\theta_j x_{ij}^2$ subject to a linear constraint. For random instances of this problem defined by Wigner inputs A, B, our analyses imply an asymptotic characterization of the minimizer X and its associated minimum objective value as $n \to \infty$.

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STRONG TOPOLOGICAL TRIVIALIZATION OF MULTI-SPECIES SPHERICAL SPIN GLASSES

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We study the landscapes of multi-species spherical spin glasses. Our results determine the phase boundary for annealed trivialization of the number of critical points and establish its equivalence with a quenched *strong topological trivialization* property. Namely, in the "trivial" regime, the number of critical points is constant, all are well conditioned, and all *approximate* critical points are close to a true critical point. As a consequence, we deduce that Langevin dynamics at sufficiently low temperature has logarithmic mixing time.

Our approach begins with the Kac–Rice formula. We characterize the annealed trivialization phase by explicitly solving a suitable multidimensional variational problem, obtained by simplifying certain asymptotic determinant formulas from (*Probab. Math. Phys.* **3** (2022) 731–789; *Ann. Inst. Henri Poincaré Probab. Stat.* **60** (2024) 636–657). To obtain more precise quenched results, we develop general purpose techniques to avoid subexponential correction factors and show nonexistence of *approximate* critical points. Many of the results are new, even in the *one*-species case.

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